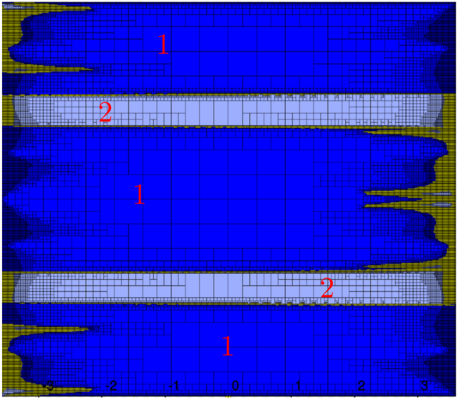
Directional inflation

Brest (virtual)
2021, July 22



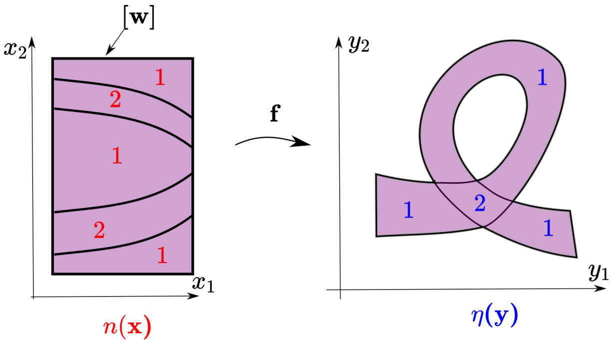
How to avoid these unclassified yellow boxes?

Problem

Given a box $[\mathbf{w}]$, a continuous function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$. We define two functions $\eta : \mathbb{R}^2 \to \mathbb{N}$, $n : \mathbb{R}^2 \to \mathbb{N}$ as

$$\eta(\mathbf{y}) = \text{card}\left\{\mathbf{f}^{-1}(\{\mathbf{y}\}) \cap [\mathbf{w}]\right\}$$

$$n(\mathbf{x}) = \eta(\mathbf{f}(\mathbf{x}))$$

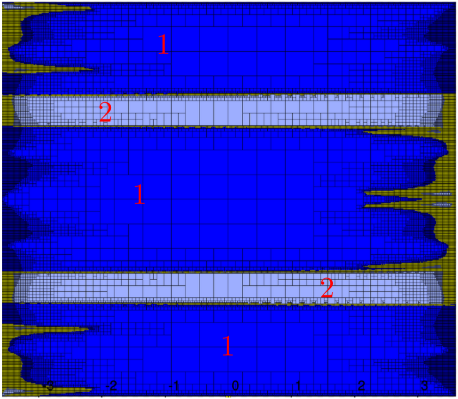

[w]
x_2
x_1
1
2
1
2
1
$n(\mathbf{x})$
f
y_2
y_1
1
1
2
1
$\eta(\mathbf{y})$

Assumptions.
Given $[\mathbf{y}]$ we can get $\eta([\mathbf{y}])$ using the winding number.
We have an inclusion function $[\mathbf{f}]$ for $\mathbf{f}$.
For all $\mathbf{x} \in [\mathbf{w}]$, $\det \mathbf{J}_{\mathbf{f}}(\mathbf{x}) > 0$ (local injectivity)

Problem. Characterize the function $n(\mathbf{x})$.

Basic inclusion. Since $n(\mathbf{x}) = \eta(\mathbf{f}(\mathbf{x}))$, we have

$$n([\mathbf{x}]) \subset \eta([\mathbf{f}]([\mathbf{x}])).$$

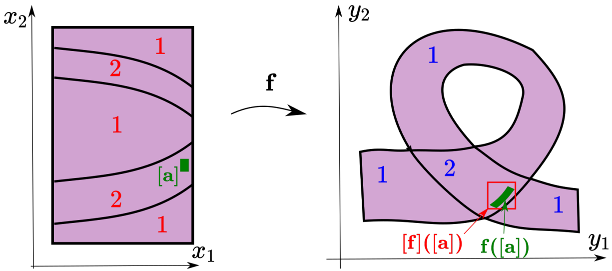

1
2
1
2
1
-3
-2
-1
0
1
2
3

Proposition. The function n changes on the set $\mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}])$.
Consequence.

