Interval methods for the detection and the capture of intruders using autonomous robots

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Abstract. The talk is motivated by the detection of submarine intruders inside the Bay of Biscay (golfe de Gascogne). In this project, we consider a group of underwater robots that are able to localize with a given accuracy using a state observer. We assume that each robot is able to detect any intruder inside a disk of a known radius. Moreover, the speed of the intruders is assumed to be bounded and is small with respect to that of our robots. The goal of this project is twofold: (1) to characterize a set for which we can guarantee that there is no intruder (this corresponds to the secure zone) (2) to find a control strategy for the group of robots in order to extend the secure zone as much as possible. We will see how interval analysis can be used to compute efficiently the sets of interest.

1. Secure a zone

INFO OBS. Un sous-marin nucléaire russe repéré dans le Golfe de Gascogne

Le navire a été repéré en janvier. Ce serait la première fois depuis la fin de la Guerre Froide qu'un tel sous-marin, doté de missiles nucléaires, se serait aventuré dans cette zone au large des côtes françaises.

Bay of Biscay 220 000 km^2

An intruder

- Several robots $\mathscr{R}_1, \dots, \mathscr{R}_n$ at positions $\mathbf{a}_1, \dots, \mathbf{a}_n$ are moving in the ocean.
- If the intruder is in the visibility zone of one robot, it is detected.

Complementary approach

- We assume that a virtual intruder exists inside $\mathbb{G}$.
- We localize it with a set-membership observer inside $\mathbb{X}(t)$.
- The secure zone corresponds to the complementary of $\mathbb{X}(t)$.

Assumptions

- The intruder satisfies

$$\dot{\mathbf{x}} \in \mathbb{F}(\mathbf{x}(t)).$$

- Each robot $\mathscr{R}_i$ has the visibility zone $\mathbb{V}(\mathbf{a}_i)$

For instance

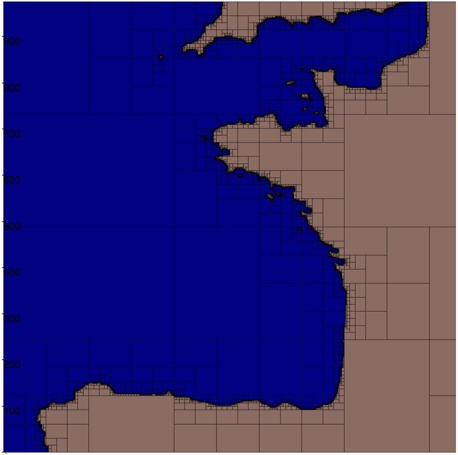
$$\mathbb{V}(\mathbf{a}_i) = \{\mathbf{x} | \text{ such that } \|\mathbf{a}_i - \mathbf{x}\| \leq 100\}$$

Theorem. An (undetected) intruder has a state vector $\mathbf{x}(t)$ inside the set

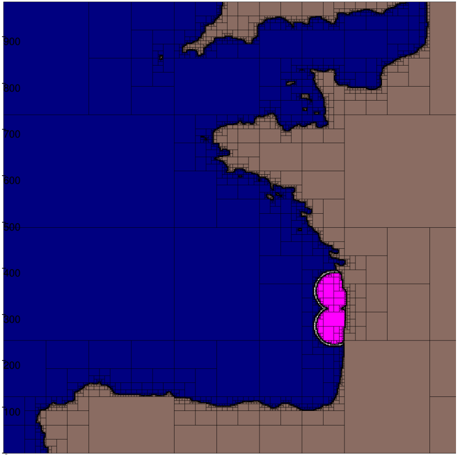
$$\mathbb{X}(t)=\mathbb{G}\cap(\mathbb{X}(t-dt)+dt\cdot\mathbb{F}(\mathbb{X}(t-dt))\cap\bigcap_i\overline{\mathbb{V}(\mathbf{a}_i)}$$

