Inner and Outer Approximations of Probabilistic Sets

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1 Probabilistic-set approach

Bounded-error estimation

$$\mathbf{y} = \boldsymbol{\psi}(\mathbf{p}) + \mathbf{e},$$

where

$\mathbf{e} \in \mathbb{E} \subset \mathbb{R}^m$ is the error vector,

$\mathbf{y} \in \mathbb{R}^m$ is the collected data vector,

$\mathbf{p} \in \mathbb{R}^n$ is the parameter vector to be estimated.

Or equivalently

$$\mathbf{e} = \mathbf{y} - \boldsymbol{\psi}(\mathbf{p}) = \mathbf{f}_{\mathbf{y}}(\mathbf{p}),$$

The *posterior feasible set* for the parameters is

$$\mathbb{P} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}\right).$$

Probabilistic set approach. We decompose the error space into two subsets: $\mathbb{E}$ on which we bet $\mathbf{e}$ will belong and $\overline{\mathbb{E}}$. We set

$$\pi = \Pr\left(\mathbf{e} \in \mathbb{E}\right)$$

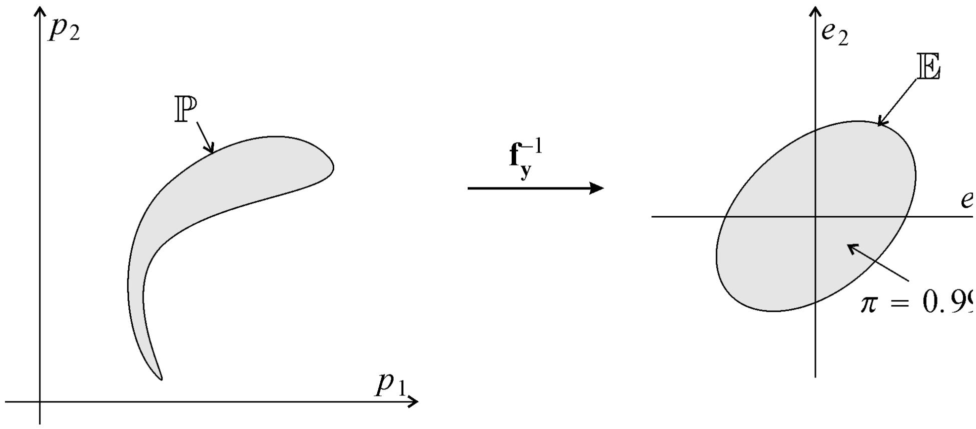
The event $\mathbf{e} \in \overline{\mathbb{E}}$ is considered as *rare*, i.e., $\pi \simeq 1$.

Once $\mathbf{y}$ is collected, we compute

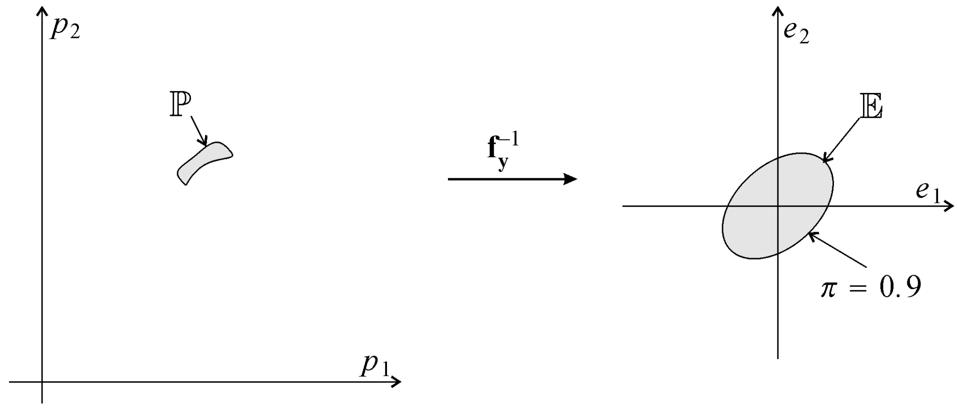
$$\mathbb{P} = \mathbf{f}_{\mathbf{y}}^{-1}(\mathbb{E}) \, .$$

