Solving geometrical constraints in space-time

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1 Contractors

The operator $\mathcal{C} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$\begin{array}{lll} x_3 = x_1 + x_2 \Rightarrow & x_3 \in & [x_3] \cap ([x_1] + [x_2]) \quad // \text{ forward} \\ x_1 = x_3 - x_2 \Rightarrow & x_1 \in & [x_1] \cap ([x_3] - [x_2]) \quad // \text{ backward} \\ x_2 = x_3 - x_1 \Rightarrow & x_2 \in & [x_2] \cap ([x_3] - [x_1]) \quad // \text{ backward} \end{array}$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

2 Interval trajectories

A trajectory is a function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^n$. For instance

$$\mathbf{f}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

is a trajectory.

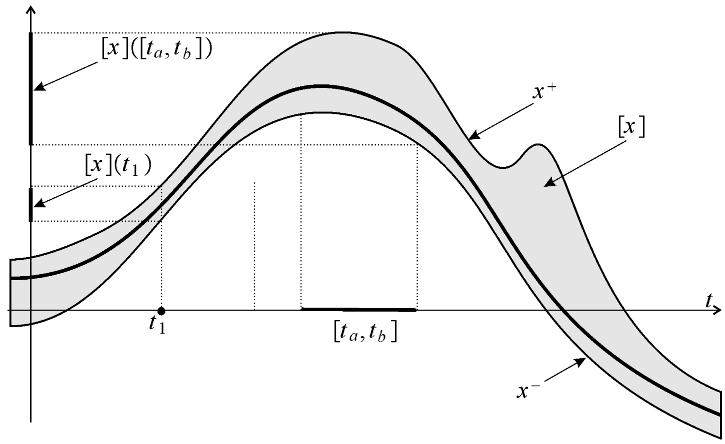
Order relation

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall t, \forall i, f_i\left(t\right) \leq g_i\left(t\right).$$

We have

$$
\begin{aligned}
\mathbf{h} &= \mathbf{f} \quad \wedge \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i\left(t\right) = \mathsf{min}\left(f_i\left(t\right), g_i\left(t\right)\right), \\
\mathbf{h} &= \mathbf{f} \quad \vee \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i\left(t\right) = \mathsf{max}\left(f_i\left(t\right), g_i\left(t\right)\right).
\end{aligned}
$$

The set of trajectories is a lattice. Interval of trajectories (tubes) can be defined.



Example.

$$[\mathrm{f}](t) = \begin{pmatrix} \cos t + \left[0, t^2\right] \\ \sin t + [-1, 1] \end{pmatrix}$$

is an interval trajectory (or tube).

3 Tube arithmetics

If $[x]$ and $[y]$ are two scalar tubes, we have

$$[z] = [x] + [y] \Rightarrow [z](t) = [x](t) + [y](t) \qquad \text{(sum)}$$
$$[z] = \mathsf{shift}_a([x]) \Rightarrow [z](t) = [x](t+a) \qquad \text{(shift)}$$
$$[z] = [x] \circ [y] \Rightarrow [z](t) = [x]([y](t)) \qquad \text{(compositi}$$
$$[z] = \int [x] \Rightarrow [z](t) = \left[\int_0^t x^-(\tau)\,d\tau, \int_0^t x^+(\tau)\,d\tau\right] \qquad \text{(integral)}$$

4 Tube contractors

Tube arithmetic allows us to build contractors.

Consider for instance the differential constraint

$$\begin{array}{rcl} \dot{x}(t) & = & x(t+1) \cdot u(t), \\ x(t) & \in & [x](t), \dot{x}(t) \in [\dot{x}](t), u(t) \in [u](t) \end{array}$$

We decompose as follows

$$\left\{ \begin{array}{rcl} x(t) & = & x(0) + \int_0^t y(\tau) d\tau \\ y(t) & = & a(t) \cdot u(t). \\ a(t) & = & x(t+1) \end{array} \right.$$

Possible contractors are

$$\left\{\begin{array}{rcl}
[x](t) & = & [x](t) \cap \left([x](0) + \int_0^t [y](\tau)\, d\tau\right) \\
[y](t) & = & [y](t) \cap [a](t) \cdot [u](t) \\
[u](t) & = & [u](t) \cap \frac{[y](t)}{[a](t)} \\
[a](t) & = & [a](t) \cap \frac{[y](t)}{[u](t)} \\
[a](t) & = & [a](t) \cap [x](t+1) \\
[x](t) & = & [x](t) \cap [a](t-1)
\end{array}\right.$$

