Explore and return for an underwater robot in a minimalist environment, with no localization system and with a few computation

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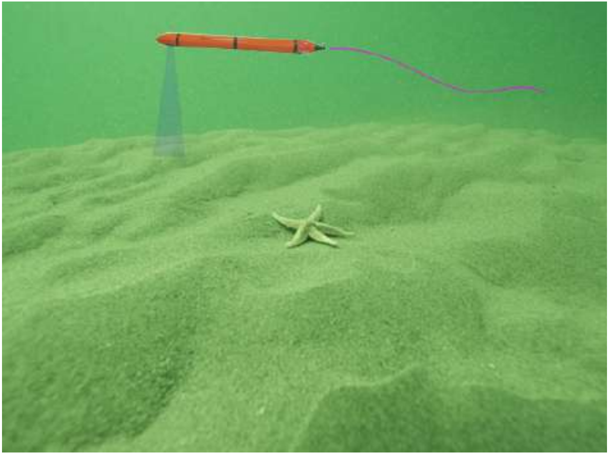
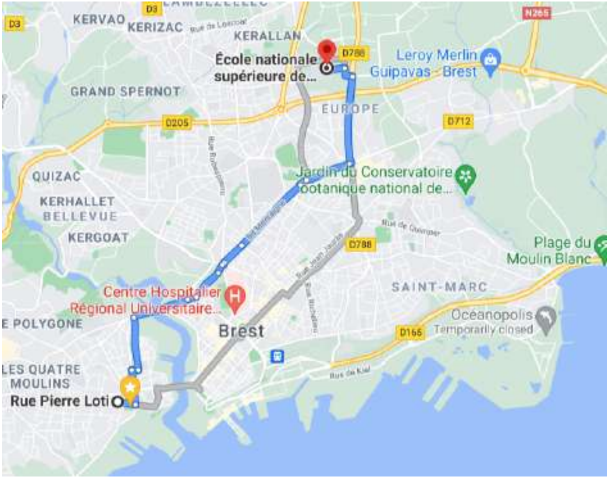
1. Underwater navigation



Explore and return in a minimalist environment

Map-based navigation

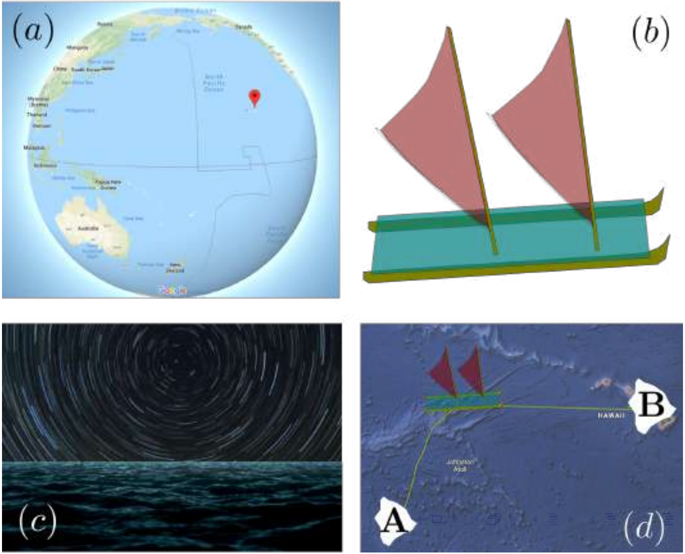


Modern navigation: high cost (computation, infrastructure)

