Actions of the hyperoctahedral group to compute minimal contractors

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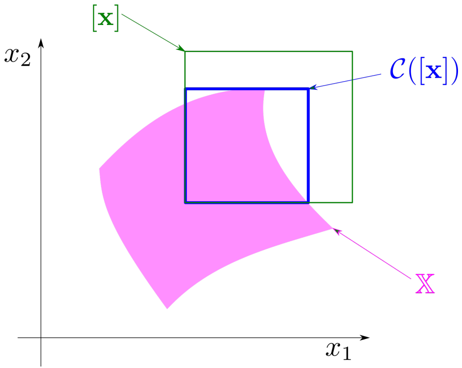
Palaiseau, Télécom

Motivation


[x]
x_2
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$
x_1

We want a procedure to generate automatically an optimal contractor for the *rotate* constraint

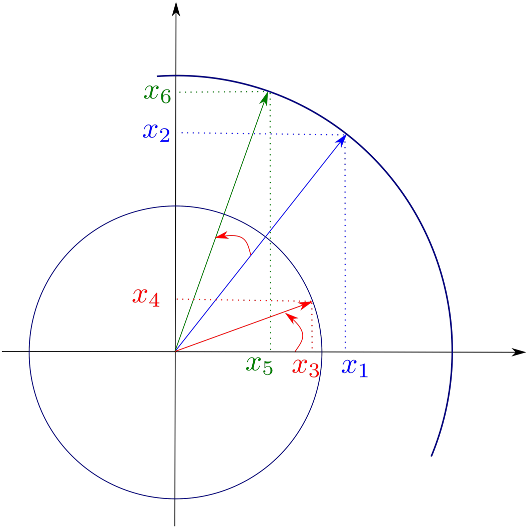
$$\mathbb{X} : \begin{cases} x_3 x_1 - x_4 x_2 - x_5 & = & 0 \\ x_4 x_1 + x_3 x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{cases}$$

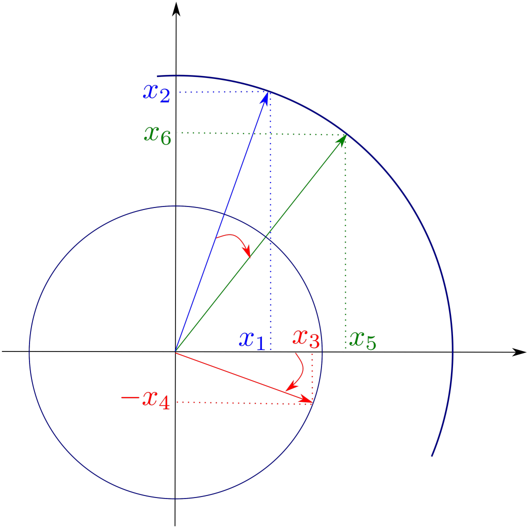
which is a generalization of the polar constraint [5]

$$\mathbb{X} : \begin{cases} x_3 x_1 - x_5 & = & 0 \\ x_4 x_1 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{cases}$$

We will take into account hyperoctahedral symmetries such as

$$\left\{ \begin{array}{rcl} x_3 x_1 - x_4 x_2 - x_5 & = & 0 \\ x_4 x_1 + x_3 x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} x_3 x_5 + x_4 x_6 - x_1 & = & 0 \\ -x_4 x_5 + x_3 x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array} \right.$$


x_6
x_2
x_4
x_5
x_3
x_1


x_2
x_6
x_1
x_3
x_5
$-x_4$

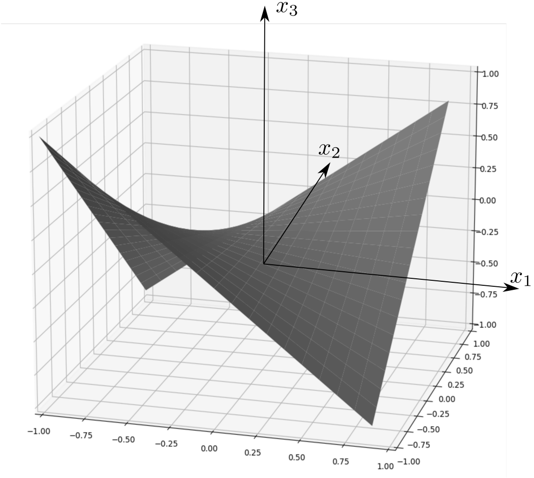
Multiplication

We consider the product

$$x_3 = x_1 \cdot x_2$$

Equivalently

$$\mathbb{X} = \{(x_1, x_2, x_3) \mid x_1 \cdot x_2 = x_3\}$$


x_3
x_2
x_1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

We have

$$x_1 \cdot x_2 = x_3 \Leftrightarrow (-x_1) \cdot x_2 = -x_3$$

We say that $x_1 \cdot x_2 = x_3$ is invariant by the symmetry

$$\sigma_1 : \begin{cases} x_1 & \mapsto & -x_1 \\ x_2 & \mapsto & x_2 \\ x_3 & \mapsto & -x_3 \end{cases}$$

Equivalently, $x_1 \cdot x_2 = x_3$ is said to be invariant by

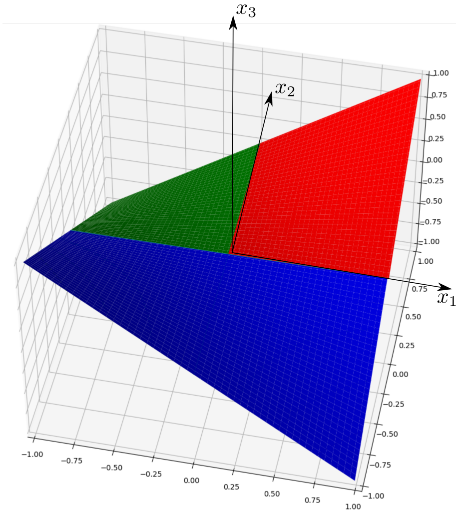
$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We also have

$$x_1 \cdot x_2 = x_3 \Leftrightarrow x_1 \cdot (-x_2) = -x_3$$

invariant with respect to

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$


x_3
x_2
x_1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

Due to the monotonicity, the minimal contractor for the box

$$[\mathbf{x}] \subset [\mathbf{a}] = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ .$$

associated to $x_1 \cdot x_2 = x_3$ is

$$\mathscr{C}_0 \begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap [\frac{x_3^-}{x_2^+}, \frac{x_3^+}{x_2^-}] \\ [x_2] \cap [\frac{x_3^-}{x_1^+}, \frac{x_3^+}{x_1^-}] \\ [x_3] \cap [x_1^- \cdot x_2^-, x_1^+ \cdot x_2^+] \end{pmatrix}.$$

Hyperoctohedral symmetries

The hyperoctahedral group B_n is the group of symmetries of the unit hypercube [4] of $\mathbb{R}^n$.
It contains $2^n \cdot n!$ elements.