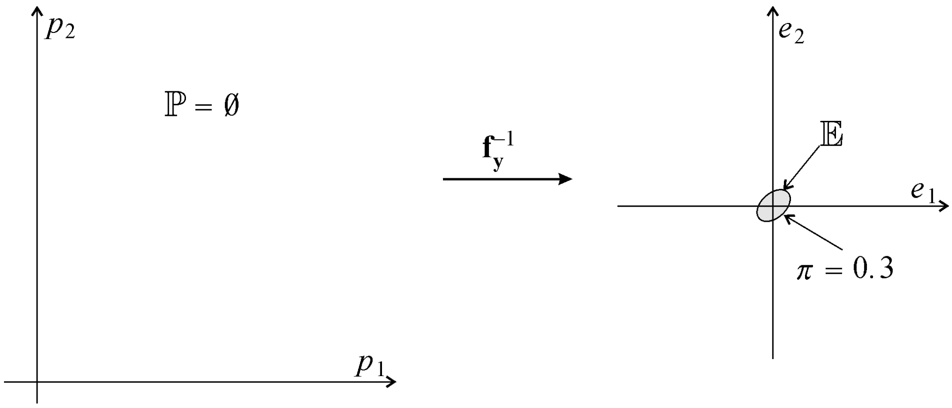
If $\mathbb{P} \neq \emptyset$, we conclude that $\mathbf{p} \in \mathbb{P}$ with a prior probability of π.

If $\mathbb{P} = \emptyset$, than we conclude the rare event $\mathbf{e} \in \overline{\mathbb{E}}$ occurred.



p_2
$\mathbb{P}$
p_1
$\mathbf{f}_{\mathbf{y}}^{-1}$
e_2
$\mathbb{E}$
e
$\pi = 0.9$


p_2
$\mathbb{P}$
p_1
$\mathbf{f}_{\mathbf{y}}^{-1}$
e_2
$\mathbb{E}$
e_1
$\pi = 0.9$



p_2
$\mathbb{P} = \emptyset$
p_1
$\mathbf{f}_{\mathbf{y}}^{-1}$
e_2
$\mathbb{E}$
e_1
$\pi = 0.3$

Example. The model is described by $y = p^2 + e$, *i.e.*,

$$e = y - p^2 = f_y(p).$$

Assume that $\Pi_e : \mathcal{N}(0,1)$. If $\mathbb{E} = [-6, 6]$ then,

$$\Pr\left(e \in \overline{\mathbb{E}}\right) = -\frac{1}{\sqrt{2\pi}} \int_{-6}^{6} \exp\left(-\frac{e^2}{2}\right) de \simeq 1.97 \times 10^{-9}.$$

If we collect $y = 10$, we have

$$\begin{aligned} \mathbb{P} &= f_y^{-1}(\mathbb{E}) = f_y^{-1}([-6, 6]) \\ &= \sqrt{10 - [-6, 6]} = \sqrt{[4, 16]} = [-4, -2] \cup [2, 4], \end{aligned}$$

with a prior probability of $1 - 1.97 \times 10^{-9}$.

If we collect $y = -10$, we get $\mathbb{P} = \emptyset$. We conclude that the rare event $\mathbf{e} \in \overline{\mathbb{E}}$ occurred.

2 Robust regression

Consider the error model

$$\underbrace{\begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}}_{=\mathbf{e}} = \underbrace{\begin{pmatrix} y_1 - \psi_1(\mathbf{p}) \\ \vdots \\ y_m - \psi_m(\mathbf{p}) \end{pmatrix}}_{=\mathbf{f_y}(\mathbf{p})}$$

The data y_i is an *inlier* if $e_i \in [e_i]$ and an *outlier* otherwise. We assume that

$$\forall i,\ \mathsf{Pr}\left(e_i \in [e_i]\right) = \pi$$

and that all e_i's are independent.

Equivalently,

$$\begin{cases} y_1 - \psi_{\mathbf{1}}(\mathbf{p}) \in [e_1] & \text{with a probability } \pi \\ \vdots & \vdots \\ y_m - \psi_m(\mathbf{p}) \in [e_m] & \text{with a probability } \pi \end{cases}$$

The probability of having k inliers is

$$\frac{m!}{k!\,(m-k)!}\pi^k.\,(1-\pi)^{m-k}\,.$$

The probability of having strictly more than q outliers is thus

$$\gamma\left(q,m,\pi\right) \stackrel{\text{def}}{=} \sum_{k=0}^{m-q-1} \frac{m!}{k!\left(m-k\right)!}\pi^{k}.\left(1-\pi\right)^{m-k}.$$

Denote by $\mathbb{E}^{\{q\}}$ the set of all $\mathbf{e} \in \mathbb{R}^m$ consistent with at least $m- q$ error intervals $[e_i]$.