Route-based navigation

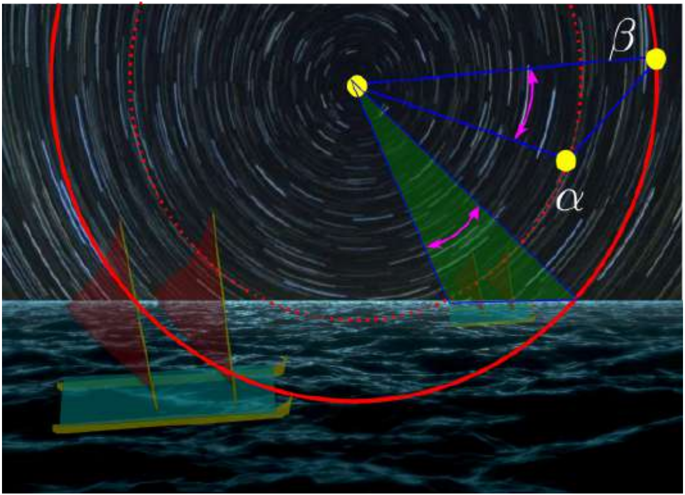
Submeeting 2018

Find the route without GPS, compass, clocks, computer with *wa'a kaulua*

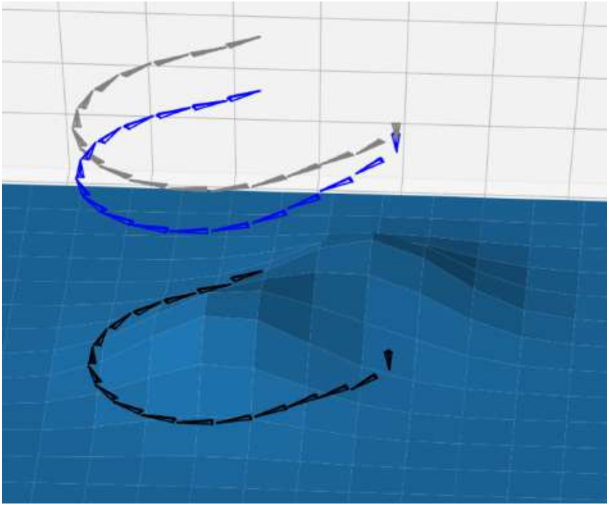

(a)
(b)
(c)
(d)
A
B
HAWAII

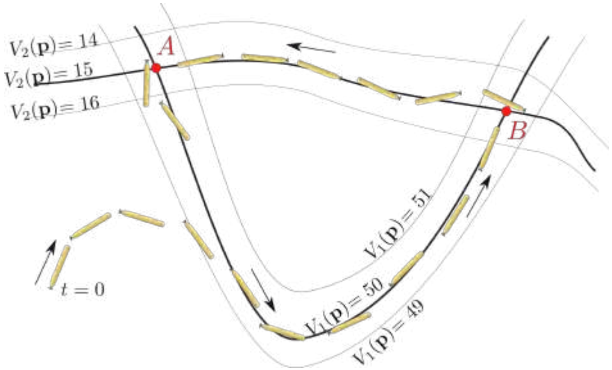
Follow a route

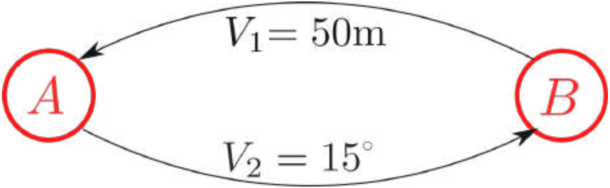
Given a function $h : \mathbb{R}^2 \mapsto \mathbb{R}$, a route in defined by $h(\mathbf{p}) = 0$.
h could be the temperature, the radiation, the pressure, the altitude, the time shift between two periodic events.



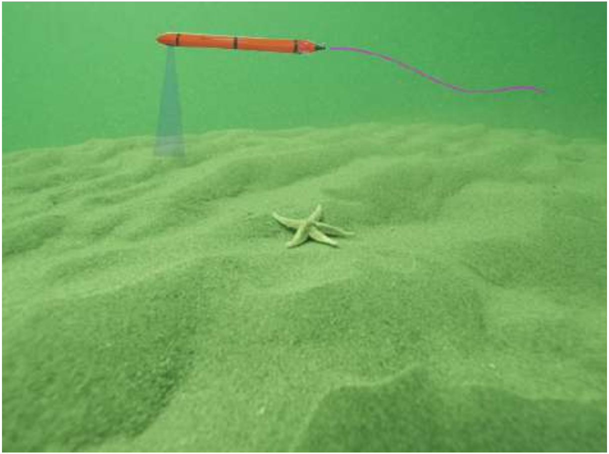
The stars α and β set simultaneously. $h(\mathbf{p}) = t_\alpha - t_\beta$