Example. Consider $x(t) \in [x](t)$ with the constraint

$$\forall t,\ x(t) = x(t+1)$$

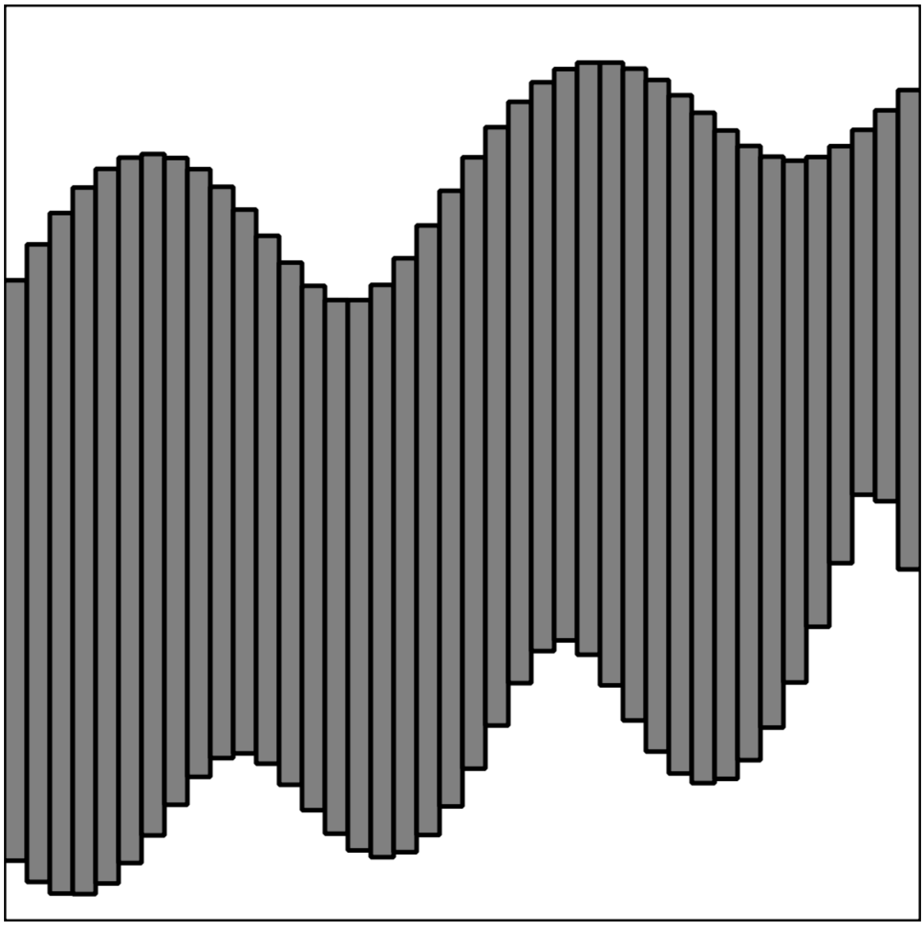
Contract the tube $[x](t)$.

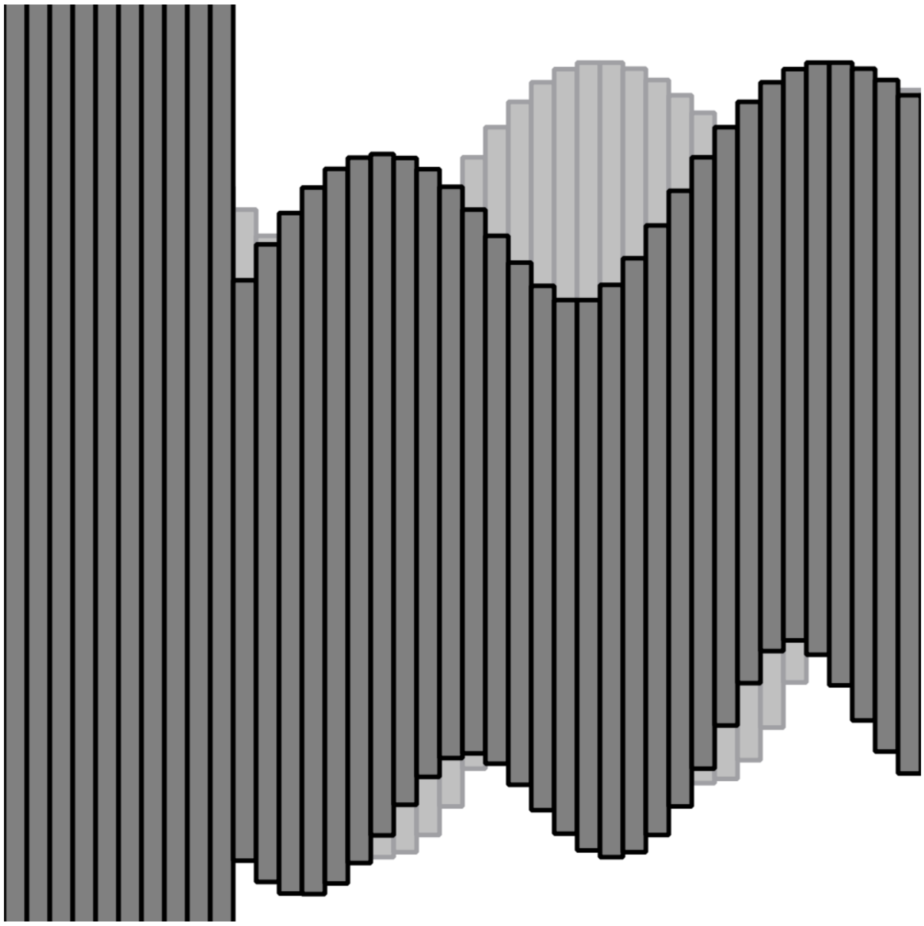
We first decompose into primitive trajectory constraints

