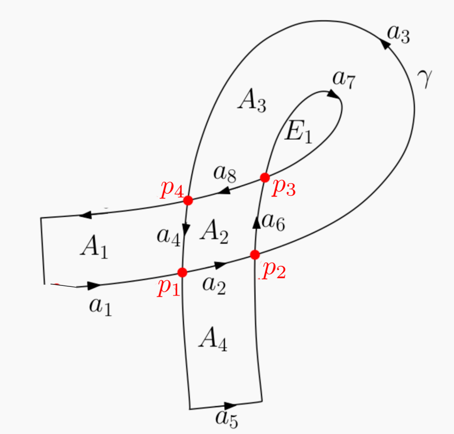
Isowinding method

Brest
2022, April 26

Introduction to homology

See the video of Wildburger : *Introduction to Homology*
https://youtu.be/ShWdSNJeuOg


a_3
γ
a_7
A_3
E_1
a_8
p_4
p_3
a_6
a_4
A_2
A_1
p_2
p_1
a_2
a_1
A_4
a_5

Remark. $a_1 + a_5 + a_6 + a_8$ is a cycle

$$\partial(a_1 + a_5 + a_6 + a_8) \quad = \quad 0$$

To compute H_1, we form the boundary equation:

$$a_1 + a_5 + a_6 + a_8 \quad = \quad 0$$

We have

$$\begin{aligned}
\partial a_1 &= p_1 - p_4 \\
\partial a_2 &= p_2 - p_1 \\
\partial a_3 &= p_4 - p_2 \\
\partial a_4 &= p_1 - p_4 \\
\partial a_5 &= p_2 - p_1 \\
\partial a_6 &= p_3 - p_2 \\
\partial a_7 &= 0 \\
\partial a_8 &= p_4 - p_3
\end{aligned}$$

$$\underbrace{\partial \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix}}_{\mathbf{a}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{\mathbf{B}} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

Since ∂ is a group homomorphism,

$$\partial\left(n_1 a_1 + n_2 a_2 + \cdots + n_6 a_6\right) = n_1 \partial a_1 + n_2 \partial a_2 + \cdots + n_6 \partial a_6$$

where $n_i \in \mathbb{N}$.

Equivalently

$$\partial(\mathbf{n}^{\mathsf{T}}\mathbf{a}) = \mathbf{n}^{\mathsf{T}} \cdot \mathbf{B} \cdot \mathbf{p}$$

The vector $\mathbf{n}$ generates a cycle if

$$\partial(\mathbf{n}^{\mathsf{T}}\mathbf{a}) = \mathbf{0} \Leftrightarrow \mathbf{n}^{\mathsf{T}} \cdot \mathbf{B} = 0$$

To get the cycles, we have to compute the *integer* left kernel $\mathbf{K}$ of $\mathbf{B}$.

https://sagecell.sagemath.org/
B = Matrix([[1,0,0,-1],[-1,1,0,0],[0,-1,0,1],[1,0,0,-1],[-1,1,0,0],[0,-1,1,0],[0,0,0,0],[0,0,-1,1]])
B.left_kernel()
We get

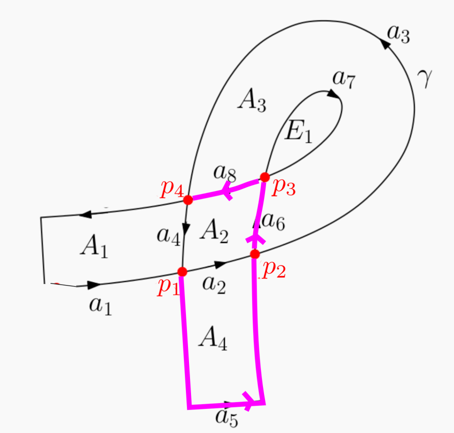
$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The generating cycle are:

$$
\begin{aligned}
&a_1 + a_5 + a_6 + a_8 \\
&a_2 - a_5 \\
&a_3 - a_6 - a_8 \\
&a_4 + a_5 + a_6 + a_8 \\
&a_7
\end{aligned}
$$

Following homology theory, we would say that
$H_1(\mathbb{X}) = \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$.
Now, what is the manifold $\mathbb{X}$ here ?

