Lie symmetries for an efficient propagation of uncertainties ; application to robot localization

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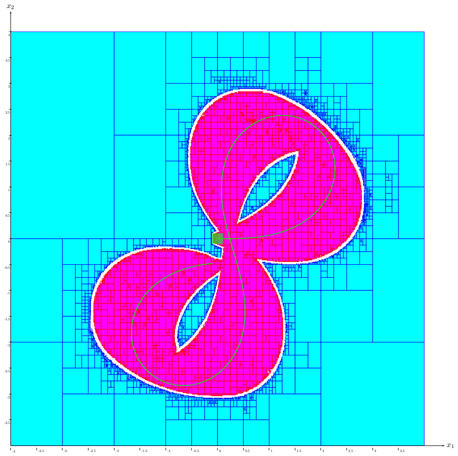
Consider the system

$$\begin{cases} \dot{x}_1 = u_1 \cdot \cos x_3 \\ \dot{x}_2 = u_1 \cdot \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}$$

where u_1, u_2 are time-dependent.



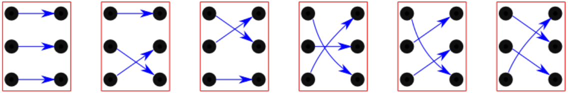
$$\operatorname{Proj}_{(x_1,x_2)} \bigcup_{t\in[0,14]} \mathbb{X}_t$$

Group action

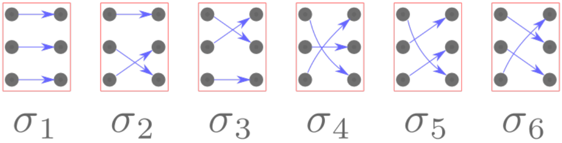
Define by $\mathbb{A} = \{a, b, c\}$.
The *symmetric group* is the set of all permutations in $\mathbb{A}$ is

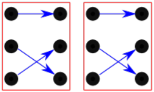
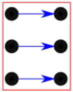
$$\begin{aligned} S_3 &= \{\sigma_1, \ldots, \sigma_6\} \\ &= \{abc \to abc, abc \to acb, abc \to bac, \\ &\quad abc \to cba, abc \to bca, abc \to cab\} \end{aligned}$$

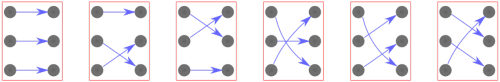
It is a group with respect to $\circ$.



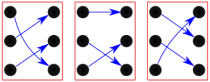
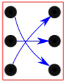
σ_1 σ_2 σ_3 σ_4 σ_5 σ_6



$\sigma_2 \circ \sigma_2 =$  $=$  $= \sigma_1$



σ_1 σ_2 σ_3 σ_4 σ_5 σ_6

$\sigma_6 \circ \sigma_2 \circ \sigma_6^{-1} =$  $=$  $= \sigma_4$

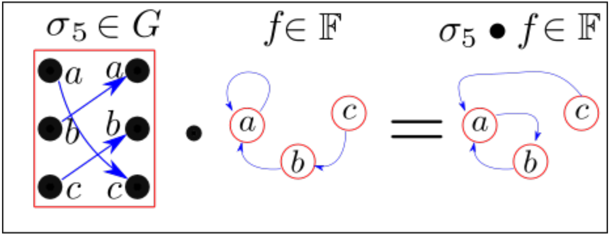
σ_6^{-1} σ_2 σ_6

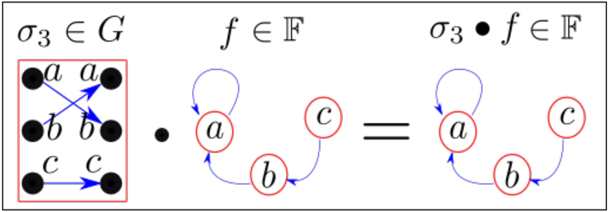
Action. Define by $\mathbb{F}$ the set of applications from $\mathbb{A}$ to $\mathbb{A}$. For instance

$$f_{aab} = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a \\ a \\ b \end{pmatrix} \right) \in \mathbb{F}$$

Given $\sigma \in S_3$, we define the *action* of σ on f as

$$\sigma \bullet f = f \circ \sigma.$$


$\sigma_5 \in G$
$f \in \mathbb{F}$
$\sigma_5 \bullet f \in \mathbb{F}$
a
a
b
b
c
c
a
b
c
a
b
c


$\sigma_3 \in G$
$f \in \mathbb{F}$
$\sigma_3 \bullet f \in \mathbb{F}$
a a
b b
c c
a
b
c
=
a
b
c