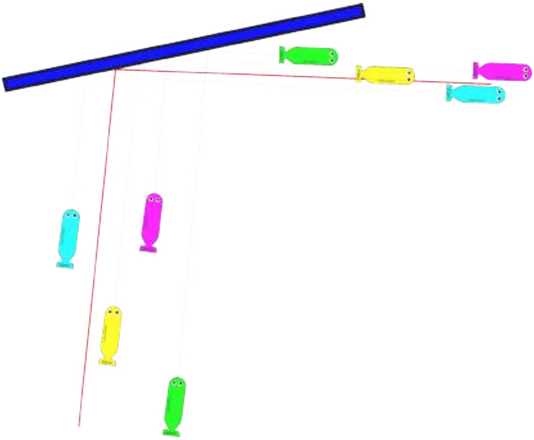

$V_2(\mathbf{p}) = 14$
$V_2(\mathbf{p}) = 15$
$V_2(\mathbf{p}) = 16$
A
B
$V_1(\mathbf{p}) = 51$
$V_1(\mathbf{p}) = 50$
$V_1(\mathbf{p}) = 49$
$t = 0$

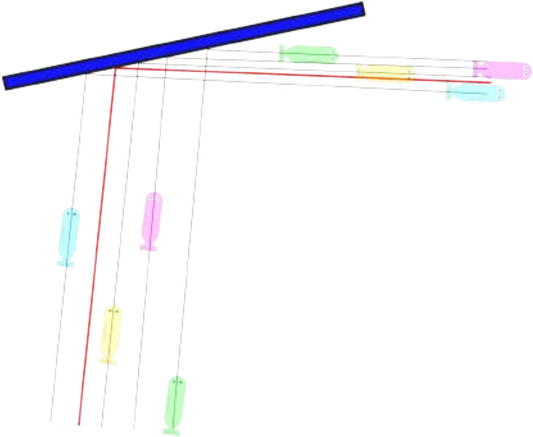

$V_1 = 50\text{m}$
A
B
$V_2 = 15^\circ$

2. Stable bouncing (phd of Quentin Brateau)

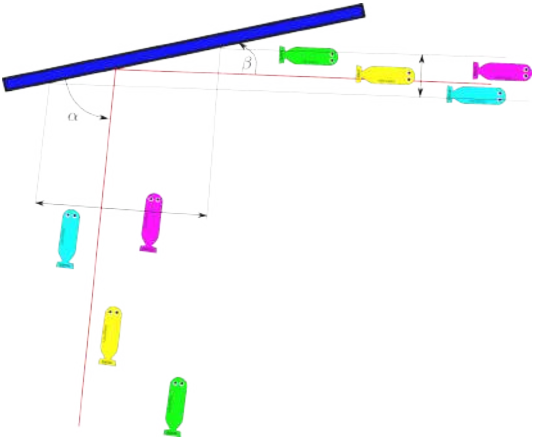


No route exists

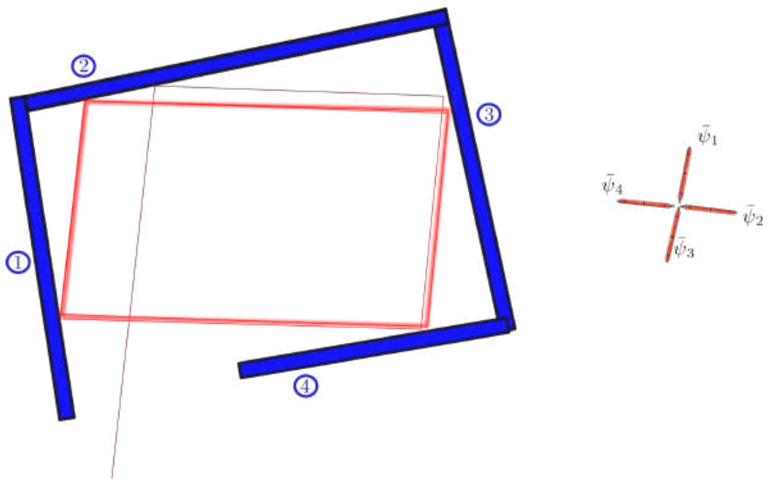


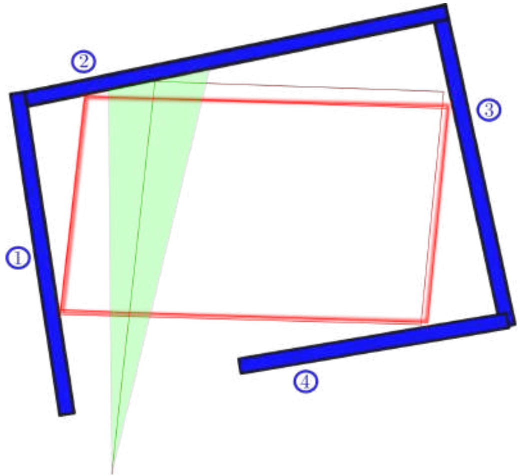


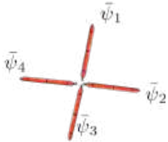
Contraction of the distance

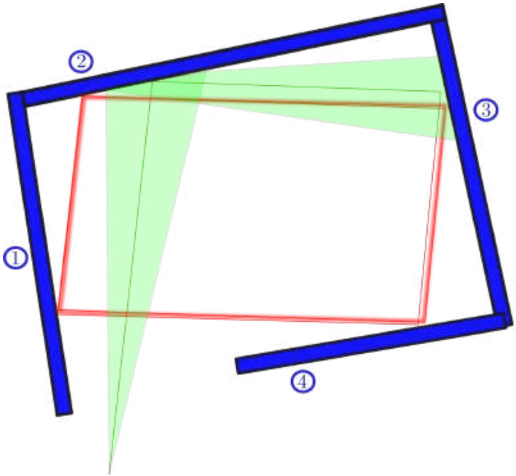


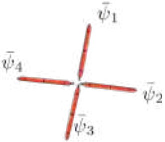
Contraction if $\frac{\sin\beta}{\sin\alpha} < 1$