For $m = 3$, we have

$$\begin{array}{rcl}
\mathbb{E}^{\{0\}} & = & [e_1] \times [e_2] \times [e_3] \\
\mathbb{E}^{\{1\}} & = & ([e_1] \cap [e_2]) \cup ([e_2] \cap [e_3]) \cup ([e_1] \cap [e_3]) \\
\mathbb{E}^{\{2\}} & = & [e_1] \cup [e_2] \cup [e_3] \\
\mathbb{E}^{\{3\}} & = & \mathbb{R}^3.
\end{array}$$

Define

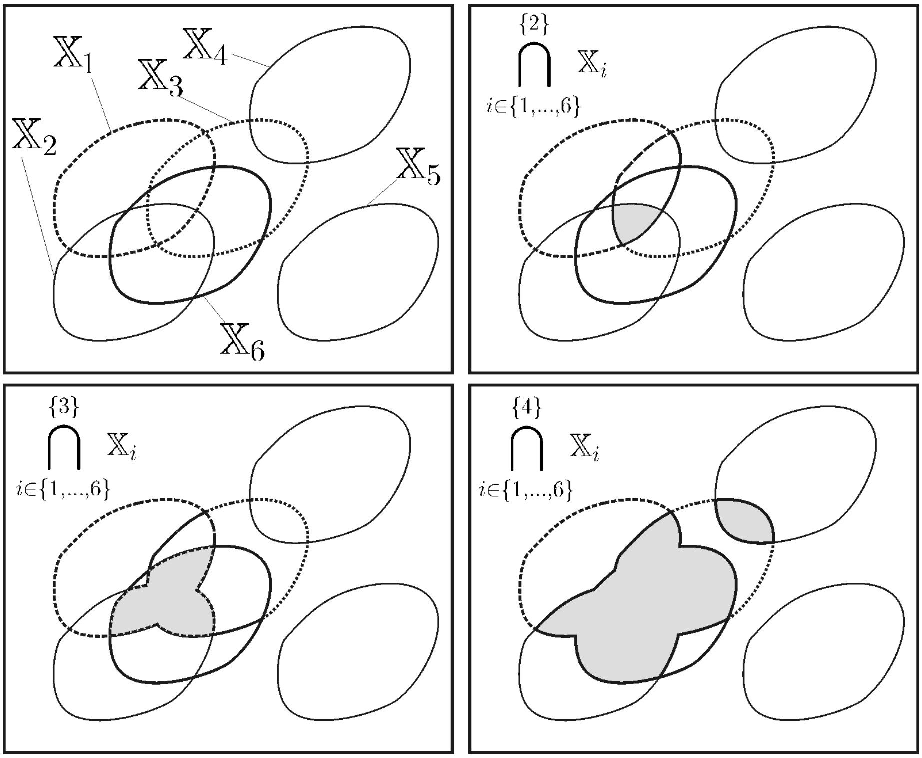
$$\mathbb{P}^{\{q\}} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}^{\{q\}}\right).$$

We have

$$\begin{aligned} \text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{q\}}\right) &= 1 - \gamma\left(q, m, \pi\right) \\ \text{prob}\left(\mathbf{p} \in \overline{\mathbb{P}^{\{q\}}}\right) &= \gamma\left(q, m, \pi\right). \end{aligned}$$

Thus $\mathbb{P}^{\{q\}}$ is the inverse of $\mathbb{E}^{\{q\}}$ and inner/outer approximations can thus be found.

3 Relaxed intersection



q-relaxed intersection

$$\mathbb{P}^{\{q\}} = \mathbf{f}_{\mathbf{y}}^{-\mathbf{1}}\left(\mathbb{E}^{\{q\}}\right) = \bigcap_{i\in\{\mathbf{1},\ldots,m\}}^{\{q\}} f_{y_i}^{-\mathbf{1}}\left([e_i]\right).$$

Proposition *(new)*. We have

$$\overline{\mathbb{P}^{\{q\}}} = \bigcap^{\{m-q-1\}} f_{y_i}^{-1}\left(\overline{[e_i]}\right).$$

This proposition allows to obtain an inner approximation of $\mathbb{P}^{\{q\}}$.

4 Application to localization

A robot measures distances to three beacons.

beacon	x_i	y_i	$[d_i]$
1	1	3	$[1,2]$
2	3	1	$[2,3]$
3	-1	-1	$[3,4]$

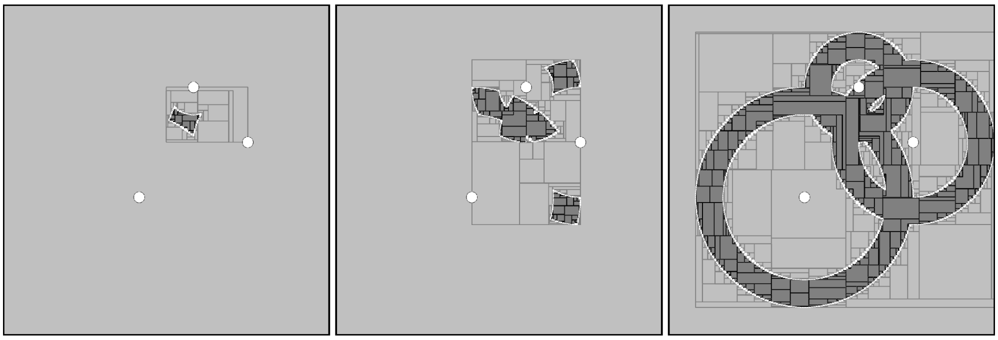
The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5] \right\},$$

where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$.