where $\mathbb{X}(0)=\mathbb{G}$. The secure zone is

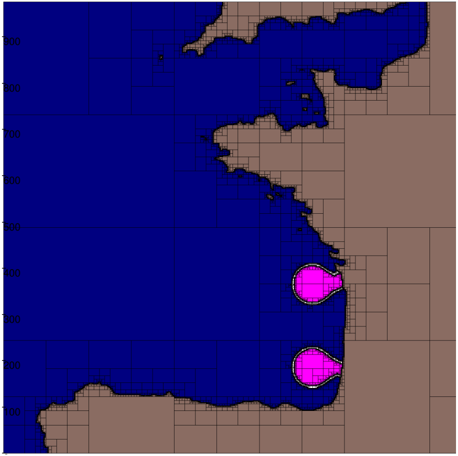
$$\mathbb{S}(t)=\overline{\mathbb{X}(t)}.$$



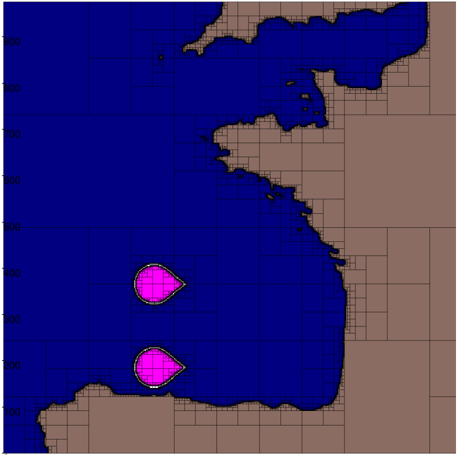
Set $\mathbb{G}$ in blue



Magenta: $\bigcup_i \mathbb{V}(\mathbf{a}_i)$ Blue: $\mathbb{G} \cap \bigcap_i \overline{\mathbb{V}(\mathbf{a}_i)}$



Blue: $\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt))) \cap \bigcap_i \overline{\mathbb{V}(\mathbf{a}_i)}$.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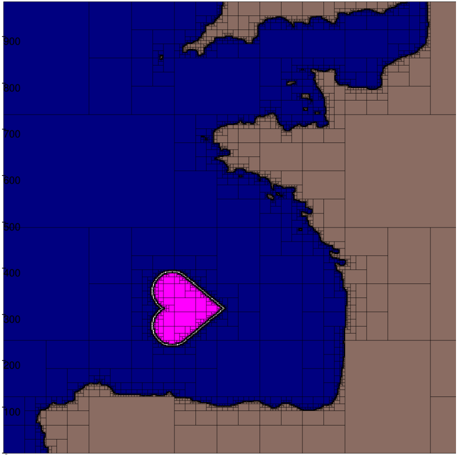
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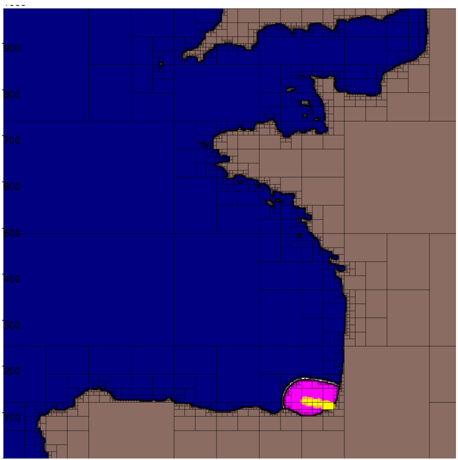
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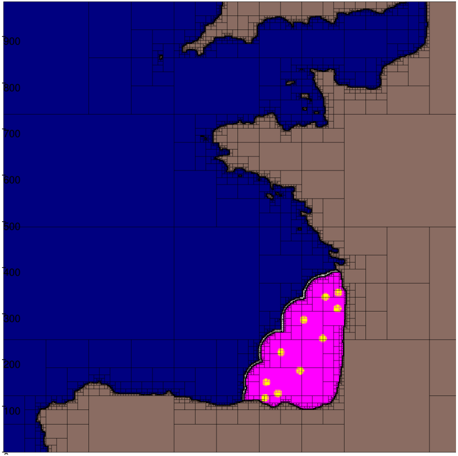
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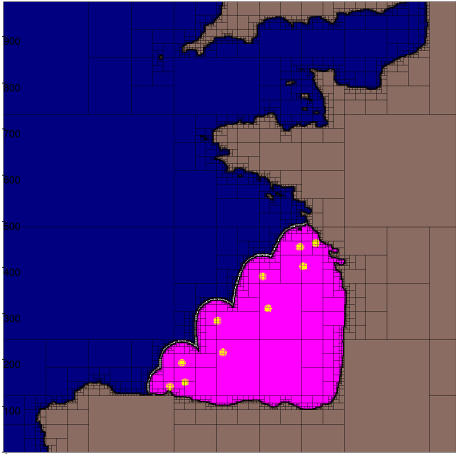
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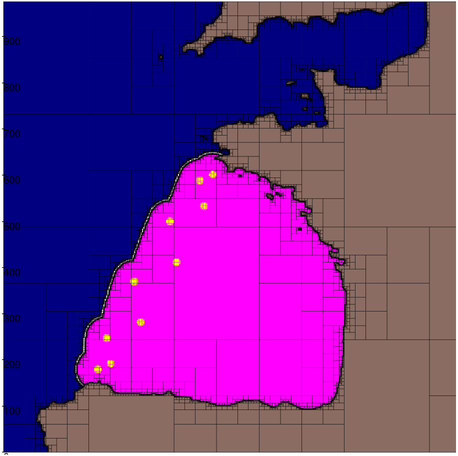
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2. Interval computation