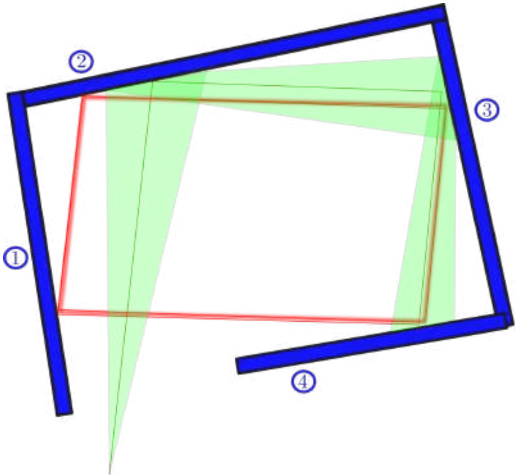

②
③
①
④
$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

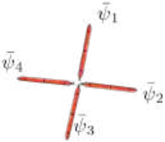

②
③
①
④

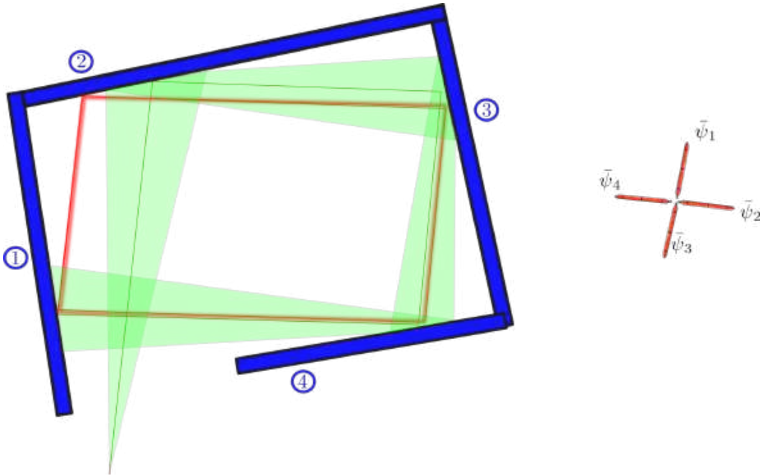

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

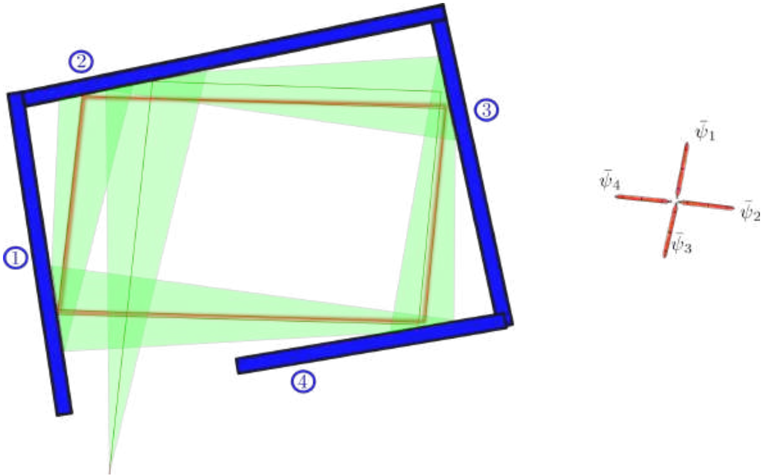

②
③
①
④

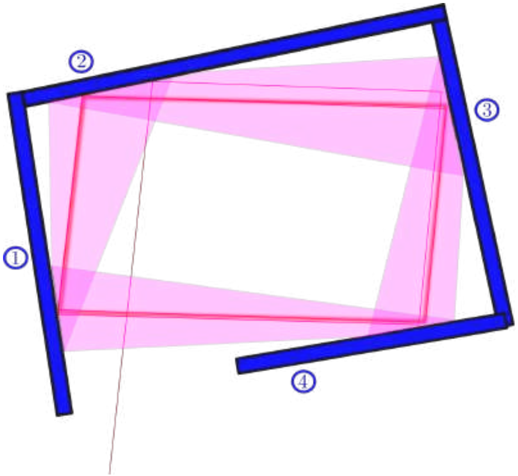

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

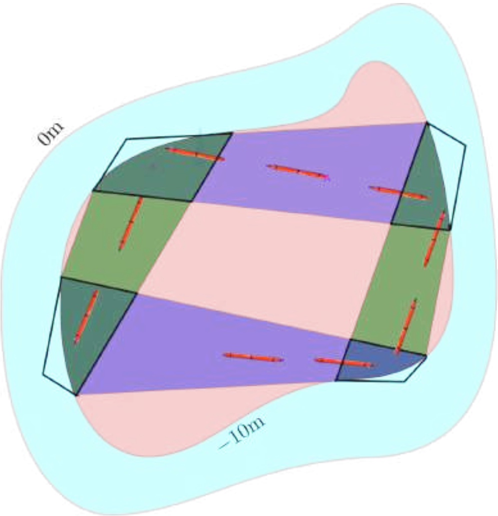

②
③
①
④

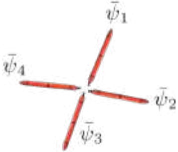

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

