Kleene algebra to compute with invariant sets of dynamical systems

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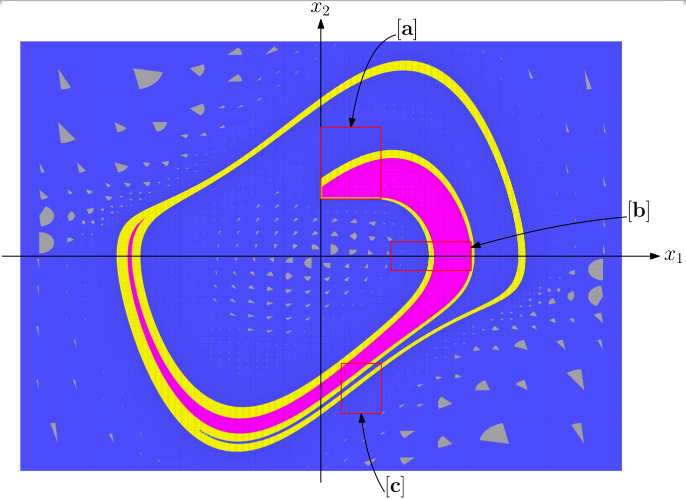
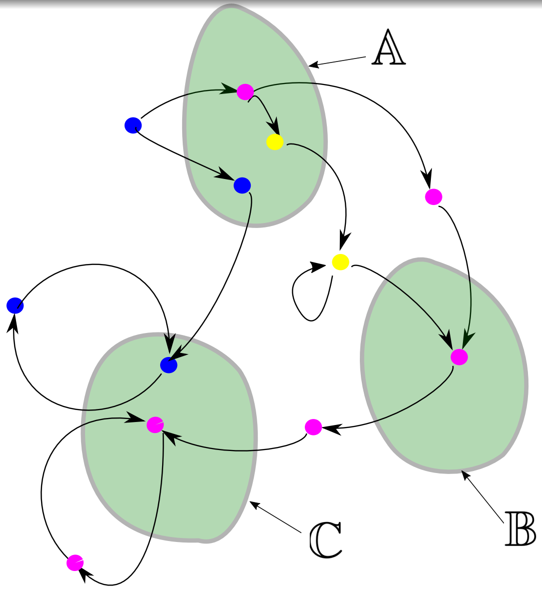
Motivation [4][3]


x_2
[a]
[b]
x_1
[c]


A
B
C

Consider the system

$$\mathscr{S} : \dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t))$$

Denote by $\varphi_\gamma(t, \mathbf{x})$ the flow map.

Positive invariant set

A set $\mathbb{A}$ is *positive invariant* [1] if

$$\mathbf{x} \in \mathbb{A}, t \geq 0 \Longrightarrow \varphi(t, \mathbf{x}) \in \mathbb{A}.$$

Or equivalently

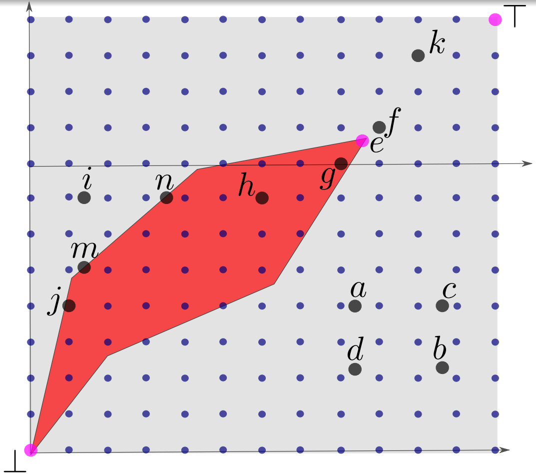
$$\varphi_\gamma([0,\infty], \mathbb{A}) \subset \mathbb{A}.$$

The set of all positive invariant sets is a complete lattice.