$(S_3, \circ, \bullet)$ is a *left group action* on $\mathbb{F}$ since

- $(S_3, \circ)$ is a group
- $\forall f \in \mathbb{F}, \sigma_1 \bullet f = f$ (identity)
- $(\sigma_i \circ \sigma_j) \bullet f = \sigma_i \bullet (\sigma_j \bullet f)$ (compatibility)

Stabilizer. For f in $\mathbb{F}$, the stabilizer group (or symmetry group) of G with respect to f is

$$G_f = Sym(f) = \{\sigma \in S_3 | \sigma \bullet f = f\}.$$

In our example we can check that

$$G_{f_{aab}} = \{\sigma_1, \sigma_3, \sigma_5\}$$

Differential group action

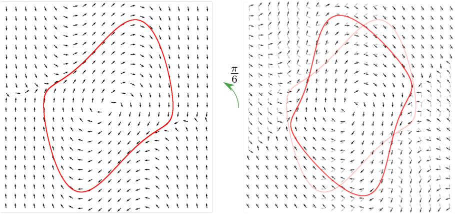
Define by $\mathbb{F}$ the set of all state equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$
The set $(\text{diff}(\mathbb{R}^n), \circ, \bullet)$ with

$$\mathbf{g} \bullet \mathbf{f} = \left(\frac{d\mathbf{g}}{d\mathbf{x}} \circ \mathbf{g}^{-1} \right) \cdot \left(\mathbf{f} \circ \mathbf{g}^{-1} \right).$$

is a left group action
Proposition. If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} = \mathbf{g}(\mathbf{x})$, we have $\dot{\mathbf{y}} = \mathbf{g} \bullet \mathbf{f}(\mathbf{y})$.

Example. If $\mathbf{g}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, we get

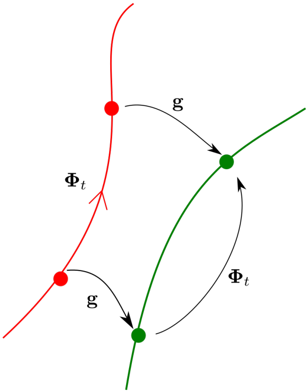
$$
\begin{aligned}
(\mathbf{g} \bullet \mathbf{f})(\mathbf{x}) &= \underbrace{\left(\frac{d\mathbf{g}}{d\mathbf{x}}\left(\mathbf{g}^{-1}(\mathbf{x})\right)\right)}_{\mathbf{A}} \cdot \left(\mathbf{f}(\mathbf{g}^{-1}(\mathbf{x})\right) \\
&= \mathbf{A} \cdot \mathbf{f}(\mathbf{A}^{-1} \cdot \mathbf{x}).
\end{aligned}
$$



A transformation $\mathbf{g}$ is a *stabilizer* of $\mathbf{f}$ if $\mathbf{g} \bullet \mathbf{f} = \mathbf{f}$.
Equivalently, $\mathbf{g} \in Sym(\mathbf{f})$.

Proposition. Define $\Phi\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ as the flow associated to $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$. We have:

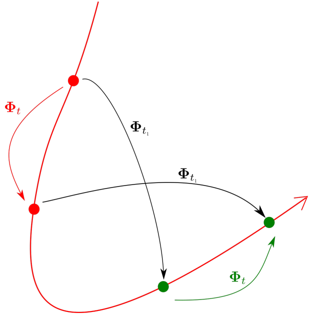
$$\mathbf{g}\bullet\mathbf{f}=\mathbf{f}\Leftrightarrow\Phi_t\circ\mathbf{g}=\mathbf{g}\circ\Phi_t.$$

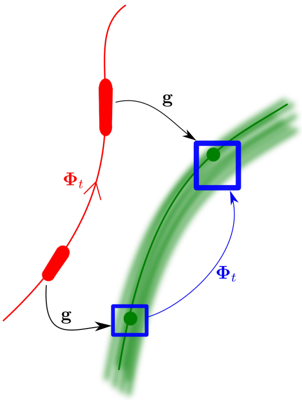


Clock symmetry. Since

$$\Phi_t \circ \Phi_{t_1} = \Phi_{t_1} \circ \Phi_t = \Phi_{t+t_1}$$

with $t_1 \in \mathbb{R}$, then $\Phi_{t_1} \bullet \mathbf{f} = \mathbf{f}$, *i.e.*, Φ_{t_1} is a stabilizer.