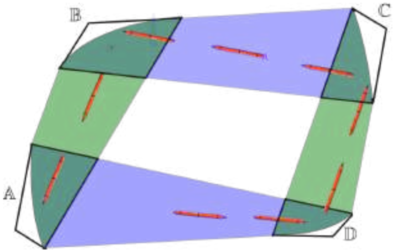

2
3
1
4
$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

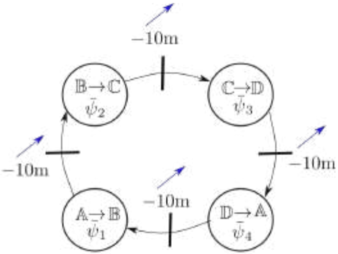

1
2
3
4
$\bar{\psi}_1$
$\bar{\psi}_2$
$\bar{\psi}_3$
$\bar{\psi}_4$

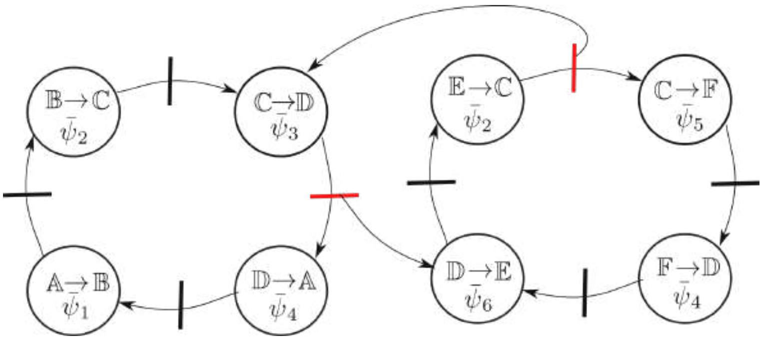

②
③
①
④


0m
−10m

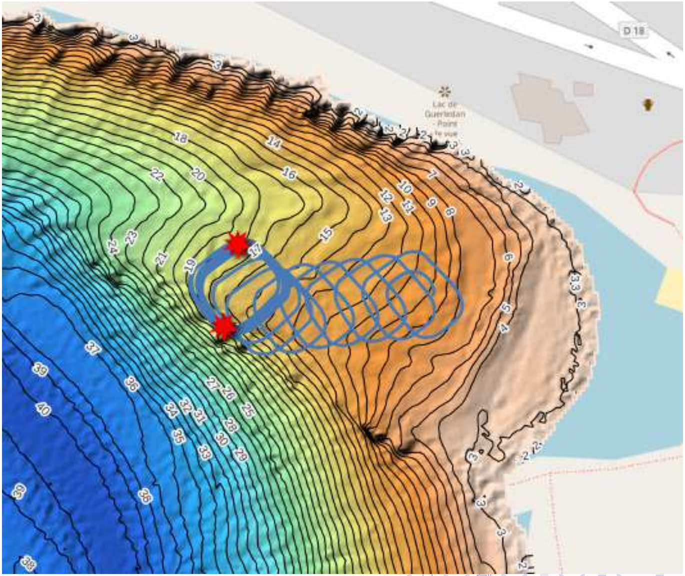

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


B
C
A
D


−10m
B→C
$\bar{\psi}_2$
C→D
$\bar{\psi}_3$
−10m
−10m
−10m
A→B
$\bar{\psi}_1$
D→A
$\bar{\psi}_4$


$\mathbb{B}\to\mathbb{C}$
$\bar{\psi}_2$
$\mathbb{C}\to\mathbb{D}$
$\bar{\psi}_3$