For $n = 2$, we have $2^2 \cdot 2! = 8$ elements:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\sigma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_7 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We will write equivalently

$$\sigma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \sigma : \begin{cases} \mathbb{R}^2 & \mapsto & \mathbb{R}^2 \\ (x_1, x_2) & \mapsto & (x_2, -x_1) \end{cases}$$

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_1	σ_1	σ_0	σ_6	σ_4	σ_3	σ_7	σ_2	σ_5
σ_2	σ_2	σ_5	σ_0	σ_7	σ_6	σ_1	σ_4	σ_3
σ_3	σ_3	σ_4	σ_7	σ_0	σ_1	σ_6	σ_5	σ_2
σ_4	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2	σ_7	σ_6
σ_5	σ_5	σ_2	σ_4	σ_6	σ_7	σ_3	σ_0	σ_1
σ_6	σ_6	σ_7	σ_1	σ_5	σ_2	σ_0	σ_3	σ_4
σ_7	σ_7	σ_6	σ_3	σ_2	σ_5	σ_4	σ_1	σ_0

Multiplication table

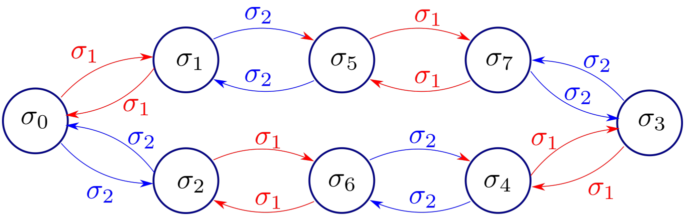
The two elements σ_1, σ_2 are generators of the group B_2 or equivalently, we write $B_2 =< \sigma_1, \sigma_2 >$.

Compute for instance $\sigma_6 \circ \sigma_7$, we get

$$\sigma_6 \circ \sigma_7 = \underbrace{\sigma_1 \circ \sigma_2}_{\sigma_6} \circ \underbrace{\sigma_1 \circ \underbrace{\sigma_2 \circ \sigma_1}_{\sigma_5}}_{\sigma_7} = \sigma_4$$

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_1	σ_1	σ_0	σ_6	σ_4	σ_3	σ_7	σ_2	σ_5
σ_2	σ_2	σ_5	σ_0	σ_7	σ_6	σ_1	σ_4	σ_3
σ_3	σ_3	σ_4	σ_7	σ_0	σ_1	σ_6	σ_5	σ_2
σ_4	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2	σ_7	σ_6
σ_5	σ_5	σ_2	σ_4	σ_6	σ_7	σ_3	σ_0	σ_1
σ_6	σ_6	σ_7	σ_1	σ_5	σ_2	σ_0	σ_3	σ_4
σ_7	σ_7	σ_6	σ_3	σ_2	σ_5	σ_4	σ_1	σ_0

To prove that $\{\sigma_1, \sigma_2\}$ is a generator pair of B_2, we build the Cayley graph.



Cayley graph associated to B_2

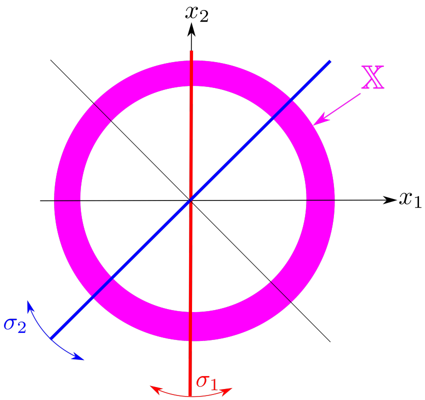
We see that $\sigma_6 \circ \sigma_7 = \underbrace{\sigma_1 \circ \sigma_2}_{\sigma_6} \circ \underbrace{\sigma_1 \circ \sigma_2 \circ \sigma_1}_{\sigma_7} = \sigma_4$

Hyperoctohedral symmetries of a set

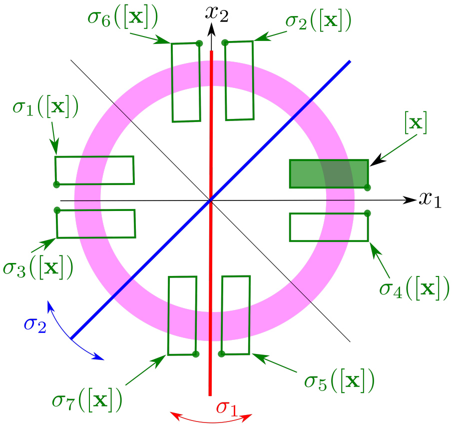
We define the *stabiliser* of B_n with respect to $\mathbb{X} \subset \mathbb{R}^n$ as

$$B_n(\mathbb{X}) = \{\sigma \in B_n \,|\, \sigma(\mathbb{X}) = \mathbb{X}\}.$$

$B_n(\mathbb{X})$ is a subgroup of B_n.



How many hyperoctahedral stabilizers for the set $\mathbb{X}$?



We have 8 hyperoctahedral stabilizers

Checking that a symmetry is a stabilizer

Checking that $\sigma \in B_n(\mathbb{X})$ *i.e.*, $\sigma(\mathbb{X}) = \mathbb{X}$ amounts to check that two polynomial equalities are equivalent.

For

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the equality $\sigma(\mathbb{X}) = \mathbb{X}$ rewrites into

$$\left\{ \begin{array}{ccc} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ccc} x_3x_5 + x_4x_6 - x_1 & = & 0 \\ -x_4x_5 + x_3x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array} \right.$$

```
from sympy import *
x1,x2,x3,x4,x5,x6=symbols('x1 x2 x3 x4 x5 x6')
S1=[x3*x1-x4*x2-x5,x4*x1+x3*x2-x6,x3**2+x4**2-1]
S2=[x3*x5+x4*x6-x1,-x4*x5+x3*x6-x2,x3**2+(-x4)**2-1]
G1=groebner(S1,x1,x2,x3,x4,x5,x6,order='lex')
G2=groebner(S2,x1,x2,x3,x4,x5,x6,order='lex')
print(G1==G2)
```

We have indeed the form

$$\left\{\begin{array}{lcl} P_1(\mathbf{x}) & = & 0 \\ P_2(\mathbf{x}) & = & 0 \\ P_3(\mathbf{x}) & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} Q_1(\mathbf{x}) & = & 0 \\ Q_2(\mathbf{x}) & = & 0 \\ Q_3(\mathbf{x}) & = & 0 \end{array}\right.$$

where $\mathbf{x} = (x_1, \ldots, x_6)$.