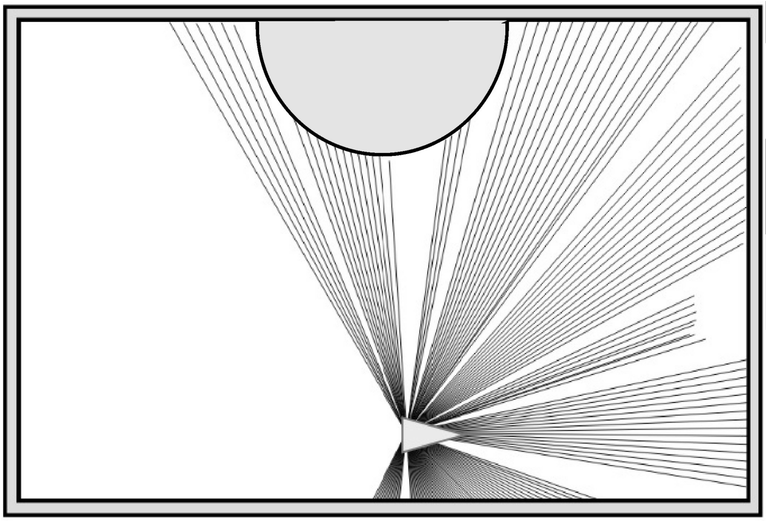
$$\begin{aligned}
\mathrm{prob}\left(\mathbf{p} \in \mathbb{P}^{\{0\}}\right) &= 0.729 \\
\mathrm{prob}\left(\mathbf{p} \in \mathbb{P}^{\{1\}}\right) &= 0.972 \\
\mathrm{prob}\left(\mathbf{p} \in \mathbb{P}^{\{2\}}\right) &= 0.999
\end{aligned}$$



Probabilistic sets $\mathbb{P}^{\{0\}}, \mathbb{P}^{\{1\}}, \mathbb{P}^{\{2\}}$.

5 With real data

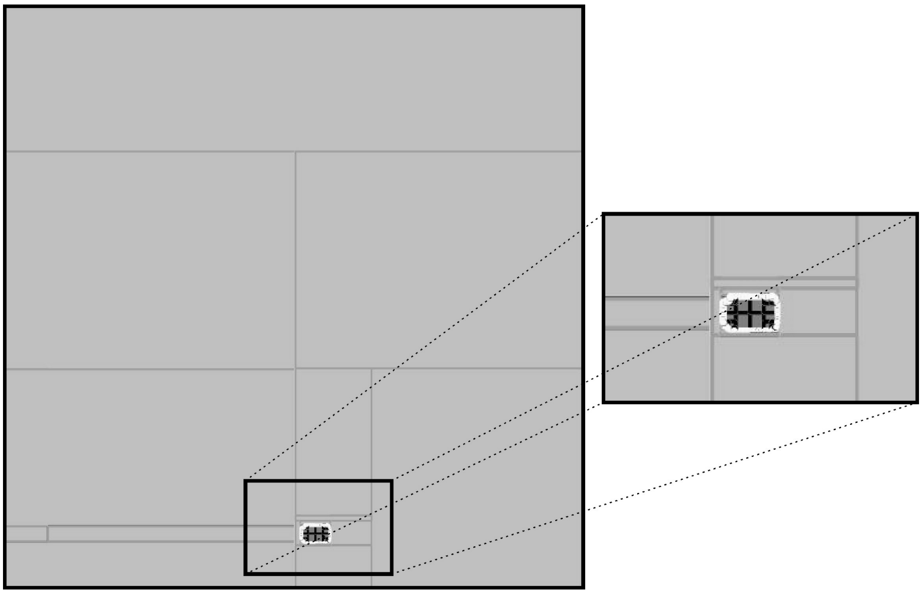
Robot equipped with a laser rangefinder and a compass.



143 distances collected by the rangefinder $\pm 10cm$

For $q = 16, m = 143, \pi = 0.95$, the probability of being wrong is

$$\alpha = \gamma\left(q, m, \pi\right) = 8.46 \times 10^{-4}.$$



$\mathbb{P}^{\{16\}}$ contains $\mathbf{p}^*$ with a probability $1 - \alpha = 0.99915$.