$\mathbb{E}\to\mathbb{C}$
$\bar{\psi}_2$
$\mathbb{C}\to\mathbb{F}$
$\bar{\psi}_5$
$\mathbb{A}\to\mathbb{B}$
$\bar{\psi}_1$
$\mathbb{D}\to\mathbb{A}$
$\bar{\psi}_4$
$\mathbb{D}\to\mathbb{E}$
$\bar{\psi}_6$
$\mathbb{F}\to\mathbb{D}$
$\bar{\psi}_4$

Experiment (phd of Quentin Brateau)

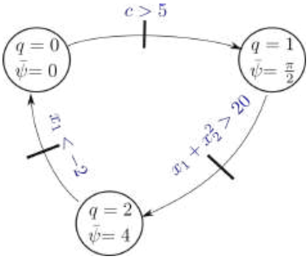


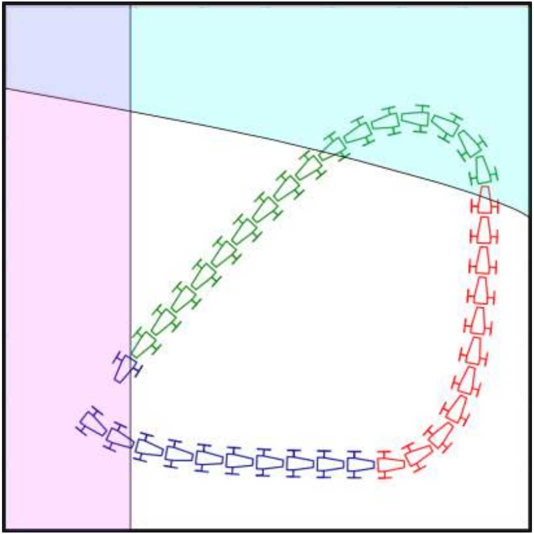
3. Proving the stability

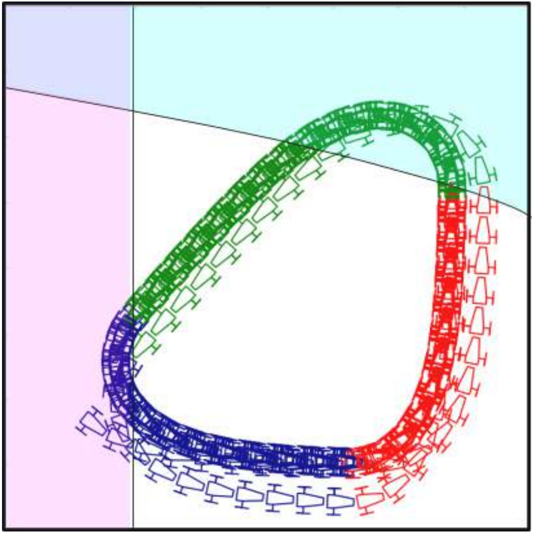
Consider the robot

$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

with the heading control $u = \sin(\bar{\psi} - x_3)$.







Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= ?, \\
[-1,3]\cdot[2,5] &= ?, \\
\mathsf{abs}\left([-7,1]\right) &= ?
\end{aligned}
$$

$$\begin{aligned} [-1,3]+[2,5] &= [1,8], \\ [-1,3]\cdot[2,5] &= [-5,15], \\ \mathsf{abs}([-7,1]) &= [0,7] \end{aligned}$$

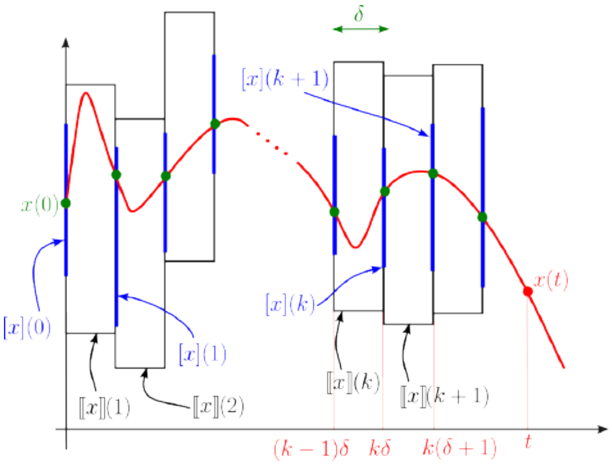
The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned}[f]([x_1],[x_2]) &= [x_1]\cdot[x_2] - ([x_1]+[x_2])\cdot\cos[x_2] \\ &\quad + \sin[x_1]\cdot\sin[x_2] + 2.\end{aligned}$$

Tubes



$$x(t) \in [x](t)$$

Stability with Poincaré map

System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

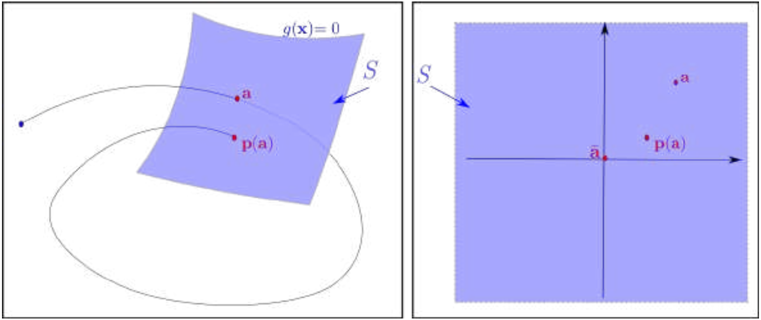
How to prove that the system has a cycle ?

System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
Poincaré section $\mathscr{G}$: $g(\mathbf{x}) = 0$

We define

$$\mathbf{p}: \begin{array}{ccc} \mathscr{G} & \rightarrow & \mathscr{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where $\mathbf{p}(\mathbf{a})$ is the point of $\mathscr{G}$ such that the trajectory initialized at $\mathbf{a}$ intersects $\mathscr{G}$ for the first time.

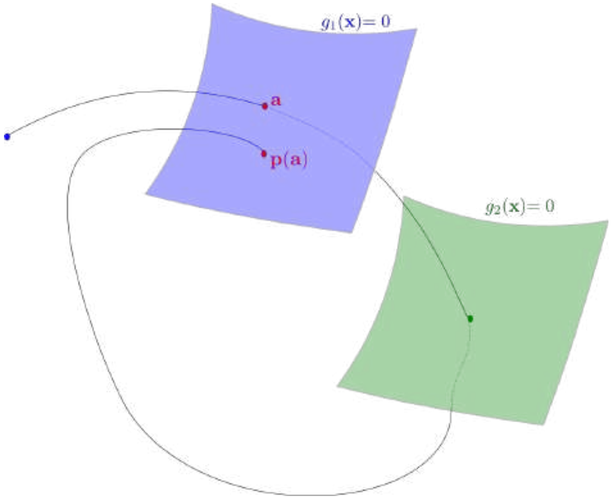

$g(\mathbf{x})=0$
S
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
S
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$\bar{\mathbf{a}}$