To prove the equivalence, we check that for all j,

$$Q_j(\mathsf{x}) = \sum_i A_{ij}(\mathsf{x}) \cdot P_i(\mathsf{x})$$

and that for all i,

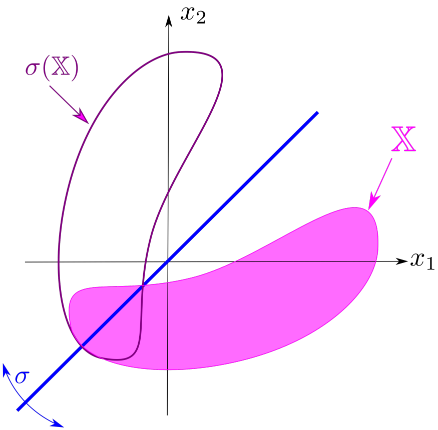
$$P_i(\mathsf{x}) = \sum_j B_{ij}(\mathsf{x}) \cdot Q_j(\mathsf{x}).$$

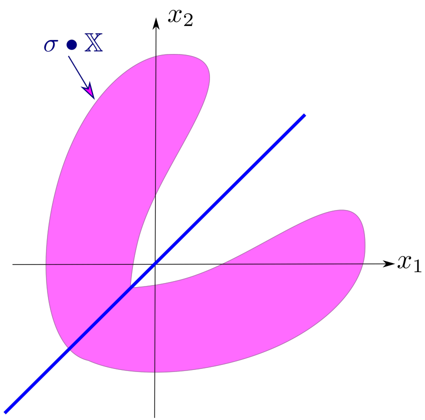
Equivalently, we check that the Grobner basis, computed by the Buchberger's algorithm, for $\{P_1, P_2, P_3\}$ and for $\{Q_1, Q_2, Q_3\}$ are the same.

Acts

For $\sigma \in B_n$, we define the *act* operator:

$$\sigma \bullet \mathbb{X} = \mathbb{X} \cup \sigma(\mathbb{X}).$$


x_2
$\sigma(\mathbb{X})$
$\mathbb{X}$
x_1
σ

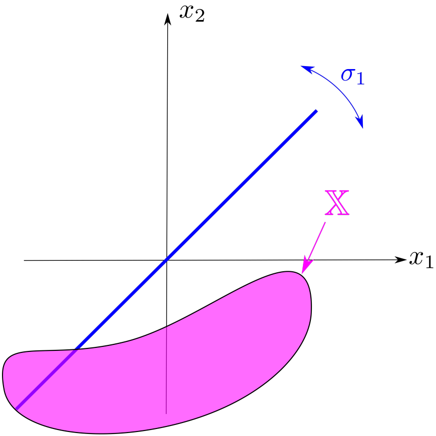


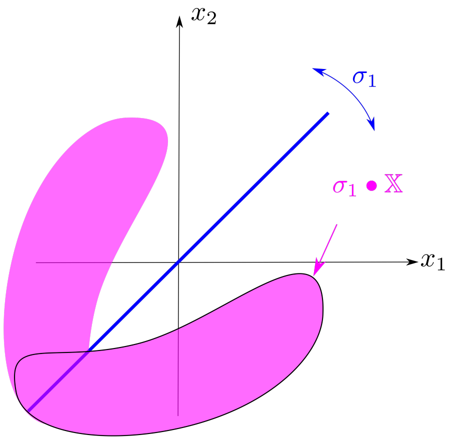
Semi-group action

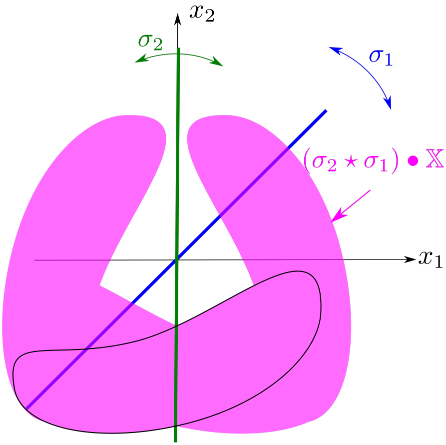
Given $\{\sigma_1, \sigma_2, \cdots\}$ in B_n, we define the operator $\star$ as follows

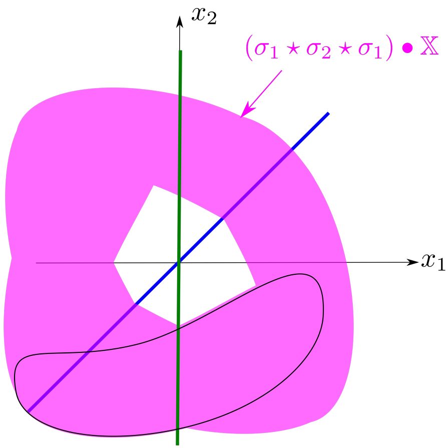
$$\begin{array}{rcl} (\sigma_2 \star \sigma_1) \bullet \mathbb{X} & = & \sigma_2 \bullet (\sigma_1 \bullet \mathbb{X}) \\ (\sigma_3 \star \sigma_2 \star \sigma_1) \bullet \mathbb{X} & = & \sigma_3 \bullet (\sigma_2 \bullet (\sigma_1 \bullet \mathbb{X})) \\ & \vdots & \end{array}$$

The algebraic structure $(\Sigma, \star)$ corresponds to a semi-group action [3].