Problem. Given $f:\mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

If $\diamond \in \{+, -, \cdot, /, \max, \min\}$, we define

$$[x] \diamond [y] \quad = \quad [\ \{x \diamond y \mid x \in [x], y \in [y]\}\ ]$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A} \subset \mathbb{R}$.

Exercise.

$$\begin{aligned}
[-1,3]+[2,5] &= [?,?] \\
[-1,3]\cdot[2,5] &= [?,?] \\
[-2,6]/[2,5] &= [?,?]
\end{aligned}$$

Solution.

$$\begin{aligned} [-1,3]+[2,5] &= [1,8] \\ [-1,3].[2,5] &= [-5,15] \\ [-2,6]/[2,5] &= [-1,3] \end{aligned}$$

Exercise. Compute

$$[-2,2]/[-1,1] \quad = [?,?]$$

Solution.

$$[-2,2]/[-1,1] \quad = [-\infty,\infty]$$

$$
\begin{aligned}
[x^-,x^+]+[y^-,y^+] = \quad & [x^-+y^-,x^++y^+] \\
[x^-,x^+]\cdot[y^-,y^+] = \quad & [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+, \\
& x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+]
\end{aligned}
$$

If $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}]$$

Exercise.

$$\begin{array}{rcl}
\sin([0,\pi]) & = & ? \\
\mathsf{sqr}([-1,3]) & = & [-1,3]^2 = ? \\
\mathsf{abs}([-7,1]) & = & ? \\
\mathsf{sqrt}([-10,4]) & = & \sqrt{[-10,4]} = ? \\
\log([-2,-1]) & = & ?
\end{array}$$

Solution.

$$
\begin{aligned}
\sin([0,\pi]) &= [0,1] \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = [0,9] \\
\mathsf{abs}([-7,1]) &= [0,7] \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = [0,2] \\
\log([-2,-1]) &= \emptyset
\end{aligned}
$$

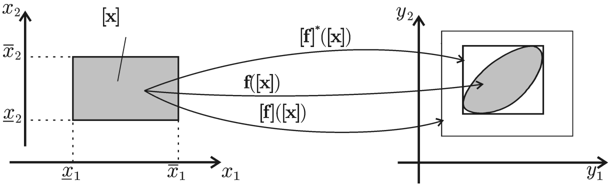
A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* of $\mathbf{f}$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \ \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4$$

For $[x] = [-3,4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3, 4]$, we have

$$\begin{aligned} [f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\ &= [0,16] + [-6,8] + 4 \\ &= [-2,28] \end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

3. Computing with sets

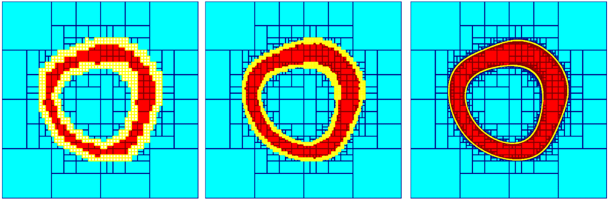
A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$. Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}$$

are approximated by $\mathbb{X}^-$ and $\mathbb{X}^+$ for different accuracies.