Spanning tree method

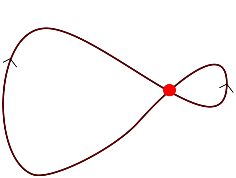
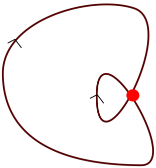

a_3
γ
a_7
A_3
E_1
a_8
p_4
p_3
a_6
a_4
A_2
A_1
p_2
p_1
a_2
a_1
A_4
a_5

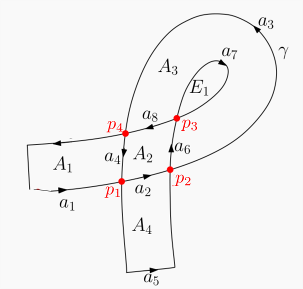
We read directly the generating cycle are:

$$a_1 + a_5 + a_6 + a_8$$
$$a_2 - a_5$$
$$a_3 - a_6 - a_8$$
$$a_4 + a_5 + a_6 + a_8$$
$$a_7$$

The number of generating cycles is
8 edges - (4 vertices-1) $= 5$

Problem with this approach : We loose necessary features for the winding numbers.
Is it a problem in our sonar context?

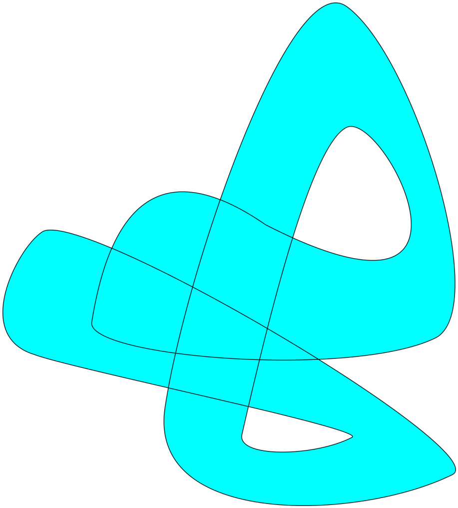
 

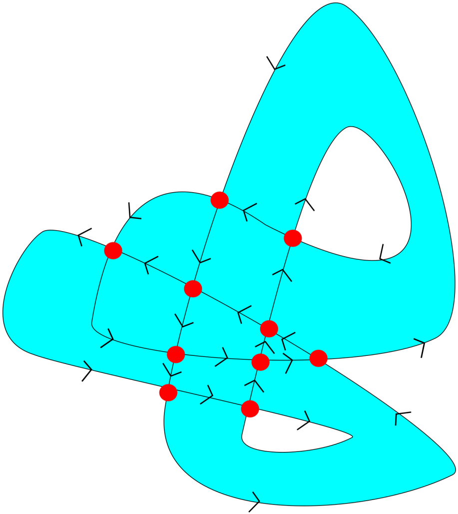


Moreover, two cells are of interest : A_1 and $A_1 \cup A_2 \cup A_3 \cup A_4$.

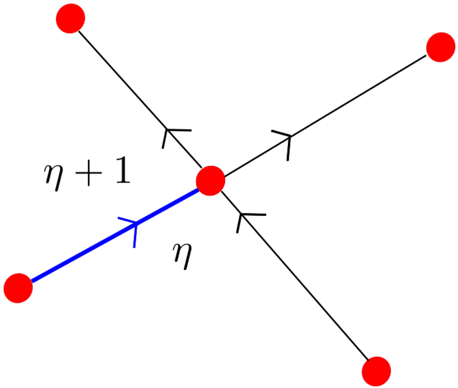
Isowinding

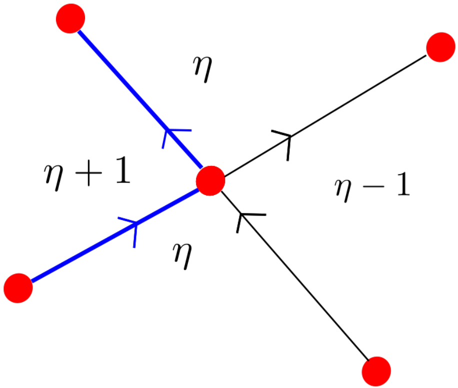
We search for isowinding cycles.