Kleene algebra

Lattice

A *lattice* $(\mathscr{L}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds [2].
A *machine lattice* $(\mathscr{L}_M, \leq)$ of $\mathscr{L}$ is complete sublattice of $(\mathscr{L}, \leq)$ which is finite. Moreover both $\mathscr{L}$ and $\mathscr{L}_M$ have the same top and bottom.



Machine lattice

Kleene algebra

Kleene algebra	$(\mathscr{K},+,\cdot,*)$
Addition	$a+b$
Product	$a\cdot b$
Associativity	$a+(b+c)=(a+b)+c$
	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$
Commutativity	$a+b=b+a$
Distributivity	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
	$(b+c)\cdot a=(b\cdot a)+(c\cdot a)$
zero	$a+0=a$
One	$a\cdot 1=1\cdot a=a$
Annihilation	$a\cdot 0=0\cdot a=0$
Idempotence	$a+a=a$
Partial order	$a\leq b\Leftrightarrow a+b=b$
Kleene star	$a^{*}=1+a+a\cdot a+a\cdot a\cdot a+\ldots$

Proposition. $+, \cdot$ are monotonic

Proof. Assume that $a_1 \geq a$, i.e., $a_1 = a + a_1$.

(i) $a_1 + b \geq a + b \Leftrightarrow a_1 + b = a_1 + b + a + b \Leftrightarrow \text{true}$.

(ii) $a_1 \cdot b \geq a \cdot b \Leftrightarrow a_1 \cdot b = a_1 \cdot b + a \cdot b \Leftrightarrow a_1 \cdot b = (a_1 + a) \cdot b \Leftrightarrow \text{true}$.

Proposition. We have

$$(1+a)^{\infty} = a^*.$$

Proof.

$$(1+a)^2 \quad = \quad (1+a)\cdot(1+a) = 1 + 1\cdot a + a\cdot 1 + a^2 = 1 + a + a^2$$

and recursively:

$$(1+a)^{\infty} = 1 + a + a^2 + a^3 \cdots = a^*.$$

A Kleene algebra $\mathscr{K}$ $(\leq, +, \cdot, *, 0, 1)$ is a lattice with respect to the relation order $\leq$.
We can also define the machine Kleene algebra $(\mathscr{K}_M, \leq)$ of $\mathscr{K}$.

An interval of $\mathscr{K}(\leq,+,\cdot,*,0,1)$ as a subset $[a]$ of $\mathscr{K}$ which can be written as

$$[a] = [a^-, a^+] = \left\{ a \in \mathscr{K} \mid a^- \leq a \leq a^+ \right\}$$

where a^-, a^+ belong to $\mathscr{K}_M$.
Note that both $\emptyset$ and $\mathscr{K}$ are intervals of $\mathscr{K}$.

If $a \in [a] = [a^-, a^+], b \in [b] = [b^-, b^+]$, we have

$$\begin{array}{rcl} [a^-, a^+]^* & = & \left[(a^-)^*, (a^+)^*\right] \\ [a^-, a^+] + [b^-, b^+] & = & [a^- + b^-, a^+ + b^+] \\ [a^-, a^+] \cdot [b^-, b^+] & = & [a^- \cdot b^-, a^+ \cdot b^+]. \end{array}$$

Automorphism

Given a lattice $(\mathscr{L}, \wedge, \vee, \perp, \top)$, an *automorphism* of $\mathscr{L}$ is a function $f : \mathscr{L} \to \mathscr{L}$ such that

$$\begin{array}{lccc} \text{(i)} & f(\top) & = & \top \\ \text{(ii)} & f(a \wedge b) & = & f(a) \wedge f(b) \end{array}$$

We denote by $\mathscr{A}(\mathscr{L})$ the set of automorphism of $\mathscr{L}$.

