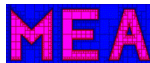


# Guaranteed characterization of trajectories of dynamical systems using contractors

Workshop on the Validation of Cyberphysic systems, Halmstad.

T. Le Mézo, L. Jaulin, B. Zerr

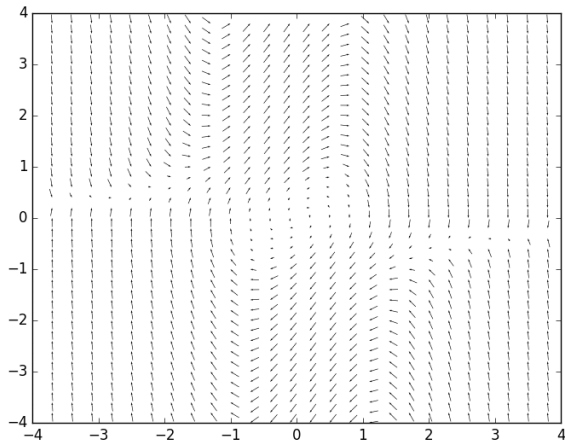


# Capture set

We consider a state equation  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ .

**Example:** The Van der Pol system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2) \cdot x_2 - x_1 \end{cases}$$

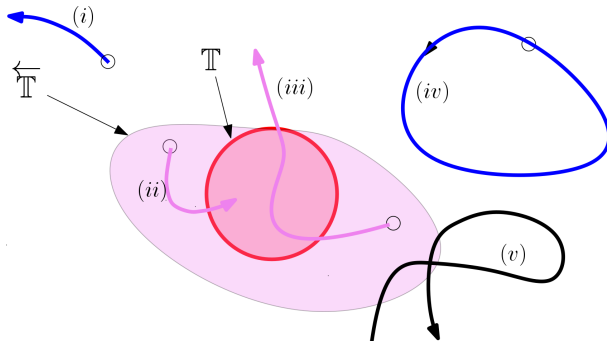


Let  $\varphi$  be the flow map.

The *capture* set of the *target*  $\mathbb{T} \subset \mathbb{R}^n$  is:

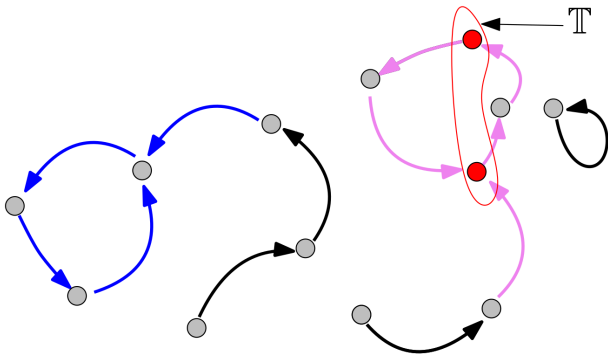
$$\overleftarrow{\mathbb{T}} = \{\mathbf{x}_0 \mid \exists t \geq 0, \varphi(t, \mathbf{x}_0) \in \mathbb{T}\}.$$

To each state, we associate a path.

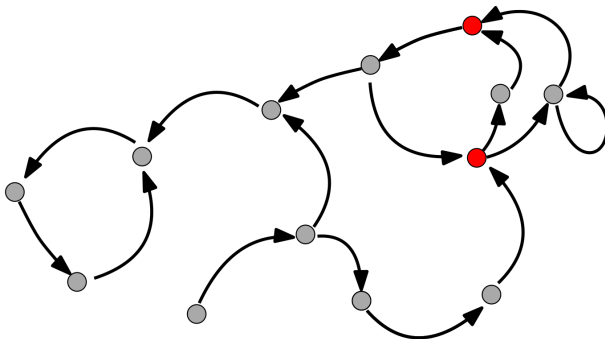


# Graph analogy

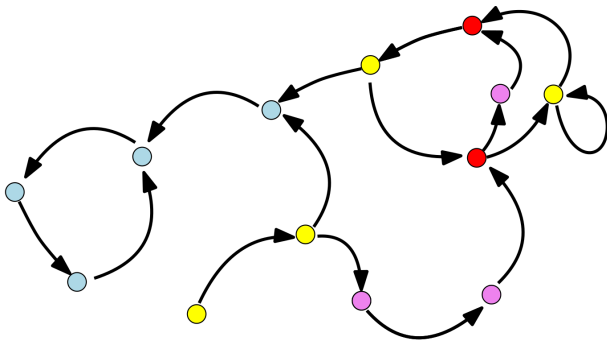
A deterministic graph  $\mathcal{G}_1$  with a target  $\mathbb{T}$  (red), a dead path (blue).