Set operations such as $\mathbb{Z} := \mathbb{X} + \mathbb{Y}$, $\mathbb{X} := \mathbf{f}^{-1}(\mathbb{Y}), \mathbb{Z} := \mathbb{X} \cap \mathbb{Y} \dots$ can be approximated by subpaving operations.

Set inversion

Given a set $\mathbb{Y}$, we want to compute

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

$$
\begin{array}{lll}
\text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow \quad [\mathbf{x}] \subset \mathbb{X} \\
\text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow \quad [\mathbf{x}] \cap \mathbb{X} = \emptyset.
\end{array}
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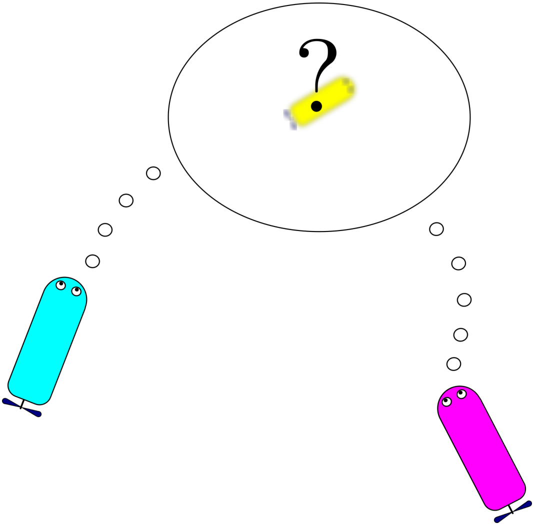
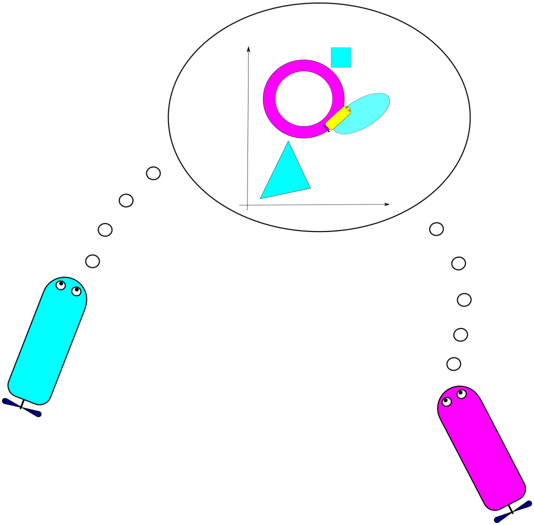
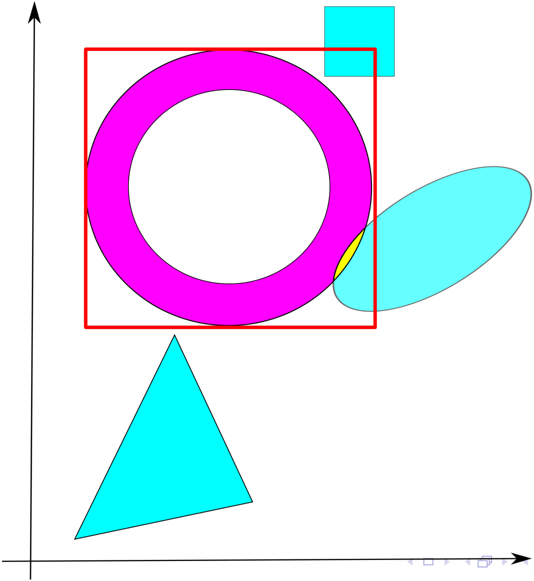
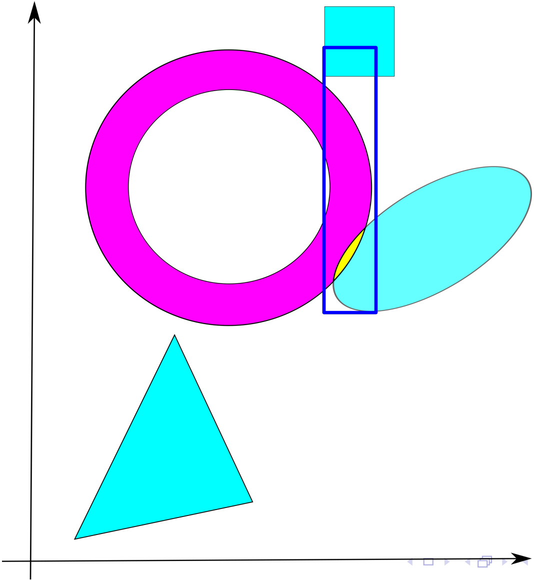
Boxes for which these tests failed, will be bisected, except if they are too small.