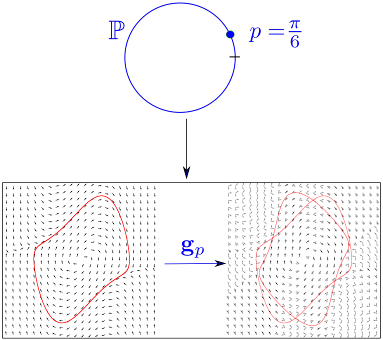



g
Φ_t
Φ_t
g

Lie group of symmetries

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and a manifold $\mathbb{P}$.
A Lie group $G_{\mathbf{p}}$ of symmetries is a family of transformations $\mathbf{g}_{\mathbf{p}}$, $\mathbf{p} \in \mathbb{P}$ such that

- $(G_{\mathbf{p}}, \circ)$ is a Lie group
- $\forall \mathbf{p} \in \mathbb{P}, \mathbf{g}_{\mathbf{p}} \bullet \mathbf{f} = \mathbf{f}$.



Here, $(G_p, \circ)$ is a Lie group but $\mathbf{g}_p \bullet \mathbf{f} \neq \mathbf{f}$

Transport function.

Given a Lie group of symmetries $G_{\mathbf{p}}$.

A *transport function* $\mathbf{h}(\mathbf{x},\mathbf{a})$ returns $\mathbf{p}$ so that $\mathbf{g}_{\mathbf{p}}(\mathbf{a})=\mathbf{x}$:

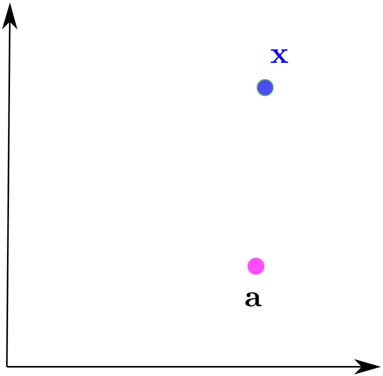
$$\mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a})}(\mathbf{a})=\mathbf{x}$$

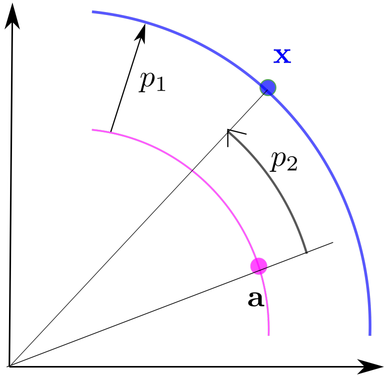
Example. Consider the Lie group of symmetries:

$$G_{\mathbf{p}} = \left\{ \mathbf{g}_{\mathbf{p}} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to p_1 \cdot \begin{pmatrix} \cos p_2 & -\sin p_2 \\ \sin p_2 & \cos p_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

To get the transport function $\mathbf{h}(\mathbf{x},\mathbf{a})$, we use the equivalence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{p_1 \cdot \begin{pmatrix} \cos p_2 & -\sin p_2 \\ \sin p_2 & \cos p_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\mathbf{g}_{\mathbf{p}}(\mathbf{a})} \Leftrightarrow \mathbf{p} = \mathbf{h}(\mathbf{x},\mathbf{a})$$


x
a



$$\mathbf{p} = \mathbf{h}(\mathbf{x}, \mathbf{a}) \text{ and } \mathbf{g_p}(\mathbf{a}) = \mathbf{x}$$

Interval integration

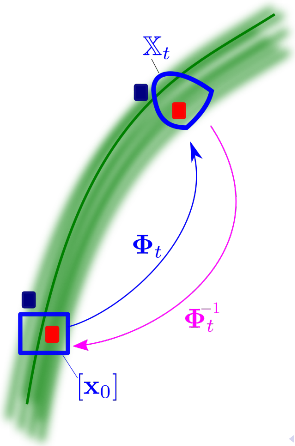
Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x}_0 \in [\mathbf{x}_0]$.
Φ_t is the flow associated to $\mathbf{f}$

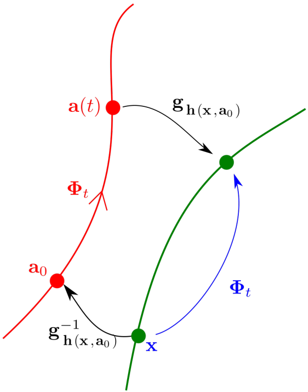
The set of all states $\mathbf{x}$ at time t consistent with the initial box $[\mathbf{x}_0]$ is [4]

$$\mathbb{X}_t = \mathbf{\Phi}_{-t}^{-1}([\mathbf{x}_0])$$

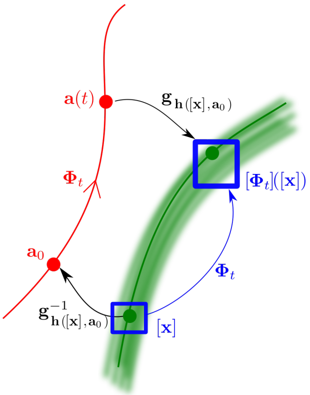


Theorem. We have one reference $\mathbf{a}(t) = \Phi_t(\mathbf{a}_0)$.
If $\mathbf{h}(\mathbf{x},\mathbf{a})$ is a transport function for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then

$$\Phi_t(\mathbf{x}) = \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t).$$