Proposition. If f, g are in $\mathscr{A}(\mathscr{L})$, then:

$$
\begin{array}{lc}
\text{(i)} & a \leq b \Rightarrow f(a) \leq f(b) \\
\text{(ii)} & f \wedge g \in \mathscr{A}(\mathscr{L}) \\
\text{(iii)} & f \circ g \in \mathscr{A}(\mathscr{L}) \\
\text{(iv)} & \mathrm{Id} \wedge f \wedge f^2 \wedge f^3 \wedge \cdots \in \mathscr{A}(\mathscr{L})
\end{array}
$$

Proposition. If $f^* = \mathrm{Id} \wedge f \wedge f^2 \wedge f^3 \wedge \dots$, the set $(\mathscr{A}(\mathscr{L}), \wedge, \circ, *)$ is a Kleene algebra.

Kleene algebra	$(\mathscr{A}(\mathscr{L}),\wedge,\circ,*)$
Addition	$f\wedge g$
Product	$f\circ g$
Associativity	$f\wedge(g\wedge h)=(f\wedge g)\wedge h$
	$f\circ(g\circ h)=(f\circ g)\circ h$
Commutativity	$f\wedge g=f\wedge g$
Distributivity	$f\circ(g\wedge h)=(f\circ g)\wedge(f\circ h)$
	$(g\wedge h)\circ f=(g\circ f)\wedge(h\circ f)$
zero	$f\wedge\top=f$
One	$f\circ\mathrm{Id}=\mathrm{Id}\circ f=f$
Annihilation	$f\circ\top=\top$
Idempotency	$f\wedge f=f$
Partial order	$f\geq g\Leftrightarrow f\wedge g=g$
Kleene star	$f^*=\mathrm{Id}\wedge f\wedge f^2\wedge f^3\wedge\dots$

Proof. For instance, to prove the distributivity $f \circ (g \wedge h) = (f \circ g) \wedge (f \circ h)$ we proceed as follows:

$$\begin{aligned} f \circ (g \wedge h)(a) &= f \circ (g(a) \wedge h(a)) \\ &= (f \circ g)(a) \wedge (f \circ h)(a). \end{aligned}$$

We have $(\mathrm{Id} \wedge f)^{\infty} = f^{*}$, , *i.e.*,

$$\mathrm{Fix}\,(f^{*}) = \{a \,|\, f^{*}(a) = a\} = \mathrm{Fix}\,(\mathrm{Id} \wedge f)$$

Factorization

We want to compute expressions, such as

$$f^*(a) \wedge (g^*(b) \vee h^*(a))^*.$$

We have

$$f^* \wedge f^* = f^*$$
$$(f^*)^* = f^*$$
$$(f^* \wedge g^*)^* = (f \wedge g)^*$$
$$f^* \circ (f \circ g^*)^* = (f \wedge g)^*$$

but we can do more.

Assume for instance that we have to compute

$$f^*(a) \wedge g^*(b).$$

Intervals

Given a lattice $(\mathscr{L}, \wedge, \circ)$ and an automorphism $f \in \mathscr{A}(\mathscr{L})$, we want to compute $f^*(a)$ where $a \in \mathscr{L}$.
We consider a machine sublattice $\mathscr{L}_M$ of $\mathscr{L}$.
Since $\mathscr{A}(\mathscr{L})$ is a Kleene algebra, we can define intervals.

An interval of $\mathscr{A}(\mathscr{L})$ is a subset $[f]$ of $\mathscr{A}(\mathscr{L})$ which can be written as

$$[f] = [f^-, f^+] = \left\{ f \in \mathscr{A}(\mathscr{L}) \,|\, f^- \leq f \leq f^+ \right\}$$

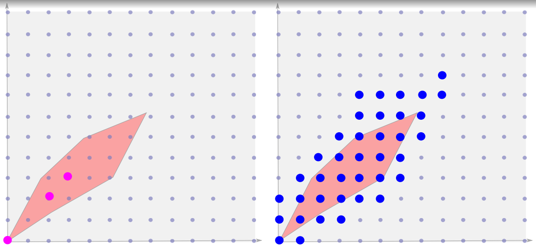
where f^-, f^+ belong to $\mathscr{A}(\mathscr{L}_M)$.