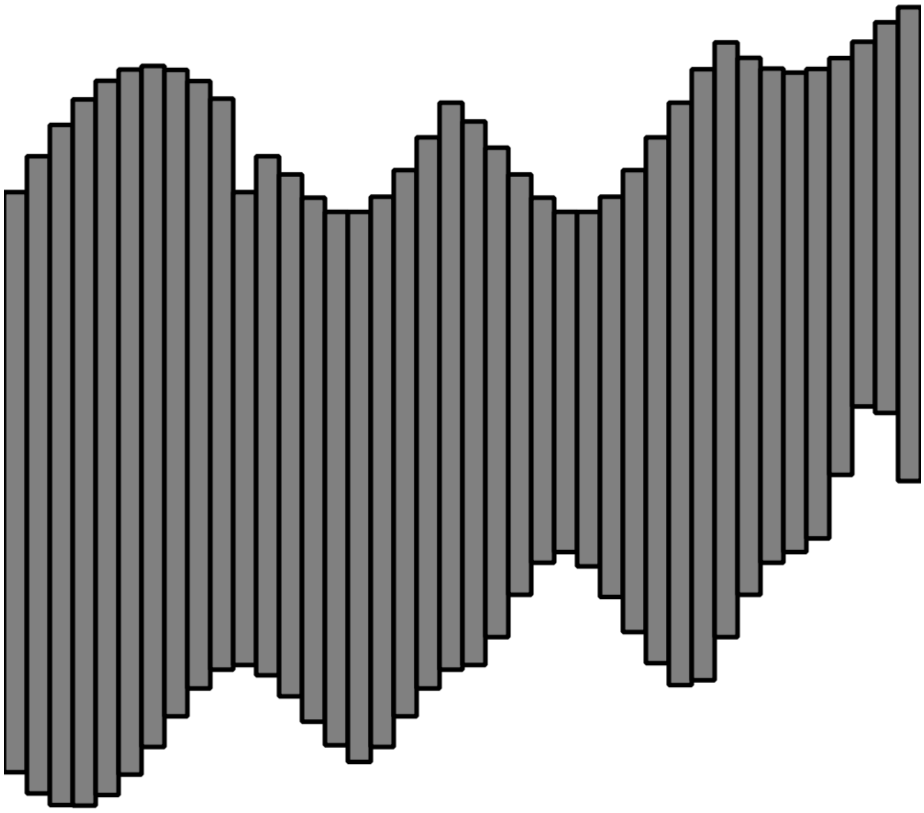
$$\begin{aligned} x(t) &= a(t+1) \\ x(t) &= a(t). \end{aligned}$$