x_2
σ_1
$\mathbb{X}$
x_1


x_2
σ_1
$\sigma_1 \bullet \mathbb{X}$
x_1


x_2
σ_2
σ_1
$(\sigma_2 \star \sigma_1) \bullet \mathbb{X}$
x_1


x_2
$(\sigma_1 \star \sigma_2 \star \sigma_1) \bullet \mathbb{X}$
x_1

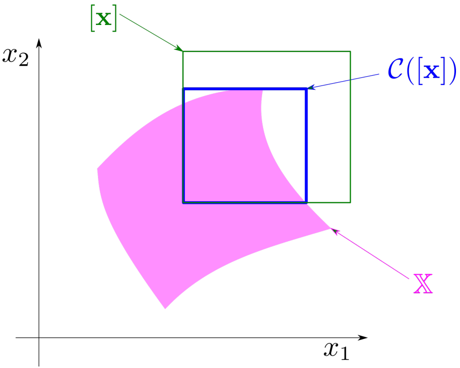
Contractor Acts

A *contractor* $\mathscr{C}$ for a set $\mathbb{X} \subset \mathbb{R}^n$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$\begin{array}{ll} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ [\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}]). & \text{(monotonicity)} \\ \mathscr{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(consistency)} \end{array}$$

There exists a is a minimal contractor for $\mathbb{X}$, given by

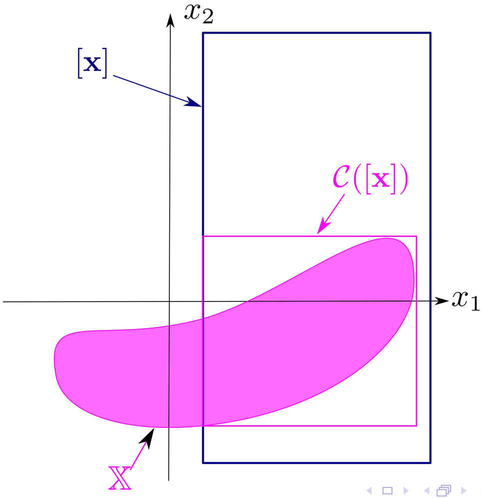
$$\mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$$

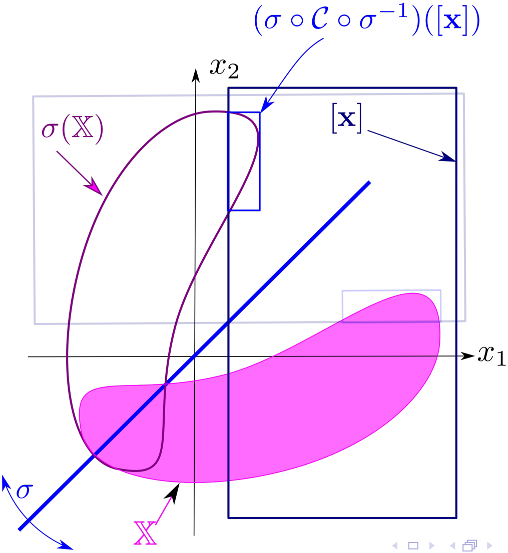


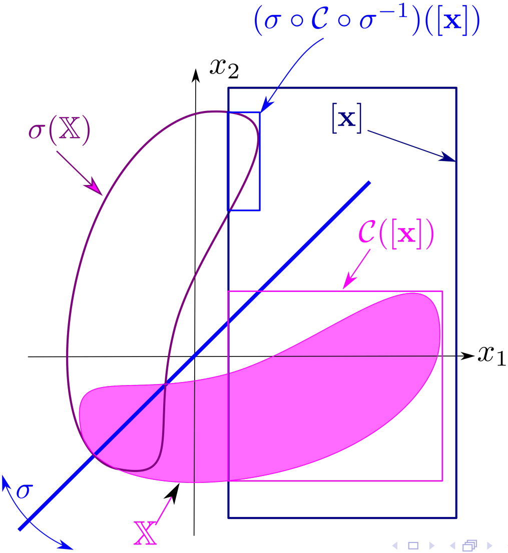
Minimal contractor $\mathscr{C}$ for the set $\mathbb{X}$

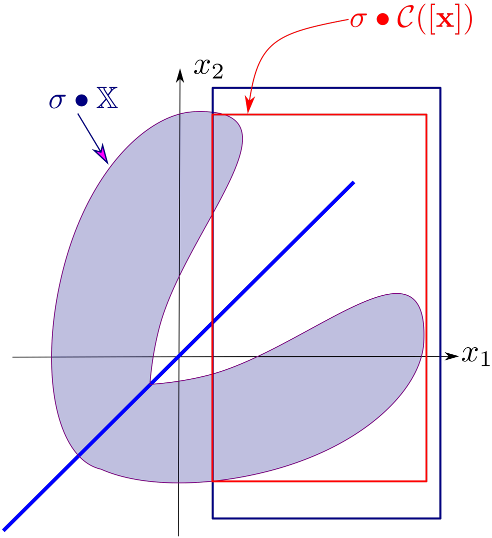
If $\mathscr{C}$ is a contractor in $\mathbb{R}^n$, and $\sigma \in B_n$, we define the *contractor act* of σ on $\mathscr{C}$ as

$$\sigma \bullet \mathscr{C} = \mathscr{C} \sqcup \sigma \circ \mathscr{C} \circ \sigma^{-1}.$$


x_2
$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
x_1
$\mathbb{X}$


$(\sigma \circ \mathcal{C} \circ \sigma^{-1})([\mathbf{x}])$
x_2
$\sigma(\mathbb{X})$
$[\mathbf{x}]$
x_1
σ
$\mathbb{X}$


$(\sigma \circ \mathcal{C} \circ \sigma^{-1})([\mathbf{x}])$
x_2
$\sigma(\mathbb{X})$
$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
x_1
σ
$\mathbb{X}$


$\sigma \bullet \mathcal{C}([\mathbf{x}])$
x_2
$\sigma \bullet \mathbb{X}$
x_1

Proposition. If $\mathscr{C}$ is a minimal contractor for $\mathbb{X}$ and if $\sigma \in B_n$ then $\sigma \bullet \mathscr{C}$ is a minimal contractor for $\sigma \bullet \mathbb{X}$, i.e.,

$$\sigma \bullet \mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \sigma \bullet \mathbb{X}]\!].$$

Corollary. If $\mathscr{C}$ is a minimal contractor for $\mathbb{X}$ and if $\sigma_1, \ldots, \sigma_k$ are in B_n then $(\sigma_k \star \cdots \star \sigma_1) \bullet \mathscr{C}$ is a minimal contractor for $(\sigma_k \star \cdots \star \sigma_1) \bullet \mathbb{X}$, i.e.,