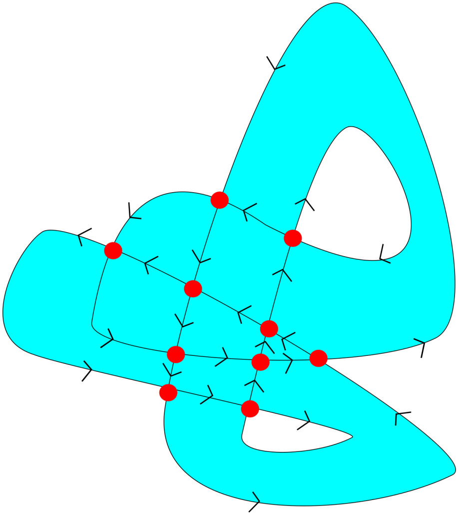


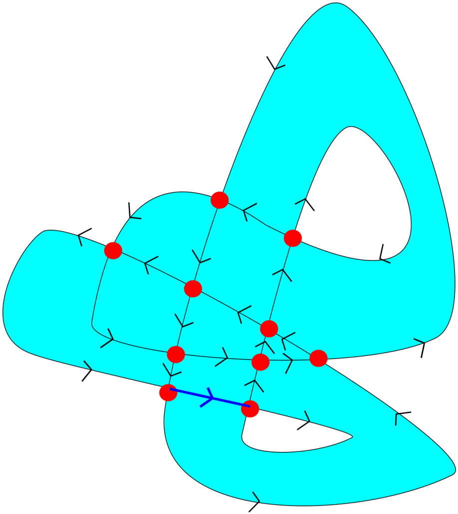
20 edges, 10 vertices
$20-(10-1)=11$ independent cycles.
We do not need all 11 cycles for our sonar goal.
5 cycles are enough !