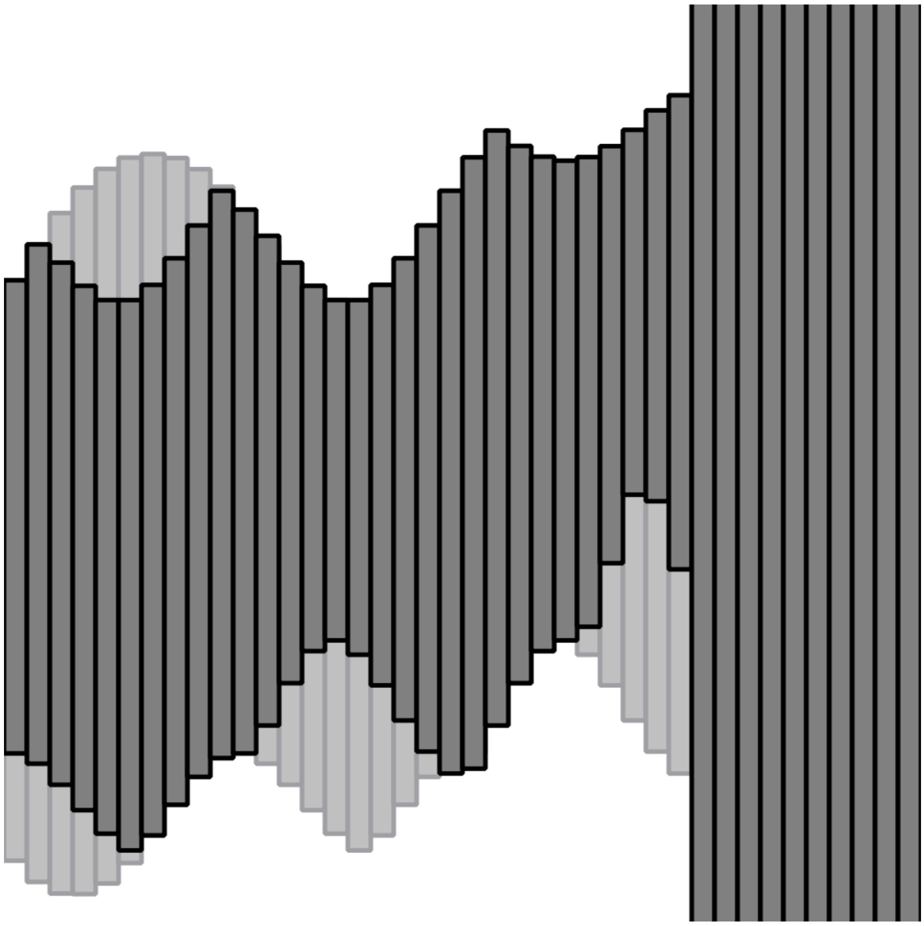
Contractors

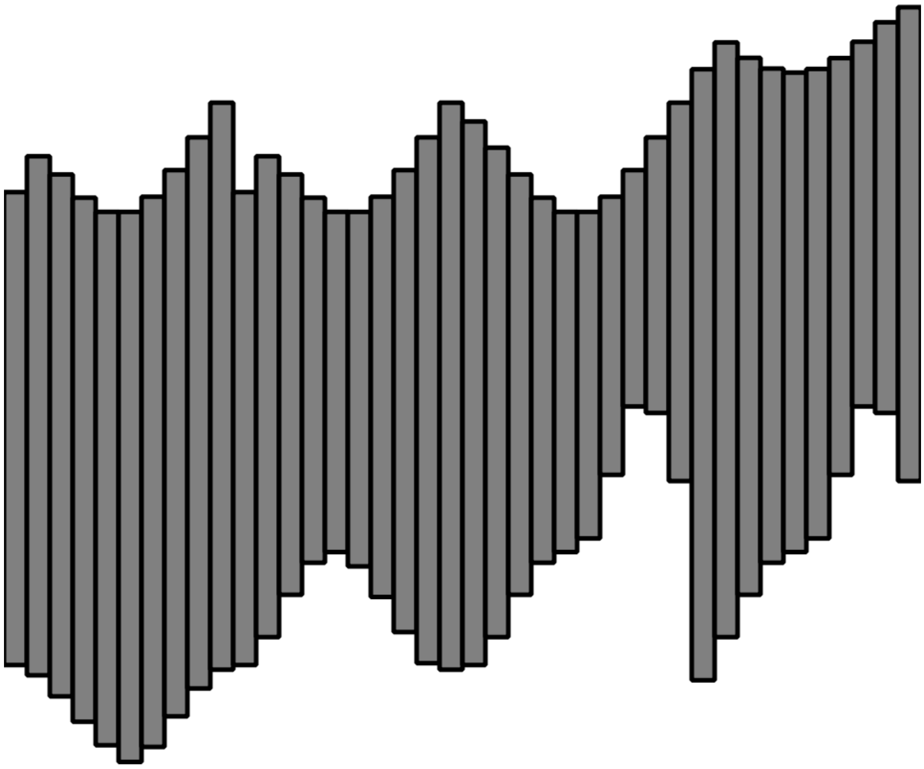
$$
\begin{aligned}
[x](t) &:= [x](t) \cap [a](t+1) \\
[a](t) &:= [a](t) \cap [x](t-1) \\
[x](t) &:= [x](t) \cap [a](t) \\
[a](t) &:= [a](t) \cap [x](t)
\end{aligned}
$$