Expansion theorem

Theorem. Consider a sequence $\{\sigma_0, \sigma_1, \dots, \sigma_{\bar{k}}\}$ of $B_n(\mathbb{X})$. Define the sequences

$$\begin{array}{rcl}\mathbb{X}(0) & = & [\mathbf{a}] \cap \mathbb{X} \\ \mathbb{X}(k+1) & = & \sigma_k \bullet \mathbb{X}(k)\end{array}$$

and

$$\begin{array}{rcl}\mathbb{A}(0) & = & [\mathbf{a}] \\ \mathbb{A}(k+1) & = & \sigma_k \bullet \mathbb{A}(k)\end{array}$$

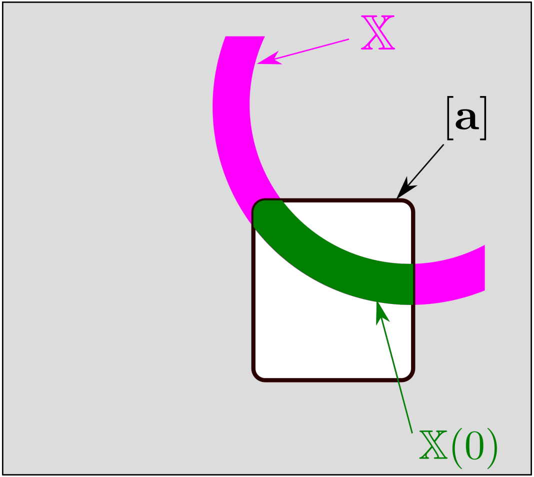
We have

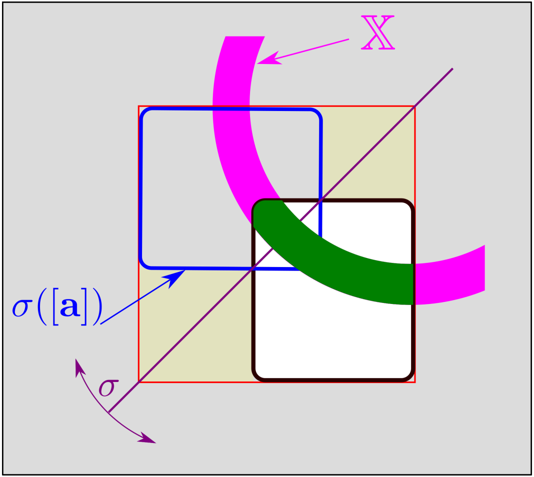
$$\mathbb{X} \subset \mathbb{A}(\bar{k}) \Rightarrow \mathbb{X}(\bar{k}) = \mathbb{X}.$$

Moreover, if $\mathbb{X} \subset \mathbb{A}(\bar{k})$, the contractor $\mathscr{C}_{\bar{k}}$ defined by the sequence

$$\begin{array}{rcl}\mathscr{C}_0([\mathbf{x}]) & = & [\![[\mathbf{x}] \cap \mathbb{X}(0)]\!] \\ \mathscr{C}_{k+1} & = & \sigma_k \bullet \mathscr{C}_k\end{array}$$

is the minimal contractor for $\mathbb{X}$.


$\mathbb{X}$
$[\mathbf{a}]$
$\mathbb{X}(0)$


$\mathbb{X}$
$\sigma([\mathbf{a}])$
σ

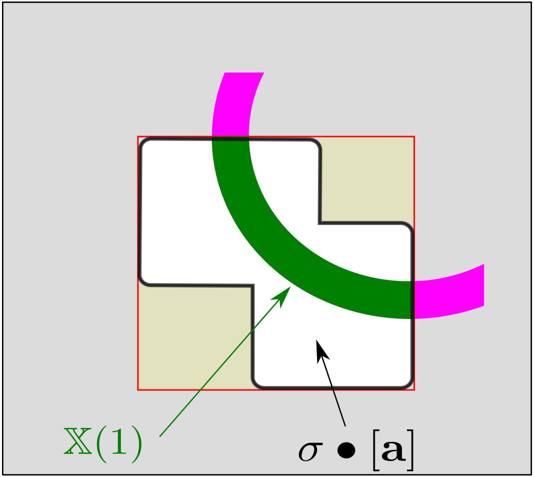
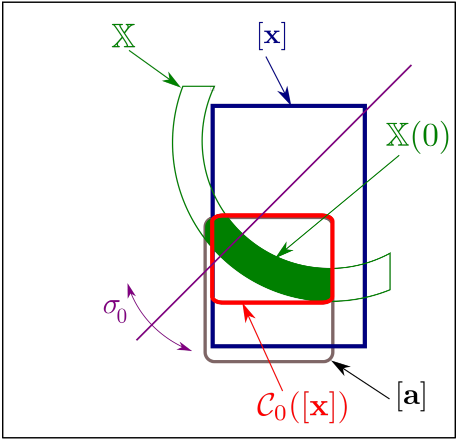
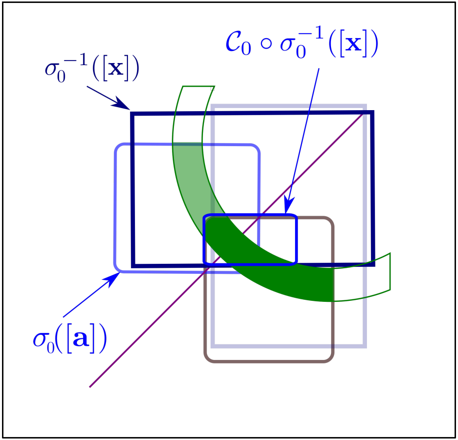
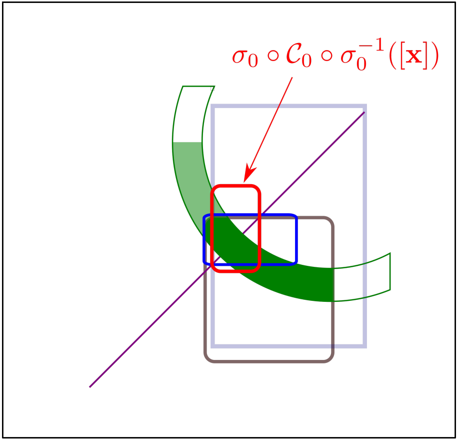
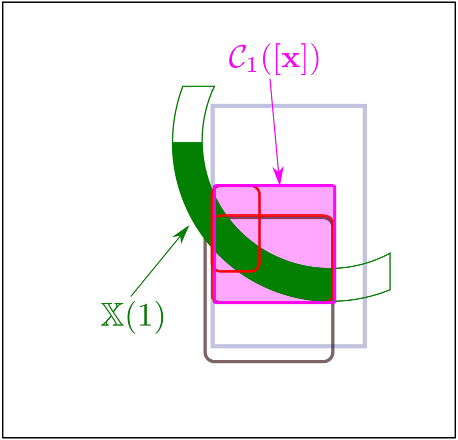

