Backward reach set

Test-cases for the backward reach set
Given $x(k+1) = f(x(k))$ and $\mathbb{Y}$. Set

$$\varphi^k = f \circ f \circ \cdots \circ f,$$

$k$ times

For a given $\bar{k}$, compute

$$X_0 = (\varphi^{\bar{k}})^{-1}(\mathbb{Y}).$$
Equivalently, solve the chain

\[ X_0 = f^{-1}(X_1), X_1 = f^{-1}(X_2), \ldots, X_k = Y \]
Robust case
With an input vector \( u(k) \in [u] \):

\[
x(k + 1) = f(x(k), u(k)).
\]

The initial feasible set \( \mathbb{X}_0 \) becomes also uncertain: it depends on \( u(k) \).
Define \( \varphi^k \) as:

\[
\begin{align*}
\varphi^1(x(0), u(0)) &= f(x(0), u(0)) \\
\varphi^{k+1}(x(0), u(0:k)) &= f(\varphi^k(x(0), u(0:k-1)), u(k))
\end{align*}
\]
Define

\[ X_0^c = \{ x_0 | \forall u(0 : \bar{k} - 1) \in [u]_{\bar{k}}, \varphi^k(x(0), u(0 : \bar{k} - 1)) \in Y \} \]
\[ X_0^\supset = \{ x_0 | \exists u(0 : \bar{k} - 1) \in [u]_{\bar{k}}, \varphi^k(x(0), u(0 : \bar{k} - 1)) \in Y \}, \]

we have

\[ X_0^c \subset X_0 \subset X_0^\supset. \]

The sets \( X_0^c \) and \( X_0^\supset \) are the minimal and the maximal backward reach set.
Thick set inversion
Set inversion problem

\[ X = f^{-1}(Y) \]

If \( f_p \) depends on a parameter vector \( p \in \mathbb{R}^q \).

\[ X \subset X(p) \subset X^\supset, \]

where

\[ X^\subset = \bigcap_{p \in [p]} f^{-1}_p(Y) = \{ x | \forall p \in [p], f_p(x) \in Y \} \]
\[ X^\supset = \bigcup_{p \in [p]} f^{-1}_p(Y) = \{ x | \exists p \in [p], f_p(x) \in Y \} \].
The pair $[X] = [X^C, X^\supset]$ is a *thick set*. It partitions $\mathbb{R}^n$ into three zones: the clear zone $X^C$, the penumbra $X^\partial = X^\supset \setminus X^C$ and the dark zone $\mathbb{R}^n \setminus X^\supset$. 
Notation. The *thick set inversion problem* is denoted by

\[ [X] = f^{-1}_{[p]}(Y). \]
One box in the penumbra. Consider the thick set inversion problem $[X] = f^{-1}_{[p]}([y])$ where

$$f_p(x) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
Take \([p] = [2,3] \times [2,5] \times [4,5] \times [-6,-1], \ [y] = [5,19] \times [-7,11]\)
and \(x \in [x] = [0,1] \times [2,3]\).

\[
\begin{align*}
\mathbf{f}^- (x, [p]) \in \left[ \mathbf{f}_{[p]}^- \right] ([x]) &= \left( \begin{array}{c} 2 \cdot [0,1] + 2 \cdot [2,3] \\ 4 \cdot [0,1] - 6 \cdot [2,3] \end{array} \right) = \left( \begin{array}{c} [4,8] \\ [-18,-8] \end{array} \right) \\
\mathbf{f}^+ (x, [p]) \in \left[ \mathbf{f}_{[p]}^+ \right] ([x]) &= \left( \begin{array}{c} 3 \cdot [0,1] + 5 \cdot [2,3] \\ 5 \cdot [0,1] - 1 \cdot [2,3] \end{array} \right) = \left( \begin{array}{c} [10,18] \\ [-3,3] \end{array} \right).
\end{align*}
\]
We conclude that $[x]$ is inside the penumbra.
Backward reach set
Thick set inversion
Links and chains
Test-cases for the backward reach set

B. Desrochers, L. Jaulin
Chain of set inversion problems; Application to reachability
Links and chains

B. Desrochers, L. Jaulin
If, for a box \([p] \subset \mathbb{R}^q\), the set

\[
 f_{[p]}(x) = f(x, [p]) = \{ y \in \mathbb{R}^p \mid \exists p \in [p], y = f_p(x) \}
\]

is a box, then \(f\) is said to be a \textit{link}.

Links will be composed later to build a \textit{chain}.
Due to their specific box-shaped structure, link can be inverted with respect to $x$ without bisections on the $p$-space.
Example. The function

\[ f(x, p) = 20e^{-x_1 p} - 8e^{-x_2 p}. \]

is a scalar link function.
Proposition. Consider $\ell$ scalar link functions, $f_i(x, p_i)$, where vectors $\{p_1, p_2, \ldots, p_\ell\}$ are vector components of the vector $p$. The function

$$f(x, p) = \begin{pmatrix} f_1(x, p_1) \\ \vdots \\ f_\ell(x, p_\ell) \end{pmatrix}$$

is a link.
Example. The function

\[ f(x, p) = \begin{pmatrix} p_1 x_1 x_2 + p_1 p_2 \sin(x_2) \\ p_3 x_1 + p_3 p_4 \cos(x_1) x_2 \end{pmatrix} \]

is a link.
Consequence.

If $f_p(x)$ is a link, for all $x$, the set $f_{[p]}(x)$ is a box.