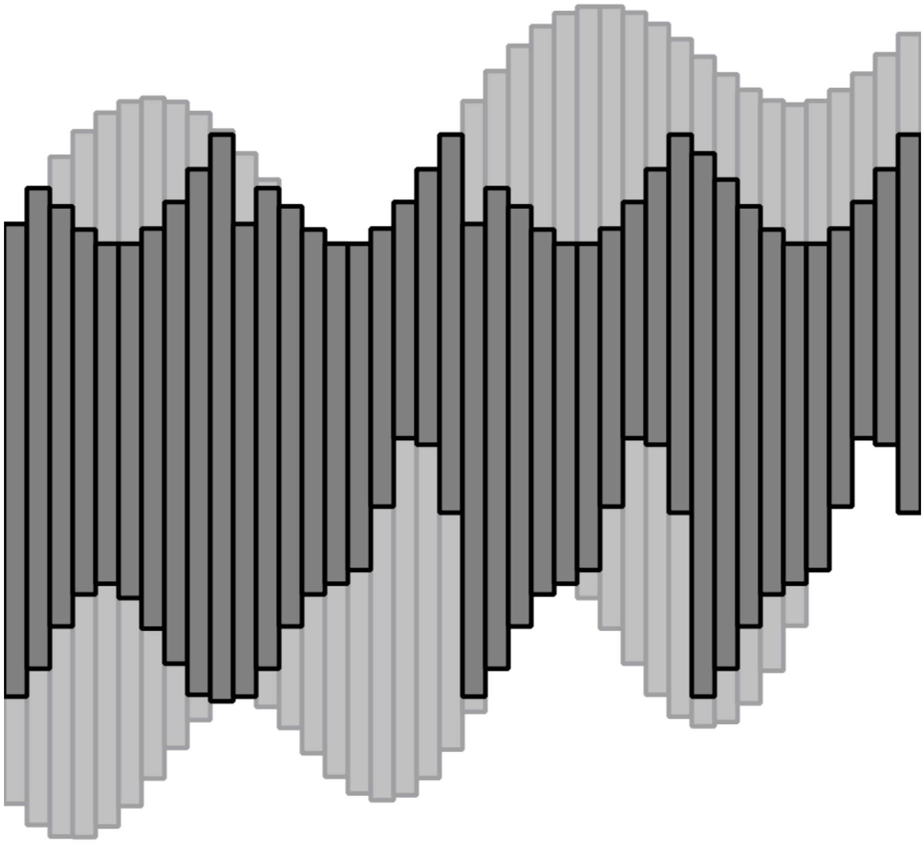


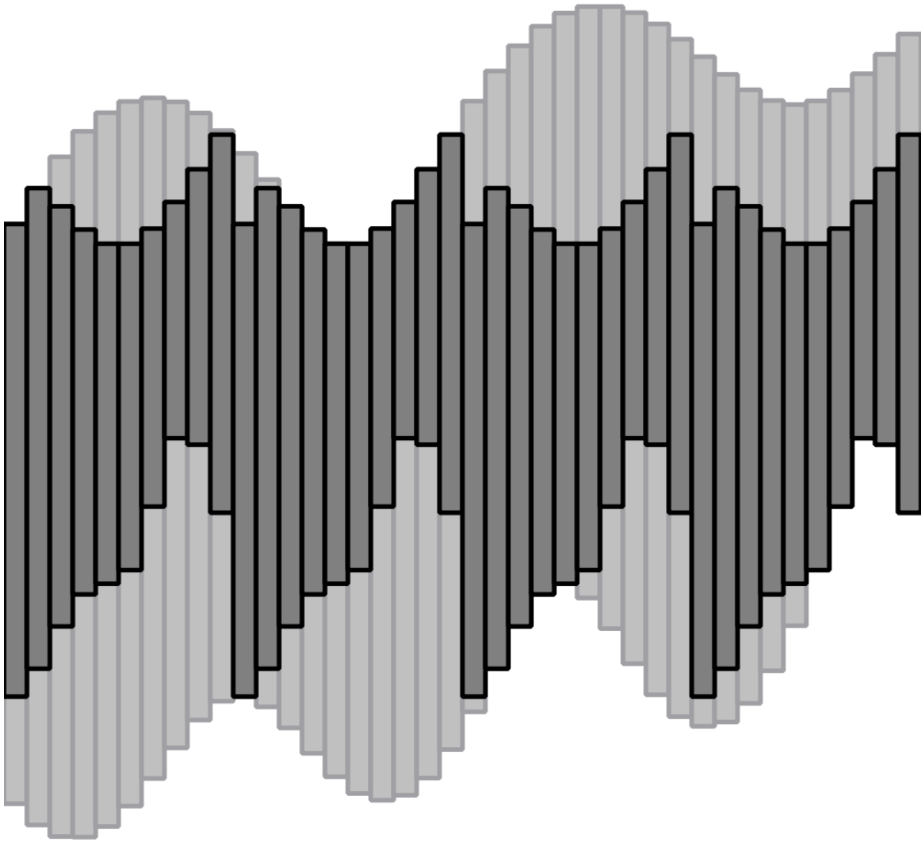












5 Time-space estimation

Classical state estimation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} = \mathbf{g}(\mathbf{x}(t), t) & t \in \mathbb{T} \subset \mathbb{R}. \end{cases}$$

Space constraint $\mathbf{g}(\mathbf{x}(t), t) = 0$.