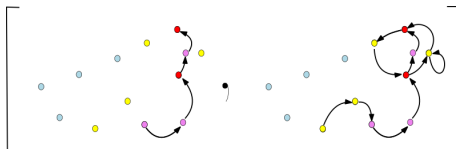
It can be approximated by a non deterministic graph  $\mathcal{G}_2$ :



Using a backward method, we can enclose  $\overleftarrow{\mathbb{T}}$ .



This corresponds to an interval of graphs:



Our new approach: bracket  $\overleftarrow{\mathbb{T}}$ , we search for paths not for states.

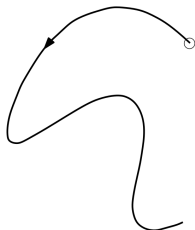
# Maze

An *interval* is a domain which encloses a real number.

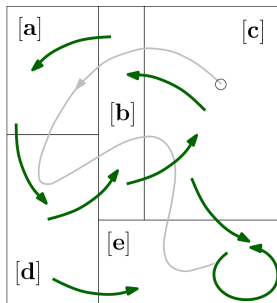
A *polygon* is a domain which encloses a vector of  $\mathbb{R}^n$ .

A *maze* is a domain which encloses a path.

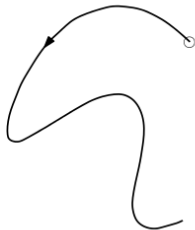
A maze is a set of paths.



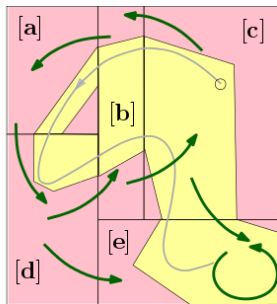
$\in$



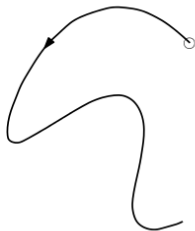
Mazes can be made more accurate by adding polygons.



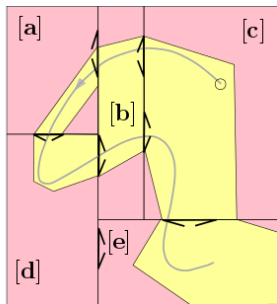
$\in$



Or using doors instead of a graph



$\in$



Here, a **maze**  $\mathcal{L}$  is composed of

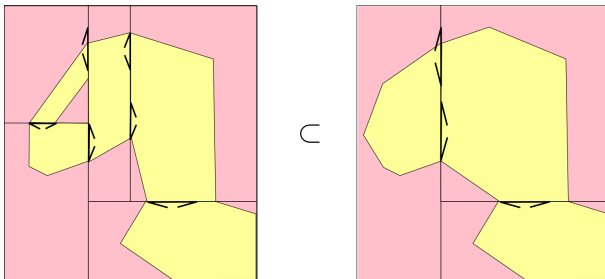
- A paving  $\mathcal{P}$
- A polygon for each box of  $\mathcal{P}$
- Doors between adjacent boxes

The set of mazes forms a lattice with respect to  $\subset$ .

$\mathcal{L}_a \subset \mathcal{L}_b$  means :

- the boxes of  $\mathcal{L}_a$  are subboxes of the boxes of  $\mathcal{L}_b$ .
- The polygons of  $\mathcal{L}_a$  are included in those of  $\mathcal{L}_b$
- The doors of  $\mathcal{L}_a$  are thinner than those of  $\mathcal{L}_b$ .

The left maze contains less paths than the right maze.



Note that yellow polygons are convex.

# Inner approximation of

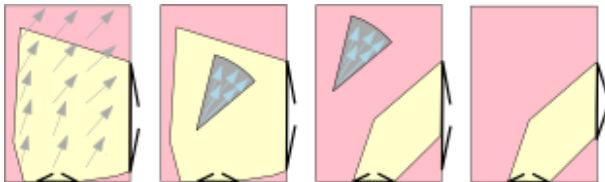
**Main idea:** Compute an outer approximation of the complementary of  $\mathbb{T}$  :

$$\overleftarrow{\mathbb{T}} = \{\mathbf{x}_0 \mid \forall t \geq 0, \varphi(t, \mathbf{x}_0) \notin \mathbb{T}\}$$

Thus, we search for a path that never reach  $\mathbb{T}$ .