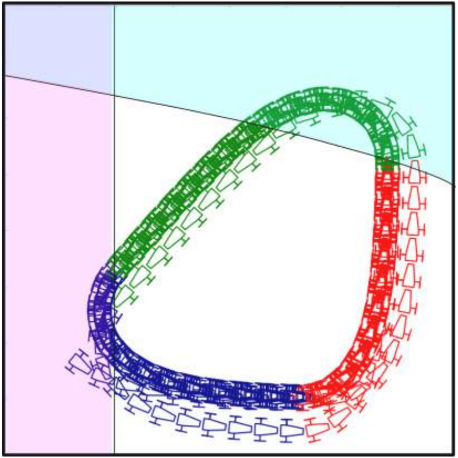
The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$

With hybrid systems

Systems: $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}), i \in \{1, \ldots, m\}$
Section i: $g_i(\mathbf{x}) = 0$


$g_1(\mathbf{x}) = 0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$g_2(\mathbf{x}) = 0$

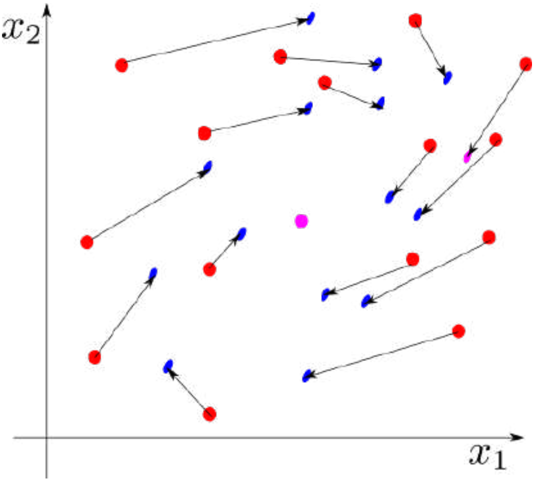


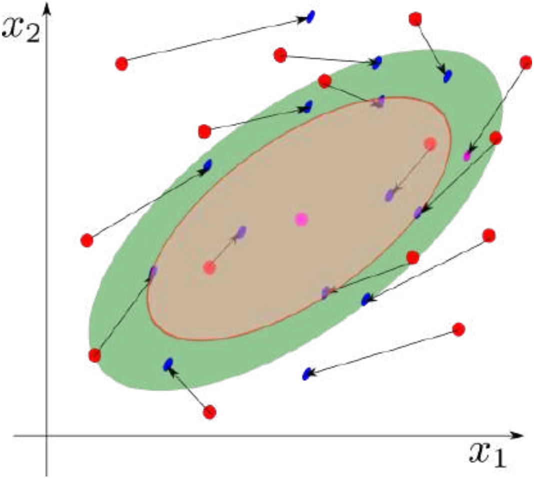
Proving the stability

Consider the discrete time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.


x_2
x_1


x_2
x_1

We have to find

$$\mathscr{E}_{\mathbf{x}} : \mathbf{x}^{\mathsf{T}} \cdot \mathbf{P} \cdot \mathbf{x} \leq \varepsilon$$

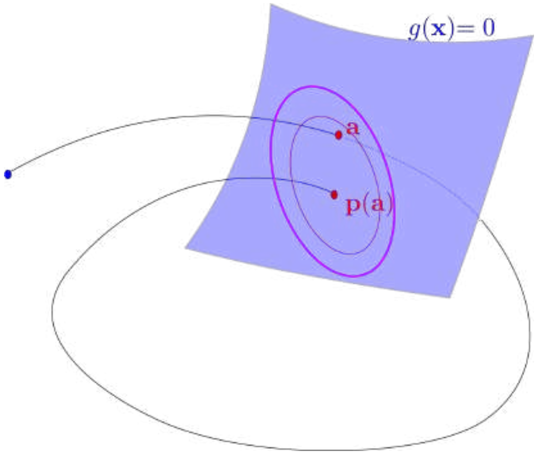
Such that

$$\mathbf{f}(\mathscr{E}_{\mathbf{x}}) \subset \mathscr{E}_{\mathbf{x}}$$

Stability of cycles

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$


$g(\mathbf{x})=0$
a
p(a)

References

1. Interval and stability [2][11]
2. Route following [4][8]
3. Navigation with stable cycles [3]
4. Tubes [10][1]
5. Integral formulation [5]
6. Ellipse and guaranteed integration [9]
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8. Axis-aligned Lyapunov equation [6]

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