An inclusion function for $\Phi_t(\mathbf{x})$ is thus $\Phi_t([\mathbf{x}]) = \mathbf{g}_{\mathbf{h}([\mathbf{x}],\mathbf{a}_0)} \circ \mathbf{a}(t)$.



Method [1]

- Enclose [6], a reference $\mathbf{a}(t)$ in a thin tube $[\mathbf{a}(t)]$.
- Find a Lie group of symmetries $G_{\mathbf{p}}$.
- Give a closed form expression for the transport function $\mathbf{h}(\mathbf{x},\mathbf{a})$.
- Solve the set inversion problem.

Using separators [5], we can compute

$$\bigcup_{t\in\mathbb{T}} \mathbb{X}_t = \bigcup_{t\in\mathbb{T}} \Phi_{-t}^{-1}([\mathbf{x}_0])$$

with $\Phi_t(\mathbf{x}) = \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t)$ and $\mathbb{T}$ is either

- a discrete set $\mathbb{T} = \{1,\ldots,m\}$;
- an interval $\mathbb{T} = [0, t_{\max}]$.

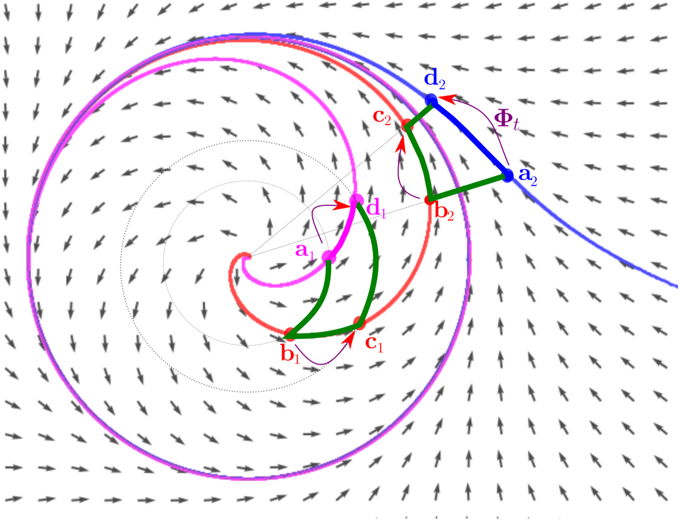
Hydon system

Example. Consider the system [2]

$$\begin{cases} \dot{x}_1 &= -x_1^3 - x_1 x_2^2 + x_1 - x_2 \\ \dot{x}_2 &= -x_2^3 - x_1^2 x_2 + x_1 + x_2 \end{cases}$$

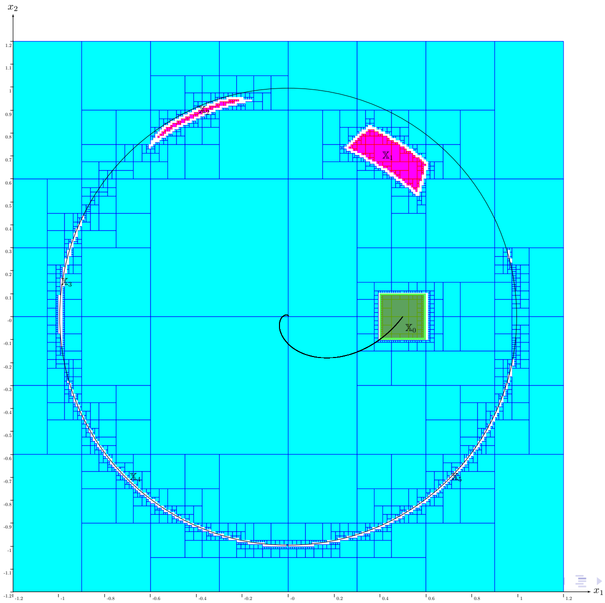
It has the following symmetry

$$\mathbf{g_p}(\mathbf{x}) = \frac{1}{\sqrt{p_2 + \left(x_1^2 + x_2^2\right)(1 - p_2)}} \cdot \begin{pmatrix} \cos p_1 & -\sin p_1 \\ \sin p_1 & \cos p_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$