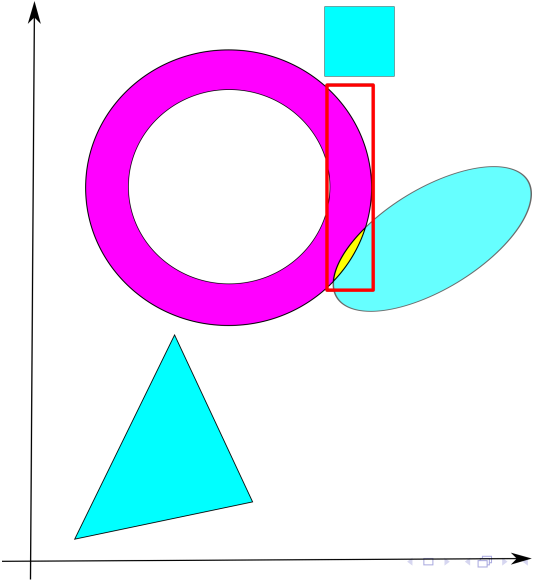
4. Distributed solving

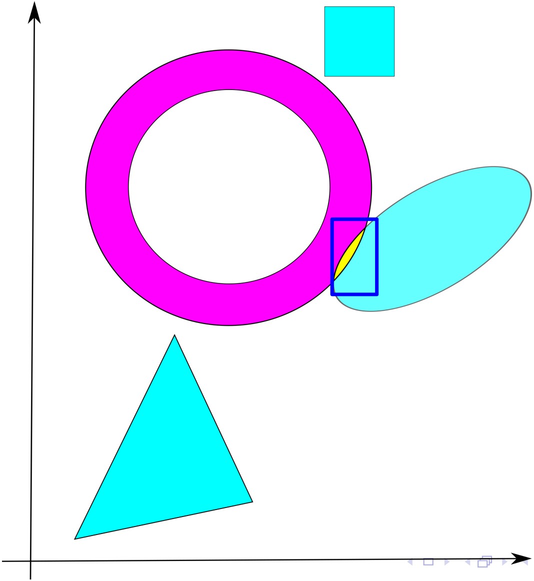












References

- The set membership to secure a zone from intruders with a group robot [5]
- Interval computation [4]
- Set inversion [2][1]
- A MOOC on interval analysis [3]

L. Jaulin.
Solution globale et garantie de problèmes ensemblistes ; application à l'estimation non linéaire et à la commande robuste.
PhD dissertation, Université Paris-Sud, Orsay, France, 1994.

L. Jaulin, M. Kieffer, O. Didrit, and E. Walter.
Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics.
Springer-Verlag, London, 2001.

L. Jaulin, O. Reynet, B. Desrochers, S. Rohou, and J. Ninin.
IaMOOC, Interval analysis with applications to parameter estimation and robot localization , `www.ensta-bretagne.fr/iamooc/`.
ENSTA-Bretagne, 2019.

R. Moore.

Methods and Applications of Interval Analysis.
Society for Industrial and Applied Mathematics, jan 1979.

K. Vencatasamy, L. Jaulin, and B. Zerr.
Secure the zone from intruders with a group robots.
Special Issue on Ocean Engineering and Oceanography, Springer, 2018.