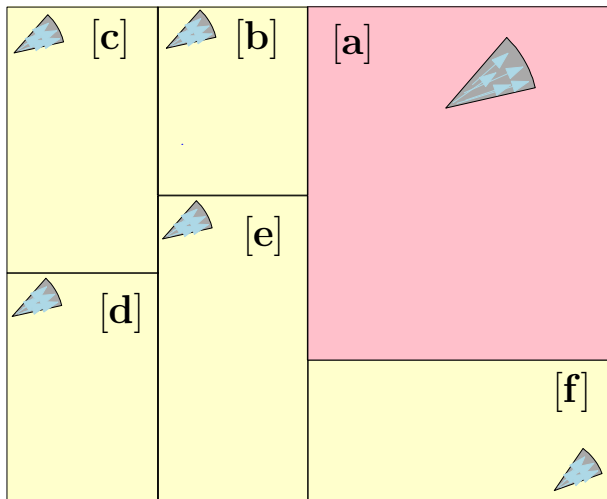
**Target contractor.** If a box  $[x]$  of  $\mathcal{P}$  is included in  $\mathbb{T}$  then remove  $[x]$  and close all doors entering in  $[x]$  .

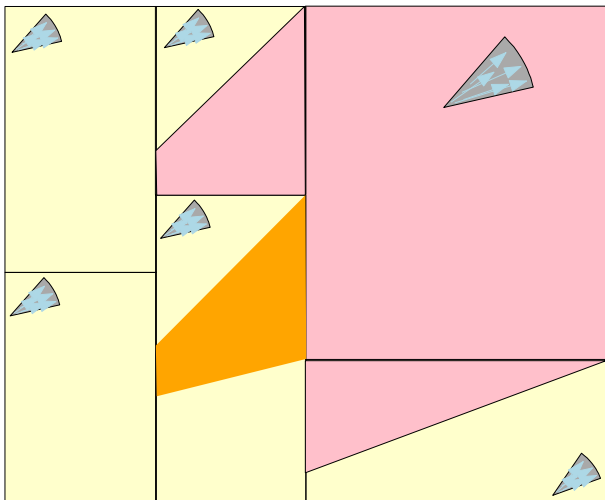
**Flow contractor.** For each box  $[x]$  of  $\mathcal{P}$ , we contract the polygon using the constraint  $\dot{x} = f(x)$ .