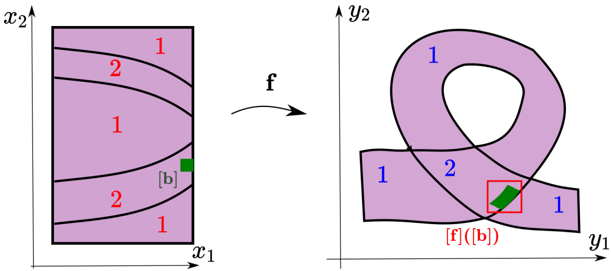
$$\left\{ \begin{array}{c} [\mathbf{x}] \cap \mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}]) = \emptyset \\ n([\mathbf{a}]) = n_a \\ [\mathbf{x}] \cap [\mathbf{a}] \neq \emptyset \end{array} \right\} \Rightarrow n([\mathbf{x}]) = n_a$$

This allows us to deduce $n([\mathbf{x}])$ from its neighbours.

Directional inflation



$$n([\mathbf{a}]) = \eta(\mathbf{f}([\mathbf{a}])) = [2,2] \subset \eta([\mathbf{f}]([\mathbf{a}])) = [1,2]$$



$$\eta(\mathbf{f}([\mathbf{b}])]) = [1,2] \subset \eta([\mathbf{f}]([\mathbf{b}])]) = [1,2]$$

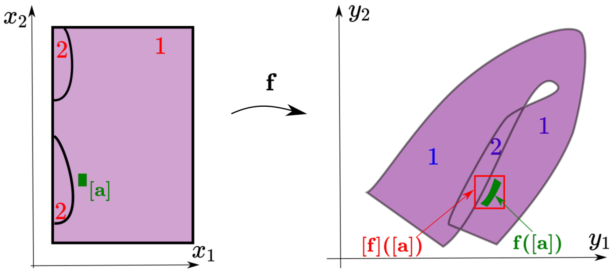
Conjecture. Assume that $[\mathbf{a}] \subset [\mathbf{w}]$. Set $[\mathbf{b}] = \text{right-inflate}([\mathbf{a}]) = [a_1^-, w_1^+] \times [a_2]$. We have

$$n([\mathbf{a}]) \quad \subset \quad (\eta([\mathbf{f}]([\mathbf{a}]))) \cap (\eta([\mathbf{f}]([\mathbf{b}])) + 1)$$

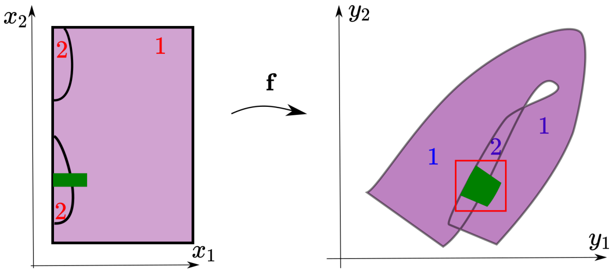
For our illustration, we have

$$\begin{array}{rcl} n([\mathbf{a}]) & \subset & (\eta([\mathbf{f}]([\mathbf{a}]))) \cap (\eta([\mathbf{f}]([\mathbf{b}])) + 1) \\ & = & [1,2] \cap ([1,2]+1) \\ & = & [1,2] \cap [2,3] \\ & = & [2,2] \end{array}$$

Counter example (Sylvie and Eric)



$$n([\mathbf{a}]) = \eta(\mathbf{f}([\mathbf{a}])) = [1,1] \subset \eta([\mathbf{f}]([\mathbf{a}])) = [1,2]$$



$$n([\mathbf{b}]) = \eta(\mathbf{f}([\mathbf{b}])]) = [1,2] \subset \eta([\mathbf{f}]([\mathbf{b}])]) = [1,2]$$

Whereas $n([\mathbf{a}]) = [1,1]$, the conjecture says:

$$\begin{aligned} n([\mathbf{a}]) &\subset (\eta([\mathbf{f}]([\mathbf{a}]))) \cap (\eta([\mathbf{f}]([\mathbf{b}])) + 1) \\ &= [1,2] \cap ([1,2] + 1) \\ &= [1,2] \cap [2,3] \\ &= [2,2] \end{aligned}$$