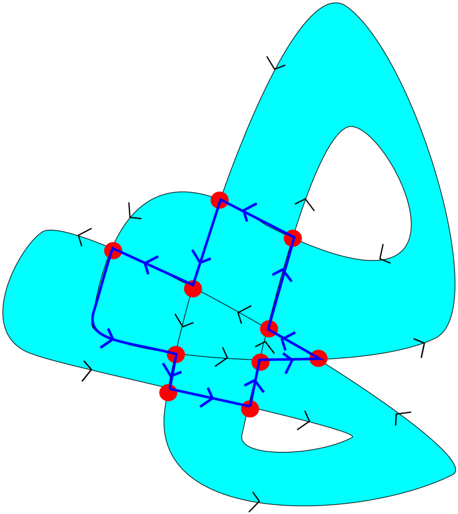

$\eta + 1$
η

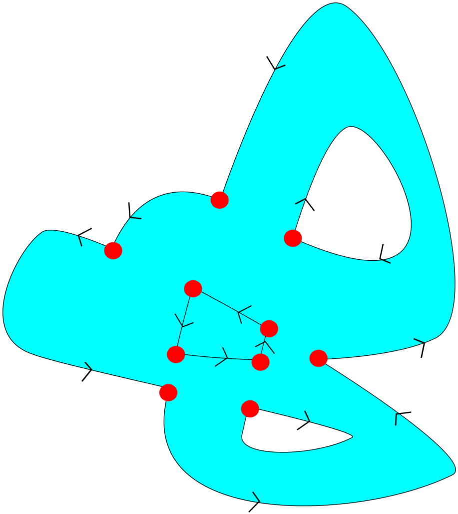

η
$\eta + 1$
$\eta - 1$
η

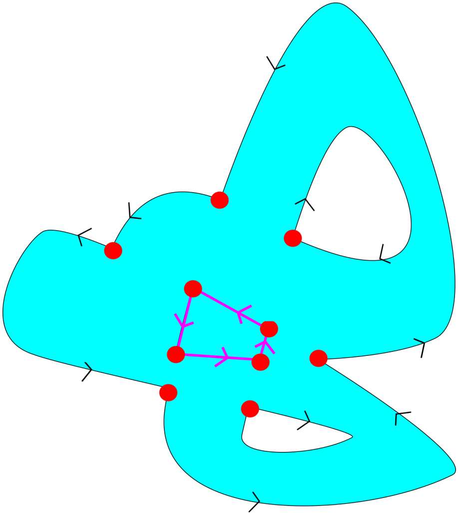
Isowinding following

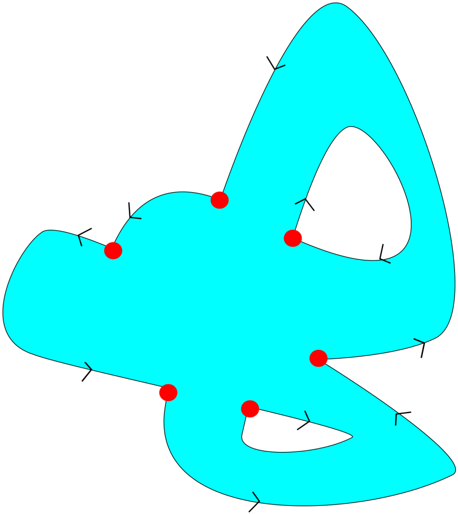


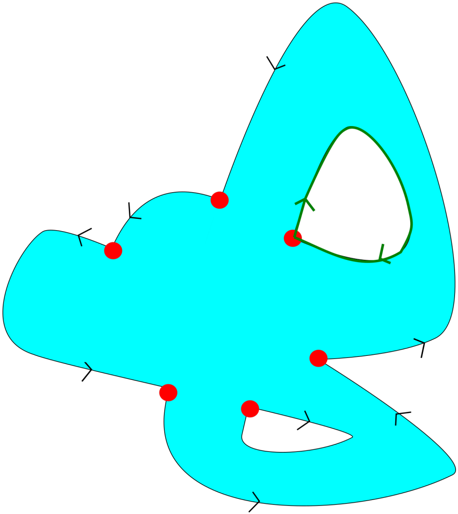


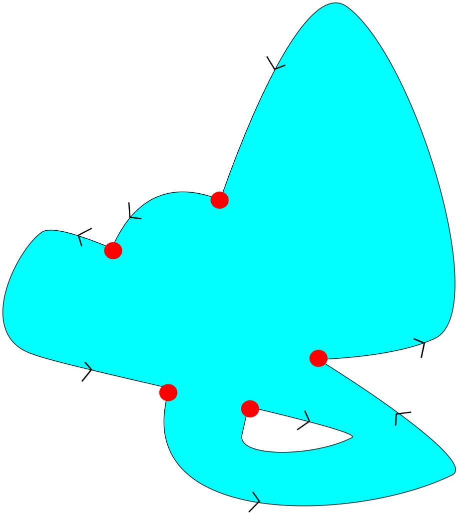






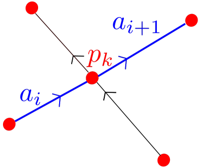




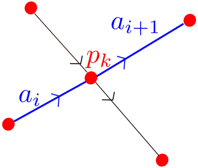


Finally, we get 5 cycles.
The complexity is the method is $O(n)$
The decomposition into cycles is unique
Sets are adapted with our sonar exploration goal