We have

$$\mathrm{Fix}\left((f^{-})^{*}\right) \subset \mathscr{L}_M \cap \mathrm{Fix}\left(f^{*}\right) \subset \mathrm{Fix}\left((f^{+})^{*}\right).$$

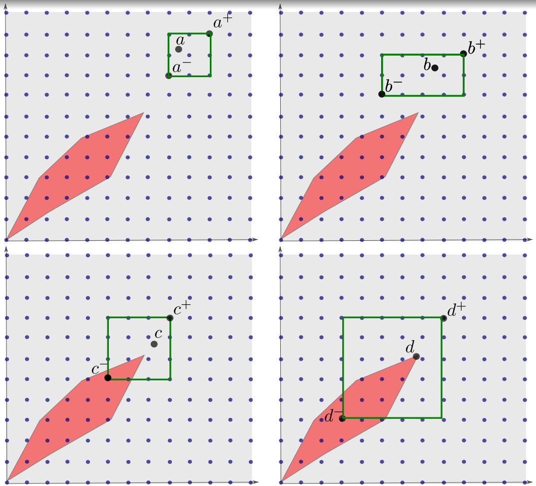
Theorem. If $a \in [a^-, a^+]$, where a^-, a^+ both belong to $\mathscr{L}_M$, then

$$\begin{array}{ll} \text{(i)} & f^*(a) \in \left[(f^-)^*(a^-), (f^+)^*(a^+)\right] \\ \text{(ii)} & f^* \circ (f^-)^*(a^-) = (f^-)^*(a^-) \\ \text{(iii)} & f^*(a) \leq (\mathrm{Id} \wedge f^+)^i(a^+),\ \forall i \geq 0 \end{array}$$



$$\mathrm{Fix}\left((f^-)^*\right) \text{ and } \mathrm{Fix}\left((f^+)^*\right)$$

Algorithm


a
a+
a−
b
b+
b−
c
c+
c−
d
d+
d−

The powerset $(\mathscr{P}(\mathbb{R}^n), \cap, \cup)$ is a lattice.
We want to find $\overrightarrow{f}$ in $(\mathscr{A}(\mathscr{P}(\mathbb{R}^n)), \cap, \circ, *)$ such that

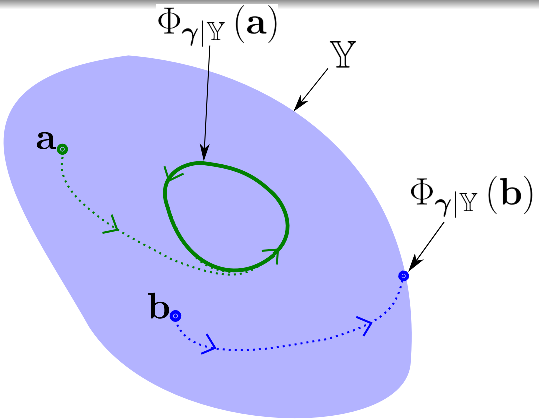
$$\overrightarrow{f}^*(\mathbb{A}) = \left\{\mathbf{a} \in \mathbb{A} \,|\, \forall t \geq 0, \varphi_\gamma(t, \mathbf{a}) \in \mathbb{A}\right\}$$

The *attractor* associated to **a** is

$$\Phi_\gamma(\mathbf{a}) = \bigcap_{t \geq 0} \varphi_\gamma([t,\infty],\mathbf{a}).$$

If $\mathbf{a} \in \mathbb{Y}$, the attractor inside $\mathbb{Y}$ is $\Phi_{\gamma|\mathbb{Y}}(\mathbf{a})$ where the function $\gamma|\mathbb{Y}$ is defined as

$$(\gamma|\mathbb{Y})(\mathbf{x}) = \begin{cases} \gamma(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbb{Y} \\ \mathbf{0} & \text{otherwize} \end{cases}$$