$\mathbb{X}(1)$
$\sigma \bullet [\mathbf{a}]$

Illustration of the contractor chain


$\mathbb{X}$
$[\mathbf{x}]$
$\mathbb{X}(0)$
σ_0
$\mathcal{C}_0([\mathbf{x}])$
$[\mathbf{a}]$


$\sigma_0^{-1}([\mathbf{x}])$
$\mathcal{C}_0 \circ \sigma_0^{-1}([\mathbf{x}])$
$\sigma_0([\mathbf{a}])$


$\sigma_0 \circ \mathcal{C}_0 \circ \sigma_0^{-1}([\mathbf{x}])$


$\mathcal{C}_1([\mathbf{x}])$
$\mathbb{X}(1)$

Corollary. If $\mathbb{X} \not\subset \mathbb{A}(\bar{k})$ then $\mathscr{C}_{\bar{k}}$ is not a contractor for $\mathbb{X}$. We do not have enough symmetries.

Algorithm

Consider a set $\mathbb{X} \subset \mathbb{R}^n$ and a box $[\mathbf{a}]$ such that $\mathscr{C}_0([\mathbf{x}]) = [\![[\mathbf{x}] \cap [\mathbf{a}] \cap \mathbb{X}]\!]$ is available.
Take $\{\sigma_0, \sigma_1, \ldots, \sigma_m\} \subset B_n(\mathbb{X})$.

ValidSequence. In: $\{\sigma_0, \sigma_1, \ldots, \sigma_m\}, [\mathbf{a}]$

$\mathbb{A}(0) = [\mathbf{a}]$
For $k \in \{0, \ldots, m\}$
    $\mathbb{A}(k+1) = \sigma_k \bullet \mathbb{A}(k)$,
If $\mathbb{X} \not\subset \mathbb{A}$ return "Fail: not enough symmetries".
Else return Success

FindMinimalSequence. In: $\mathscr{S}$,[**a**]

For $m \in \{0, 1, 2, 3, \ldots\}$
    For all sequences $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_m\}$ taken in $\mathscr{S}$
        If ValidSequence(Σ) successes return Σ

If the algorithm returns the sequence Σ, then from the expansion theorem, the minimal contractor for $\mathbb{X}$ is

$$\mathscr{C} = (\sigma_m \star \cdots \star \sigma_0) \bullet \mathscr{C}_0$$

Moreover, from the corollary, Σ is of minimal length.

Square constraint

We consider the set

$$\mathbb{X} : x_1^2 - x_2 = 0.$$

Over $[a] = \mathbb{R}^+ \times \mathbb{R}^+$, the minimal contractor is

$$\mathscr{C}_0 \begin{pmatrix} [x_1] \\ [x_2] \end{pmatrix} = \begin{pmatrix} [x_1] \cap [\sqrt{x_2^-}, \sqrt{x_2^+}] \\ [x_2] \cap [x_1^{-2}, x_1^{+2}] \end{pmatrix}.$$

The stabilizers for $\mathbb{X}$ are

$$B_2(\mathbb{X}) = \left\{ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

.

Indeed

$$x_1^2 - x_2 = 0 \Leftrightarrow (-x_1)^2 - x_2 = 0$$

ValidSequence($\{\sigma_1\}, [\mathbf{a}]$) returns True. Thus $\sigma_1 \bullet \mathscr{C}_0$ is the minimal contractor for $\mathbb{X}$.

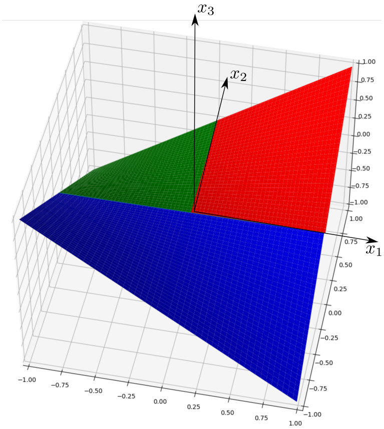
Product constraint

We consider the constraint

$$\mathbb{X}: \; x_1 x_2 = x_3.$$

i.e.

$$\mathbb{X} = \{\mathbf{x} = (x_1, x_2, x_3) | x_1 x_2 = x_3\}$$



Product constraint: $x_1 x_2 = x_3$

A minimal contractor over

$$[\mathbf{a}] = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

for the constraint $x_1 x_2 = x_3$ is:

$$\mathscr{C}_0 \begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap [\frac{x_3^-}{x_2^+}, \frac{x_3^+}{x_2^-}] \\ [x_2] \cap [\frac{x_3^-}{x_1^+}, \frac{x_3^+}{x_1^-}] \\ [x_3] \cap [x_1^- \cdot x_2^-, x_1^+ \cdot x_2^+] \end{pmatrix}.$$

B_3 has $2^3 * 3! = 48$ elements. One of them is

$$\sigma_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since

$$x_1 x_2 - x_3 = 0 \Leftrightarrow (-x_1) \cdot (-x_2) - x_3 = 0$$

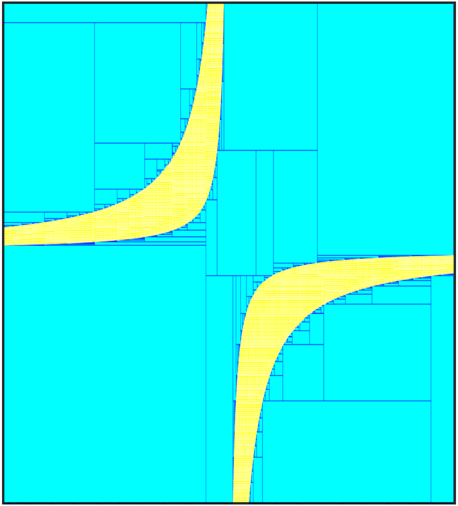
σ_0 is a stabilizer.

Other stabilizers are

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The algorithm FindMinimalSequence($\{\sigma_0, \sigma_1, \sigma_2\}, [\mathbf{a}]$) finds two expressions of minimal length for the minimal contractor for $\mathbb{X}$: $(\sigma_2 \star \sigma_0) \bullet \mathscr{C}_0$ and $(\sigma_0 \star \sigma_2) \bullet \mathscr{C}_0$.