Sign method



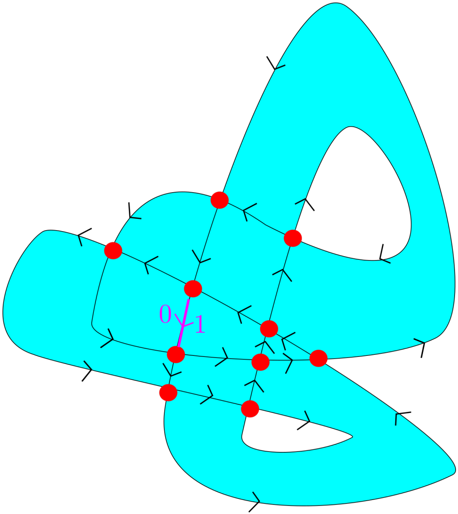
$\sigma_i = -1$

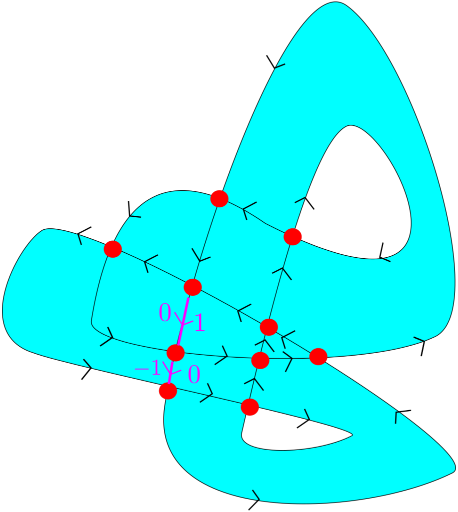


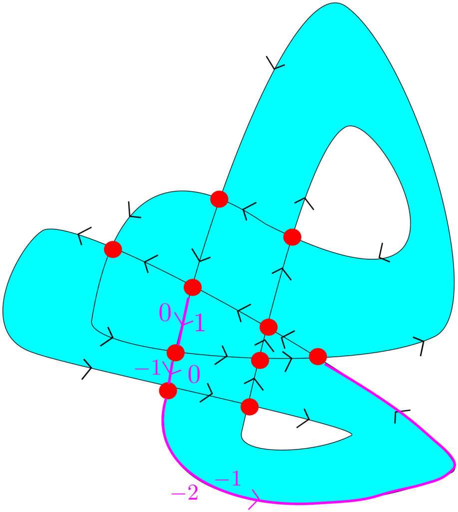
$\sigma_i = 1$

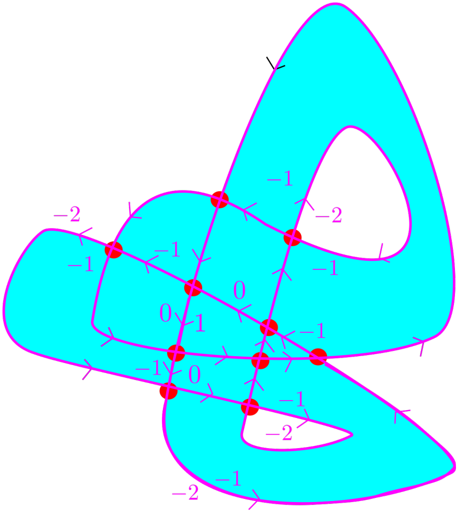
We have

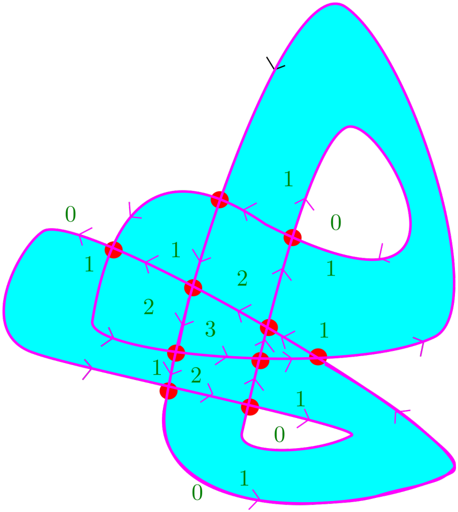
$$w(a_{i+1}) = w(a_i) + sign(p(a_i))$$


0
1


0
1
−1
0


0
1
−1
0
−1
−2


-1
-2
-2
-1
-1
-1
0
0
1
-1
-1
0
-1
-2
-1
-2


1
0
0
1
1
2
1
2
3
1
1
2
1
0
1
0