If $x \in [x]$, the two bounds of the box $f_{[p]}(x)$ are included inside the boxes $[f_{[p]}^-([x])]$ and $[f_{[p]}^+([x])]$. 
Parameter estimation with uncertain time measurement
Example. Let us estimate the parameters $q_1$ and $q_2$ of the model

$$y(q,t) = 20e^{-q_1t} - 8e^{-q_2t}.$$ 

We assume that 10 measurements $y_i$ have been collected at time $t_i$. 
### Test-cases for the backward reach set

<table>
<thead>
<tr>
<th>$i$</th>
<th>$[t_i]$</th>
<th>$[y_i]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.25, 1.25]</td>
<td>[2.7, 12.1]</td>
</tr>
<tr>
<td>2</td>
<td>[1, 2]</td>
<td>[1.04, 7.14]</td>
</tr>
<tr>
<td>3</td>
<td>[1.75, 2.75]</td>
<td>[-0.13, 3.61]</td>
</tr>
<tr>
<td>4</td>
<td>[2.5, 3.5]</td>
<td>[-0.95, 1.15]</td>
</tr>
<tr>
<td>5</td>
<td>[5.5, 6.5]</td>
<td>[-4.85, -0.29]</td>
</tr>
<tr>
<td>6</td>
<td>[8.5, 9.5]</td>
<td>[-5.06, -0.36]</td>
</tr>
<tr>
<td>7</td>
<td>[12.5, 13.5]</td>
<td>[-4.1, -0.04]</td>
</tr>
<tr>
<td>8</td>
<td>[16.5, 17.5]</td>
<td>[-3.16, 0.3]</td>
</tr>
<tr>
<td>9</td>
<td>[20.5, 21.5]</td>
<td>[-2.5, 0.51]</td>
</tr>
<tr>
<td>10</td>
<td>[24.5, 25.5]</td>
<td>[-2, 0.6]</td>
</tr>
</tbody>
</table>
Data \([t_i, y_i]\)
We are interested by the thick set

\[ \mathcal{Q} = \{ \mathbf{q} \in \mathbb{R}^2 | \forall i, y(\mathbf{q}, t_i) \in [y_i] \} . \]

If we set

\[
\begin{align*}
 f_{[t]}(\mathbf{q}) &= \begin{pmatrix}
 y(\mathbf{q}, [t_1]) \\
 \vdots \\
 y(\mathbf{q}, [t_{10}])
\end{pmatrix} \\
[y] &= [y_{10}] \times \cdots \times [y_{10}] \\
[t] &= [t_1] \times \cdots \times [t_{10}]
\end{align*}
\]

then, we have

\[ [\mathcal{Q}] = f_{[t]}^{-1}([y]). \]
Test-cases for the backward reach set

Thick set inversion

Links and chains

Thick set

\[ \mathbb{Q} \]

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Chain of set inversion problems; Application to reachability
Chain of set inversion problems; Application to reachability
Chain. The function $\varphi_p(x)$ is a *chain* if it can be written as

$$
\varphi_p(x) = f_{p_\ell} \circ \cdots \circ f_{p_2} \circ f_{p_1}(x)
$$

where $\{p_1, p_2, \ldots, p_\ell\}$ are vector components of $p$ and the $f_{p_k}$ are links.

**Consequence:** No dependency effect and no wrapping effect.
Example. The function

$$\varphi_p(x) = \left( \begin{array}{c} \sin(p_3 x_1^2) + p_4 \\ \frac{p_5 (p_1 e^{x_2} + p_1 p_2 x_1 x_2)}{p_3 x_1^2} \end{array} \right)$$

is not a link. Now, if we define

$$f_{p_1}^1(x) = \left( \begin{array}{c} p_1 e^{x_2} + p_1 p_2 x_1 x_2 \\ p_3 x_1^2 \end{array} \right)$$

$$f_{p_2}^2(z) = \left( \begin{array}{c} \sin(z_2) + p_4 \\ \frac{p_5 z_1}{z_2} \end{array} \right)$$

with $p_1 = (p_1, p_2, p_3)$ and $p_2 = (p_4, p_5)$ then, we have

$$\varphi_p(x) = f_{p_2}^2 \circ f_{p_1}^1(x).$$

Since both $f_{p_2}, f_{p_1}$ are links, $\varphi_p(x)$ is a chain.
Test-cases for the backward reach set
Test-case 1: linear system
Consider the linear system

\[ x(k + 1) = A \cdot x(k) \]

where

\[ A(k) \in \begin{pmatrix}
[2.5, 3] & [2, 3] \\
[4, 4.5] & [-3, -2]
\end{pmatrix} \]

plays the role of \( p \).
We want to reach the target

$$\bar{Y} = [4, 20] \times [-8, 12]$$

at time $\bar{k} = 3$. 

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Chain of set inversion problems; Application to reachability
\([X](3) = Y\) is thin.
\[ ([X](2) = f_{[A]}^{-1}([X](3)) \]
\[ [X](1) = f^{-1}_{[A]}([X](2)) \]
Backward reach set
Thick set inversion
Links and chains

Test-cases for the backward reach set

\[ [X](0) = f_{[A]}^{-1}([X](1)) \]
Test-case 2: nonlinear system
Nonlinear system

\[
\begin{pmatrix}
    x_1(k+1) \\
    x_2(k+1)
\end{pmatrix} = \begin{pmatrix}
    x_1(k) + x_2^2(k) \cdot u_1(k) \\
    \frac{1}{2} \cdot x_1(k) \cdot x_2(k) + u_2(k)
\end{pmatrix}
\]

with \( u_1 \in [1, 2] \) and \( u_2 \in [-2, -1] \). The set \( \mathbb{Y} \) is assumed to be a centred disk which has to be reached at time \( \bar{k} = 3 \).
\[
[X](3) = Y \text{ is thin.}
\]
\[ [X](2) = f^{-1}_{[A]}([X](3)) \]
$[X](1) = f^{-1}_{[A]}([X](2))$
Backward reach set
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\[[X](0) = f^{-1}_{[A]}([X](1))\]

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