Example.

$$
\begin{cases}
\dot{x}_1 = x_3 \cos x_4 \\
\dot{x}_2 = x_3 \cos x_4 \\
\dot{x}_3 = u_1 \\
\dot{x}_4 = u_2 \\
(x_1(5) - 1)^2 + (x_2(5) - 2)^2 - 4 = 0 \\
(x_1(7) - 1)^2 + (x_2(7) - 2)^2 - 9 = 0
\end{cases}
$$

With time-space constraints

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t), \mathbf{x}(t'), t, t') & (t, t') \in \mathbb{T} \subset \mathbb{R} \times \mathbb{R}. \end{cases}$$

Example. An ultrasonic underwater robot with state

$$\mathbf{x} = (x_1, x_2, \dots) = (x, y, \theta, v, \dots)$$

At time t the robot emits an onmidirectional sound. At time t' it receives it

$$\left(x_1 - x_1^{'}\right)^2 + \left(x_2 - x_2^{'}\right)^2 - c\left(t - t'\right)^2 = 0.$$

6 Mass spring problem

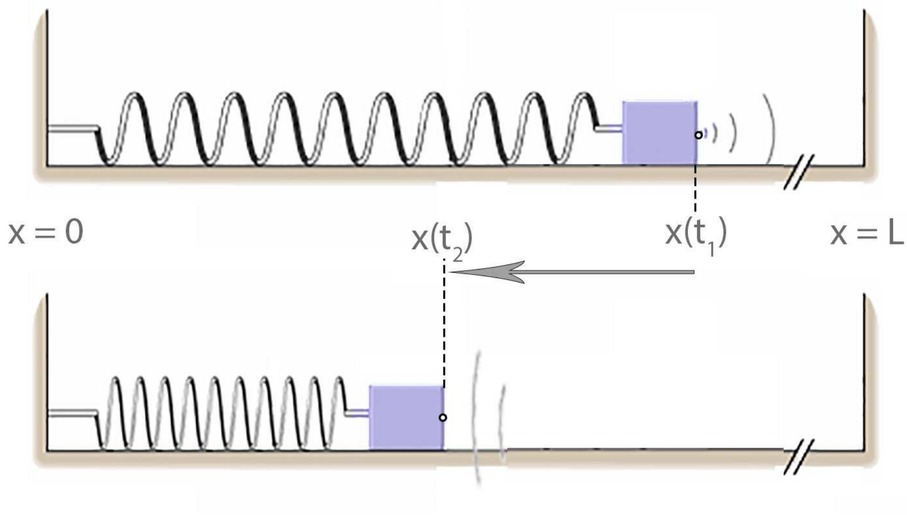
The mass spring satisfies

$$\ddot{x} + \dot{x} + x - x^3 = 0$$

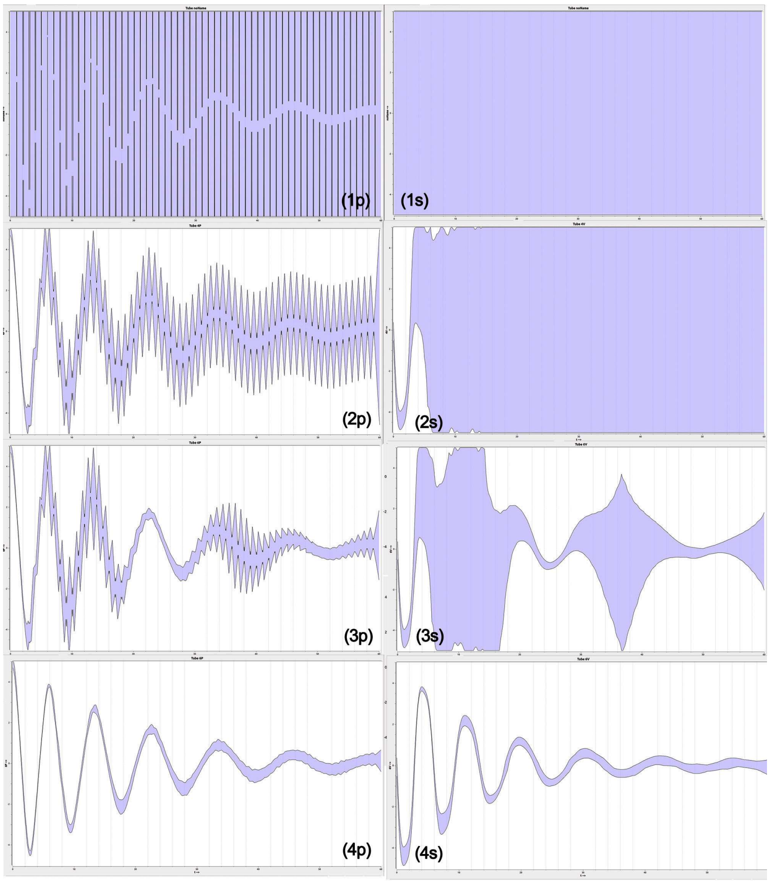
i.e.