$\Phi_{\boldsymbol{\gamma}|\mathbb{Y}}(\mathbf{a})$
$\mathbb{Y}$
$\mathbf{a}$
$\Phi_{\boldsymbol{\gamma}|\mathbb{Y}}(\mathbf{b})$
$\mathbf{b}$

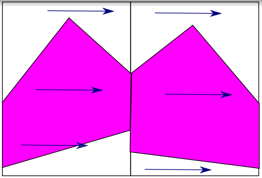
Given a paving $\mathscr{P}$, we denote by $\mathscr{P}(\mathbf{x})$ the union of all $[\mathbf{x}]$ in $\mathscr{P}$ containing $\mathbf{x}$.

Theorem. Consider the system $\mathscr{S} : \dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t))$ and a paving $\mathscr{P}$ of the state space,

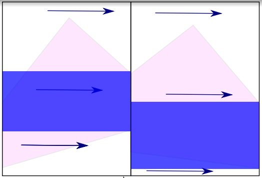
$$\overrightarrow{f}(\mathbb{A}) = \left\{\mathbf{x} \mid \Phi_{\gamma|\mathscr{P}(\mathbf{x})}(\mathbf{x}) \subset \mathbb{A}\right\}.$$

is an automorphism. Moreover,

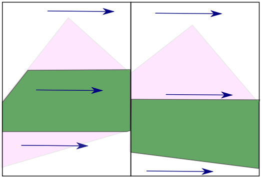
$$\overrightarrow{f}^*(\mathbb{A}) = \mathrm{Inv}^+(\gamma, \mathbb{A}) = \left\{\mathbf{a} \mid \forall t \geq 0, \varphi_\gamma(t, \mathbf{a}) \in \mathbb{A}\right\}.$$



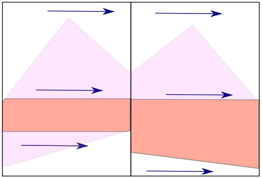
$\mathbb{A}$



$\overrightarrow{f}(\mathbb{A})$



$\mathbb{A} \cap \overrightarrow{f}(\mathbb{A})$



$\mathbb{A} \cap \overrightarrow{f}(\mathbb{A}) \cap \overrightarrow{f}^{2}(\mathbb{A})$

The set-valued function

$$\overleftarrow{f}(\mathbb{A}) = \left\{\mathbf{x} \,|\, \Phi_{-\gamma|\mathscr{P}(\mathbf{x})}(\mathbf{x}) \subset \mathbb{A}\right\}$$

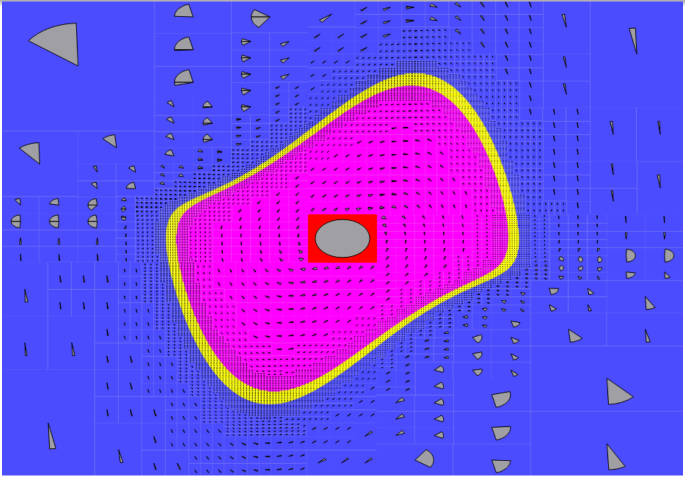
is an automorphism. Moreover $\overleftarrow{f}^*(\mathbb{A})$ corresponds to the largest negative invariant subset of $\mathbb{A}$, i.e.,

$$\overleftarrow{f}^*(\mathbb{A}) = \mathrm{Inv}^+(-\gamma, \mathbb{A}) = \left\{\mathbf{a} \mid \forall t \geq 0, \varphi_\gamma(-t, \mathbf{a}) \in \mathbb{A}\right\}.$$