Consider the set of all $\mathbf{x} \in \mathbb{R}^2$, such that $x_1 \cdot x_2 \in [-9,-2]$. We take the contractor $(\sigma_2 \star \sigma_0) \bullet \mathscr{C}_0$ inside a paver.



Rotate constraint

We consider the constraint

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\mathbf{R}_\theta} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}}$$

If $x_2 = 0$ then, we get the classical Polar constraint given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta \\ x_2 \sin\theta \end{pmatrix}.$$

We rewrite the constraint as

$$\mathbb{X}:\begin{cases} x_3x_1 - x_4x_2 - x_5 &= 0 \\ x_4x_1 + x_3x_2 - x_6 &= 0 \\ x_3^2 + x_4^2 - 1 &= 0 \end{cases}$$

Generator

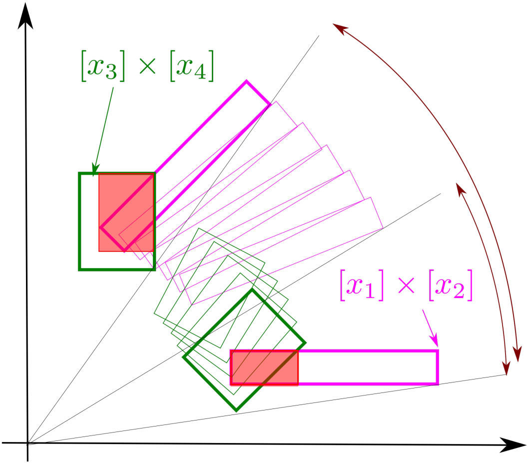
We take

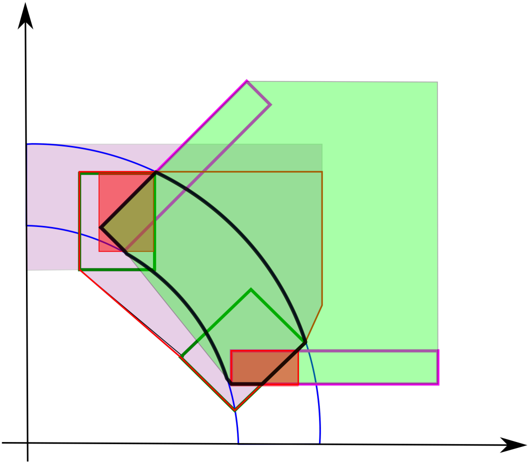
$$[\mathbf{a}] \quad = \quad \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \ .$$

To get a minimal contractor on $[\mathbf{a}]$, we need to build

$$\mathscr{C}_0 : [\mathbf{x}] \to [\![[\mathbf{x}] \cap [\mathbf{a}] \cap \mathbb{X}]\!].$$

A possibility is to use the contractor based on the monotonicity given in [2] [1].


$[x_3] \times [x_4]$
$[x_1] \times [x_2]$



Computing the stabilizers of *Rotate*

We have

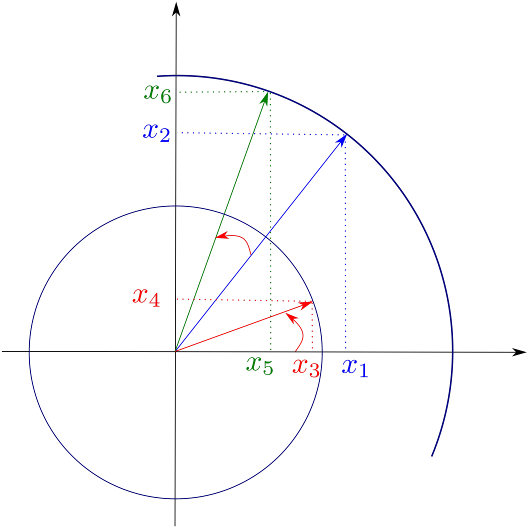
$$\mathrm{card}\,(B_6) = 2^6 \cdot 6! = 46080$$

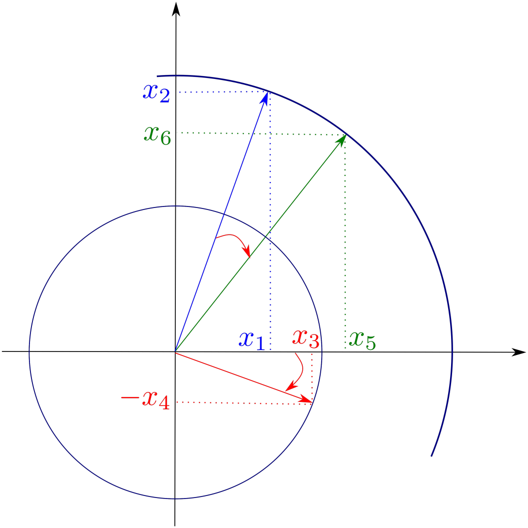
Take one of then, say

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We check that σ_0 is a stabilizer i.e.,

$$\left\{\begin{array}{ccc} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{ccc} x_3x_5 + x_4x_6 - x_1 & = & 0 \\ -x_4x_5 + x_3x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array}\right.$$


x_6
x_2
x_4
x_5
x_3
x_1


x_2
x_6
x_1
x_3
x_5
$-x_4$

Other elements of $B_6(\mathbb{X})$ could be found, at least four of them should be added to be able to generate B_6. For instance

$$\begin{aligned}
\sigma_1 : \quad (x_1,x_2,x_3,x_4,x_5,x_6) &\mapsto (x_5,x_6,x_3,-x_4,x_1,x_2)\\
\sigma_2 : \quad (x_1,x_2,x_3,x_4,x_5,x_6) &\mapsto (x_2,-x_1,-x_4,x_3,x_5,x_6)\\
\sigma_3 : \quad (x_1,x_2,x_3,x_4,x_5,x_6) &\mapsto (x_1,-x_2,x_3,-x_4,x_5,-x_6)\\
\sigma_4 : \quad (x_1,x_2,x_3,x_4,x_5,x_6) &\mapsto (-x_1,-x_2,-x_3,-x_4,x_5,x_6)\\
\sigma_5 : \quad (x_1,x_2,x_3,x_4,x_5,x_6) &\mapsto (-x_1,x_2,x_3,-x_4,-x_5,x_6)
\end{aligned}$$