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_2 - x_1 + x_1^3 \end{cases}$$

The initial state is unknown.


x = 0
x(t_2)
x(t_1)
x = L

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - x_1 + x_1^3 \\ L - x_1(t_1) + L - x_1(t_2) = c(t_2 - t_1). \end{cases}$$



(1p)
(1s)
(2p)
(2s)
(3p)
(3s)
(4p)
(4s)

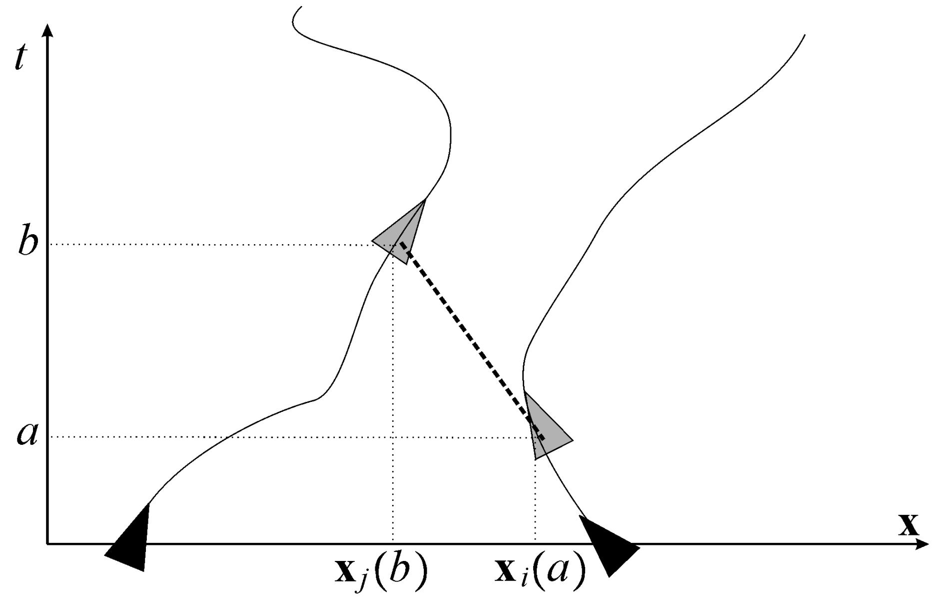
7 Swarm localization

Consider n robots $\mathcal{R}_1, \dots, \mathcal{R}_n$ described by

$$\dot{\mathbf{x}}_i = \mathbf{f}\left(\mathbf{x}_i, \mathbf{u}_i\right), \mathbf{u}_i \in \left[\mathbf{u}_i\right].$$

Omnidirectional sounds are emitted and received.

A *ping* is a 4-uple (a, b, i, j) where a is the emission time, b is the reception time, i is the emitting robot and j the receiver.


t
b
a
x
$\mathbf{x}_j(b)$
$\mathbf{x}_i(a)$

With the time space constraint

$$\dot{\mathbf{x}}_i = \mathbf{f}\left(\mathbf{x}_i, \mathbf{u}_i\right), \mathbf{u}_i \in \left[\mathbf{u}_i\right].$$
$$g\left(\mathbf{x}_{i(k)}\left(a\left(k\right)\right), \mathbf{x}_{j(k)}\left(b\left(k\right)\right), a\left(k\right), b\left(k\right)\right) = 0$$