Consider the system described by the state equation:

$$\begin{cases} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= -\left(1 - x_1^2\right) \cdot x_2 + x_1 . \end{cases}$$

To compute $\mathrm{Inv}^+(-\gamma, \mathbb{A})$, we evaluate $\left[\overrightarrow{f}^*\right](\mathbb{A})$.



The forward reach set of $\mathbb{A}$ is defined by

$$\begin{aligned}
\mathrm{Forw}(\mathbb{A}) &= \overline{\left\{\mathbf{x} \mid \exists t \geq 0, \varphi_\gamma(-t,\mathbf{x}) \in \mathbb{A}\right\}} \\
&= \overline{\left\{\mathbf{x} \mid \forall t \geq 0, \varphi_\gamma(-t,\mathbf{x}) \in \overline{\mathbb{A}}\right\}} \\
&= \overline{\overleftarrow{f}^{\,*}(\overline{\mathbb{A}})}
\end{aligned}$$

In a Boolean lattice $\mathscr{L}$, every a has a unique complement $\overline{a}$, *i.e.*, $a \vee \overline{a} = \top$ and $a \wedge \overline{a} = \bot$.
We have:

$$\begin{array}{ll} \text{(i)} & a \leq b \Leftrightarrow \overline{b} \leq \overline{a} \\ \text{(ii)} & \overline{a \vee b} = \overline{b} \wedge \overline{a} \\ \text{(iii)} & \overline{a \wedge b} = \overline{b} \vee \overline{a} \end{array}$$

Two automorphisms f, g are dual if

$$\left(\mathrm{Id} \wedge f \wedge f^2 \wedge f^3 \wedge \dots\right)(\overline{a}) = \overline{\left(\mathrm{Id} \vee g \vee g^2 \vee g^3 \vee \dots\right)(a)}.$$

and reciproquely.
The functions $\overleftarrow{f}, \overrightarrow{f}$ are dual.

Finding the smallest fixed point

Given a, we want to compute

$$x = a \vee f(a) \vee f^2(a) \vee \dots$$

using duality in Boolean lattices.

$$\begin{aligned} \overline{x} &= \overline{a \vee f(a) \vee f^2(a) \vee \dots} \\ &= \left(\mathrm{Id} \wedge g \wedge g^2 \wedge g^3 \wedge \dots\right)(\overline{a}) \\ &= g^*(\overline{a}) \end{aligned}$$

Therefore

$$x = \overline{g^*(\overline{a})}.$$

Backward reach set

The backward reach set is:

$$
\begin{aligned}
\mathrm{Back}(\mathbb{A}) &= \overline{\left\{\mathbf{x} \mid \exists t \geq 0, \varphi_\gamma(t,\mathbf{x}) \in \mathbb{A}\right\}} \\
&= \left\{\mathbf{x} \mid \forall t \geq 0, \varphi_\gamma(t,\mathbf{x}) \in \overline{\mathbb{A}}\right\} \\
&= \overline{\overrightarrow{f}^{\,*}\left(\overline{\mathbb{A}}\right)}
\end{aligned}
$$

Between reach set

Given two sets $\mathbb{A},\mathbb{B}$, the between reach set is

$$
\begin{array}{rcl}
\mathrm{Betw}(\mathbb{A},\mathbb{B}) & = & \left\{\mathbf{x} \mid \exists t_1 \geq 0, \varphi_\gamma(-t_1,\mathbf{x}) \in \mathbb{A}, \exists t_2 \geq 0, \varphi_\gamma(t,\mathbf{x}) \in \mathbb{B}\right\} \\
 & = & \mathrm{Forw}(\mathbb{A}) \cap \mathrm{Back}(\mathbb{B}) \\
 & = & \overline{\overleftarrow{f}^{*}(\overline{\mathbb{A}})} \cap \overline{\overrightarrow{f}^{*}(\overline{\mathbb{B}})}.
\end{array}
$$