If we run the algorithm FindMinimalSequence we conclude that:
There exists 0 sequence of length 4 among the 4!=144 which generates $B_6(\mathbb{X})$.
There exist 54 sequences of length 5 among the 5!=720 existing ones which generates $B_6(\mathbb{X})$.
One of them is $\sigma_5 \star \sigma_4 \star \sigma_3 \star \sigma_2 \star \sigma_1$.
A minimal contractor is given by $(\sigma_5 \star \sigma_4 \star \sigma_3 \star \sigma_2 \star \sigma_1) \bullet \mathscr{C}_0$.

```
def Crot(X1,X2,X3,X4,X5,X6):
   def C0(X1,X2,X3,X4,X5,X6): ...
   def A(C,s,_s):
      return lambda X1,X2,X3,X4,X5,X6 :
         union_tuple(C(X1,X2,X3,X4,X5,X6),
            s(*C(*_s(X1,X2,X3,X4,X5,X6))))
   def s1(X1,X2,X3,X4,X5,X6):
      return X5,X6,X3,-X4,X1,X2
   def s2(X1,X2,X3,X4,X5,X6):
      return X2,-X1,-X4,X3,X5,X6
   def _s2(X1,X2,X3,X4,X5,X6):
      return -X2, X1,X4,-X3,X5,X6
   def s3(X1,X2,X3,X4,X5,X6):
      return X1,-X2,X3,-X4,X5,-X6
   def s4(X1,X2,X3,X4,X5,X6):
      return -X1,-X2,-X3,-X4,X5,X6
```

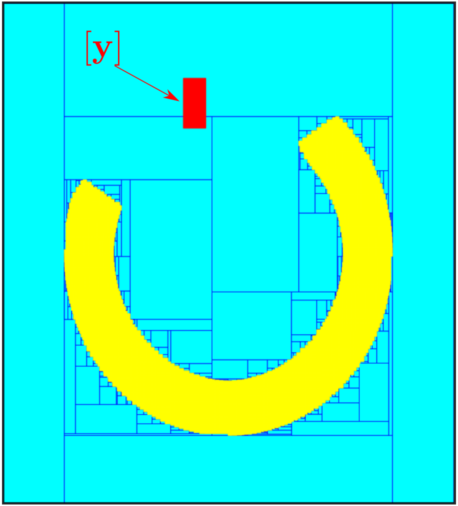
```
def s5(X1,X2,X3,X4,X5,X6):
   return -X1,X2,X3,-X4,-X5,X6
return A(A(A(A(A(C0,s1,s1),s2,_s2),s3,s3),
         s4,s4),s5,s5)(X1,X2,X3,X4,X5,X6)
```

Illustration

Consider the set of all $\mathbf{x} \in \mathbb{R}^2$, such that

$$\mathbf{y} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \mathbf{x}$$

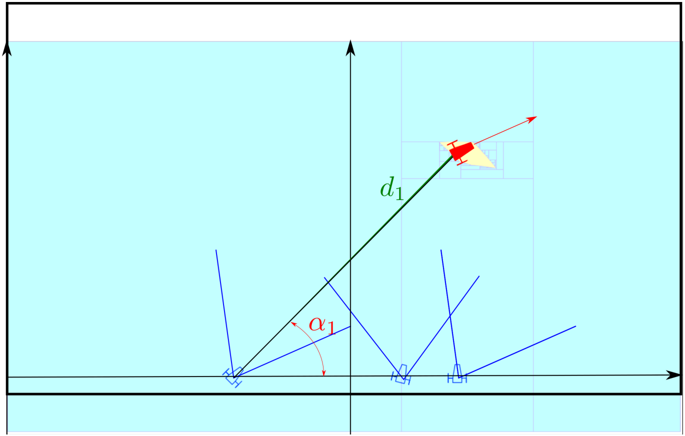
with $\theta \in [1,5]$, $\mathbf{y} \in [\mathbf{y}] = [-4,-2] \times [10,14]$. Using the contractor $(\sigma_4 \star \cdots \star \sigma_0) \bullet \mathscr{C}_0$ inside a paver.


[y]

Radar

We have three radars at location $\mathbf{a}_i$ observing a robot at position $\mathbf{x}$.
Each radar measures the distance d_i using the time of flight.
Using the Doppler effect, it measures of $\dot{d}_i$.
The angle α_i is measured with a poor accuracy (here 1rad).

i	1	2	3
$\mathbf{a}(i)$	$(-10,0)$	$(5,0)$	$(10,0$
$d(i) \in$	$[28,30]$	$[20,22]$	$[19,21]$
$\dot{d}(i) \in$	$[42,44]$	$[29,31]$	$[20,22]$
$\alpha(i) \in$	$[0,1]$	$[1,2]$	$[1,2]$



Three radars observing a robot

We have

$$\mathbf{x} - \mathbf{a}_i = d_i \cdot \begin{pmatrix} \cos\alpha_i \\ \sin\alpha_i \end{pmatrix}$$

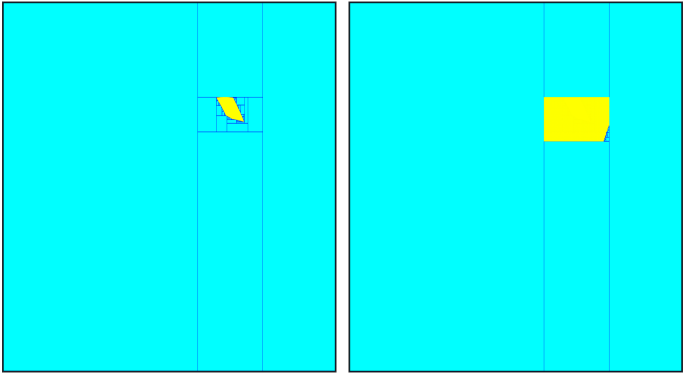
and

$$\dot{\mathbf{x}} = \begin{pmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{pmatrix} \cdot \begin{pmatrix} \dot{d}_i \\ d_i \cdot \dot{\alpha}_i \end{pmatrix}.$$

We thus get the system of constraints

$$\begin{array}{cc} \mathbf{p}_i = \mathbf{x} - \mathbf{a}_i & \forall i \in \{1, \ldots, 3\} \\ \text{rotate}(d_i, 0, c_i, s_i, p_{i1}, p_{i2}) & \\ \text{rotate}(\dot{d}_i, w_i, c_i, s_i, \dot{x}_1, \dot{x}_2) & \\ w_i = d_i \cdot \dot{\alpha}_i & \end{array}$$

We apply the corresponding contractors inside a paver.



Left: using the symmetry-based contractor; Right using classical interval contractors

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