where

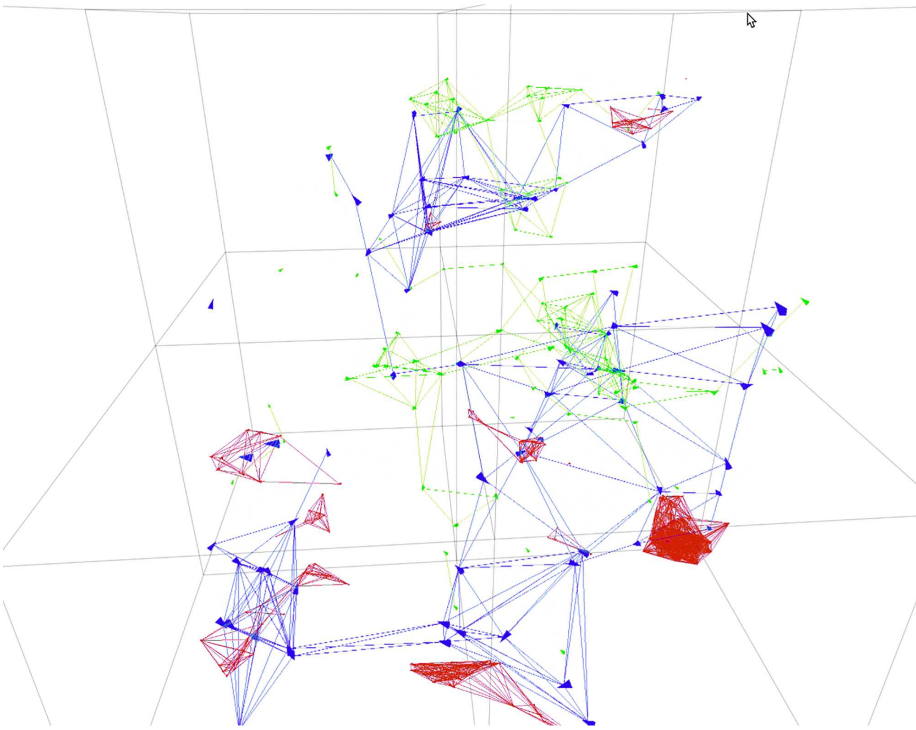
$$g\left(\mathbf{x}_i, \mathbf{x}_j, a, b\right) = \|x_1 - x_2\| - c\left(b - a\right).$$

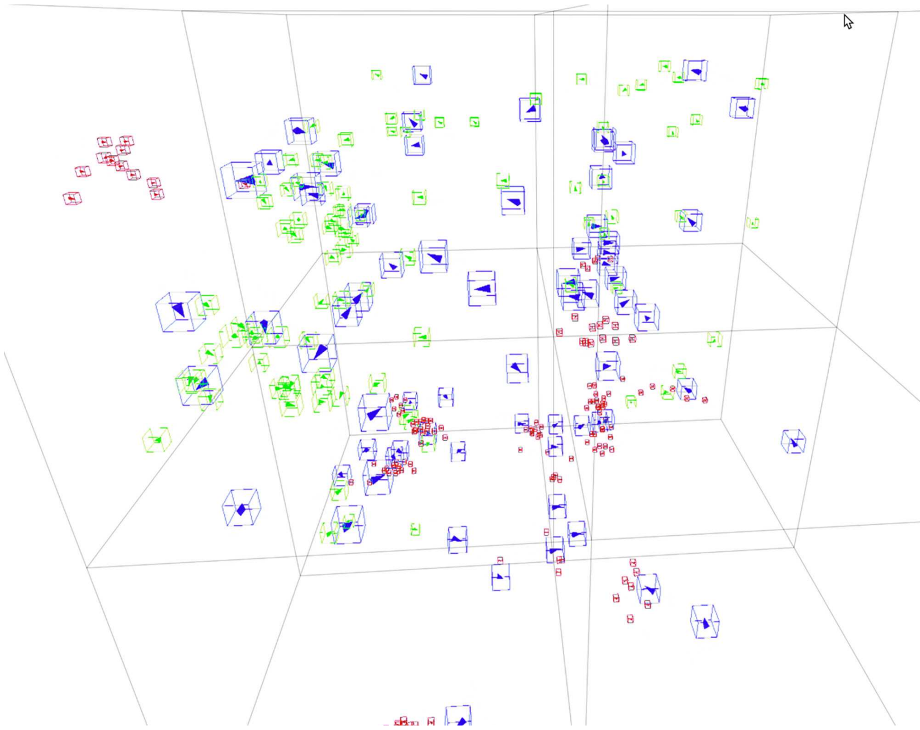
Clocks are uncertain. We only have measurements $\tilde{a}(k), \tilde{b}(k)$ of $a(k), b(k)$ thanks to clocks h_i. Thus

$$\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k))
\end{array}$$

The drift of the clocks is bounded

$$
\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k)) \\
\dot{h}_i = 1 + n_h,\ n_h \in [n_h]
\end{array}
$$





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A. Bethencourt and L. Jaulin (2013). Cooperative localization of underwater robots with unsynchronized clocks, *Journal of Behavioral Robotics*, Volume 4, Issue 4, pp 233-244, pdf.

A. Bethencourt and L. Jaulin (2014). Solving non-linear constraint satisfaction problems involving time-dependant functions. *Mathematics in Computer Science*.