Control reach set

Consider the system:

$$\mathscr{S} : \dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t), u),\ u \in \{0,1\}$$

We want to compute the largest set $\mathbb{X}$ that can be reached from the set $\mathbb{A}$, i.e.,

$$\mathbb{X} \quad = \quad (\mathrm{Forw}_{u=1} \circ \mathrm{Forw}_{u=0})^{\infty}(\mathbb{A}).$$

It is the limit of

$$\begin{array}{rcl} \mathbb{X}(k+1) & = & \mathrm{Forw}_{u=1}\left(\mathrm{Forw}_{u=0}\left(\mathbb{X}(k)\right)\right) \\ \mathbb{X}(0) & = & \mathbb{A} \end{array}$$

Thus

$$\begin{array}{rcl} \overline{\mathbb{X}}(k+1) & = & \overleftarrow{f_1}^{*}\left(\overline{\mathrm{Forw}_{u=0}\left(\mathbb{X}(k)\right)}\right) \\ & = & \overleftarrow{f_1}^{*}\left(\overline{\overline{\overleftarrow{f_0}^{*}\left(\overline{\mathbb{X}(k)}\right)}}\right) \\ & = & \overleftarrow{f_1}^{*}\circ\overleftarrow{f_0}^{*}\left(\overline{\mathbb{X}(k)}\right) \end{array}$$

Therefore

$$\overline{\mathbb{X}} = \lim_{k\to\infty} \overline{\mathbb{X}}(k) = \left(\overleftarrow{f_1}^* \circ \overleftarrow{f_0}^*\right)^* \left(\overline{\mathbb{A}}\right) = \left(\overleftarrow{f_1} \circ \overleftarrow{f_0}\right)^* \left(\overline{\mathbb{A}}\right).$$

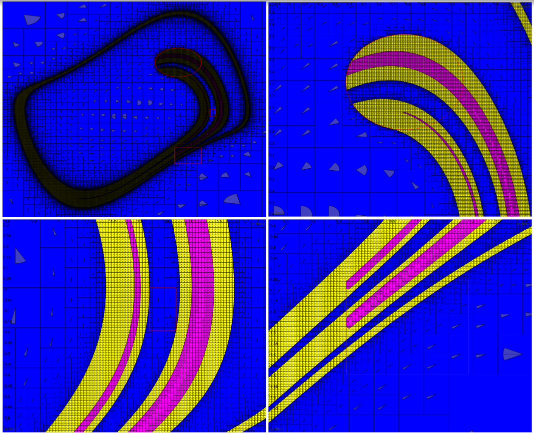
Finally

$$\mathbb{X} = \overline{\left(\overleftarrow{f_1} \circ \overleftarrow{f_0}\right)^* \left(\overline{\mathbb{A}}\right)}$$

Path planning reach set

We want the set $\mathbb{X}$ of all paths that start in $\mathbb{A}$, avoid $\mathbb{B}$ and reach $\mathbb{C}$:

$$
\begin{aligned}
\mathbb{X} &= \mathrm{Betw}_{\gamma|\overline{\mathbb{B}}}(\mathbb{A},\mathbb{C}) \\
&= \left(\overline{\overleftarrow{f_{\gamma|\overline{\mathbb{B}}}}^{*}\left(\overline{\mathbb{A}}\right)} \cap \overline{\overrightarrow{f_{\gamma|\overline{\mathbb{B}}}}^{*}\left(\overline{\mathbb{C}}\right)}\right)
\end{aligned}
$$



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