$\mathbf{d}_2$
$\mathbf{c}_2$
Φ_t
$\mathbf{a}_2$
$\mathbf{b}_2$
$\mathbf{d}_1$
$\mathbf{a}_1$
$\mathbf{c}_1$
$\mathbf{b}_1$

The reference $\mathbf{a}(t)$ is obtained with $\mathbf{a}_0 = (\frac{1}{2}, 0)^\mathsf{T}$.
We have

$$\begin{aligned}\Phi_t(\mathbf{x}) &= \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t) \\ &= \frac{\sqrt{3}\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \cdot \mathbf{a}(t)}{\sqrt{1-\|\mathbf{a}(t)\|^2+(4\|\mathbf{a}(t)\|^2-1)\|\mathbf{x}\|^2}}\end{aligned}$$


x_2
x_1
$\mathbb{X}_0$
$\mathbb{X}_1$
$\mathbb{X}_3$
$\mathbb{X}_4$

Dead reckoning

Consider the system [3]

$$\begin{cases} \dot{x}_1 = u_1 \cdot \cos x_3 \\ \dot{x}_2 = u_1 \cdot \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}$$

To avoid the time dependence in $\mathbf{u}$, we rewrite the system into

$$\begin{cases} \dot{x}_1 = u_1(x_4) \cdot \cos x_3 \\ \dot{x}_2 = u_1(x_4) \cdot \sin x_3 \\ \dot{x}_3 = u_2(x_4) \\ \dot{x}_4 = 1 \end{cases}$$

where x_4 is the clock variable.

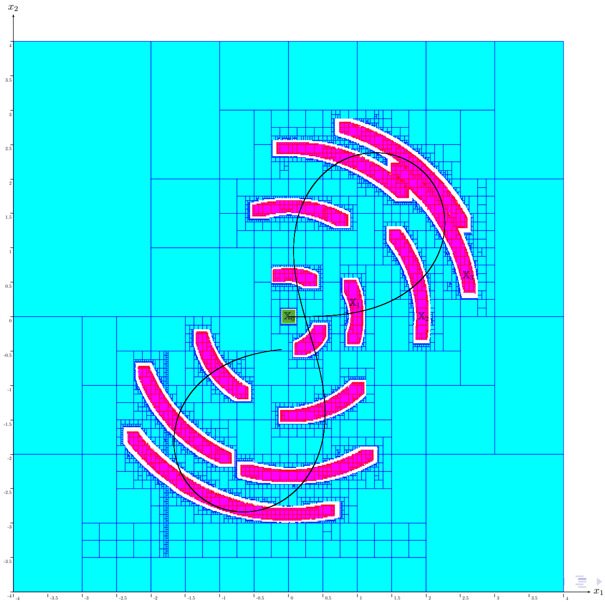
The symmetry is

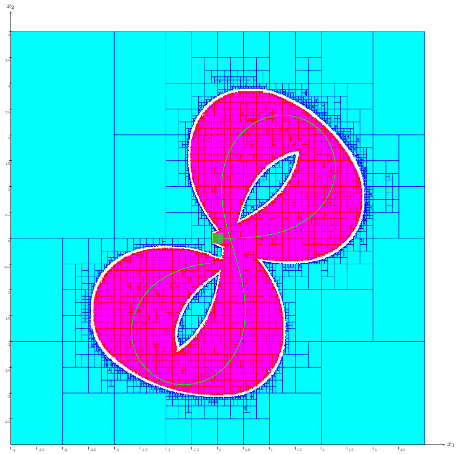
$$\mathbf{g_p}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \mathbf{R}_{p_3} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x_3 + p_3 \\ x_4 \end{pmatrix} \circ \Phi_{p_4}(\mathbf{x})$$

corresponds to the direct Euclidian group $SE(2)$.

We get

$$\begin{aligned}\Phi_t(\mathbf{x}) &= \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{0})} \circ \mathbf{a}(t) \\ &= \begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbf{R}_{x_3 - a_3(x_4)} \cdot \begin{pmatrix} a_1(t+x_4) - a_1(x_4) \\ a_2(t+x_4) - a_2(x_4) \end{pmatrix} \\ x_3 + a_3(t+x_4) - a_3(x_4) \\ a_4(t+x_4) \end{pmatrix}\end{aligned}$$


x_2
x_1
X_0
X_1
X_2
X_3



$$\mathrm{Proj}_{(x_1,x_2)} \bigcup_{t\in[0,14]} \mathbb{X}_t$$

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Guaranteed interval integration for large initial boxes.
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