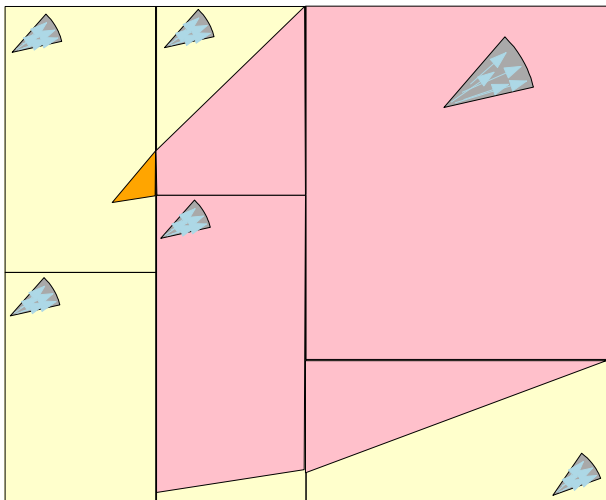


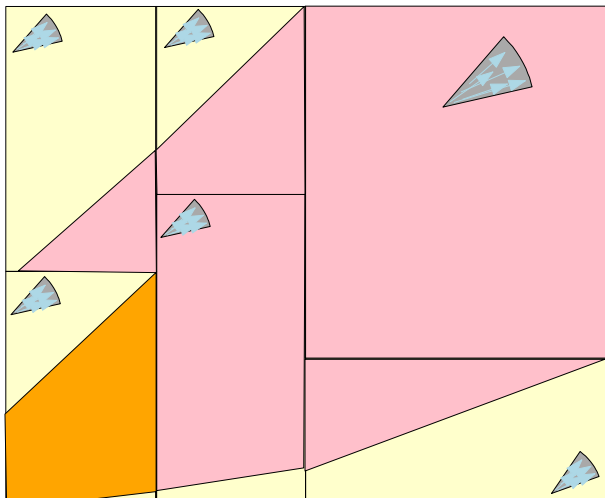
# Inner propagation

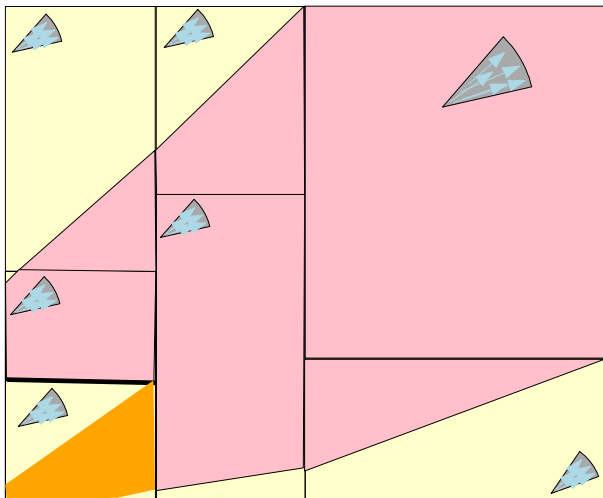


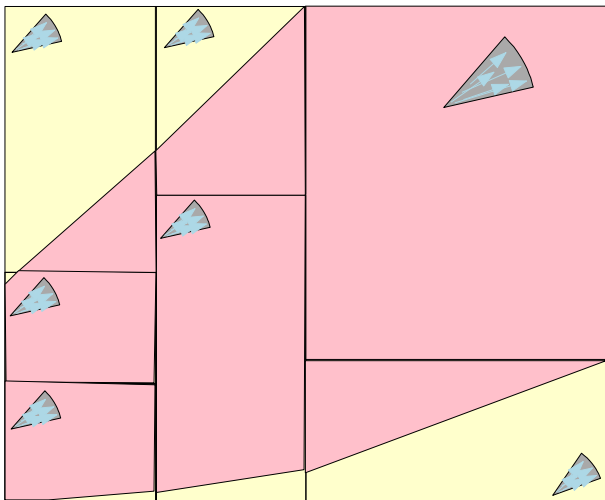


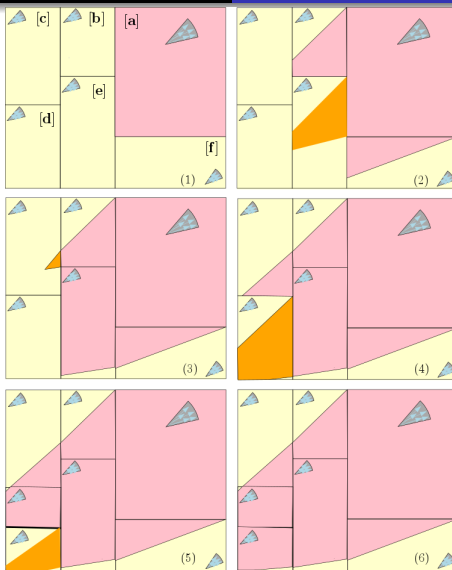




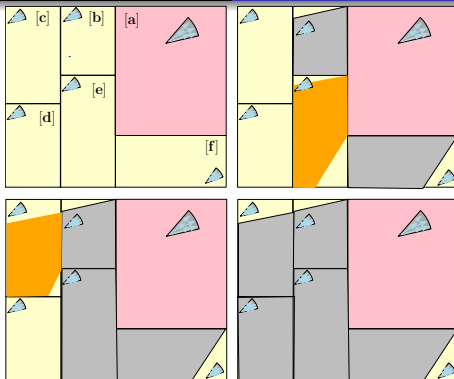








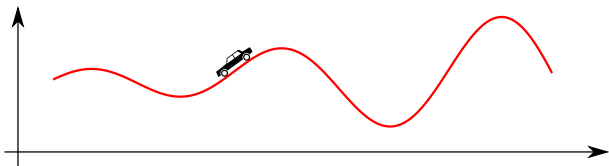
# Outer propagation

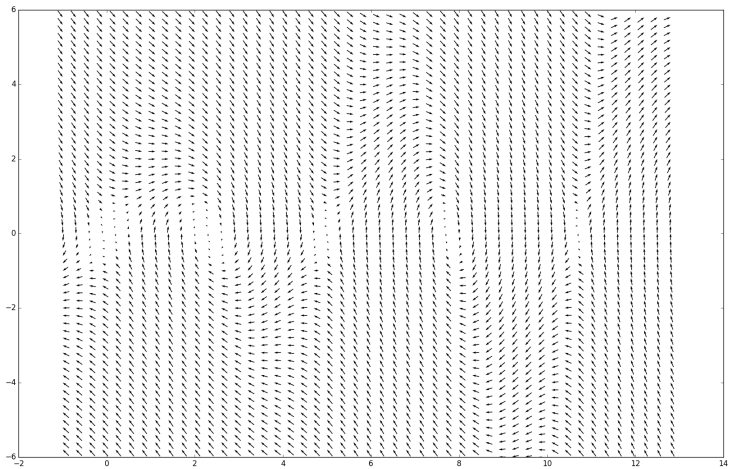


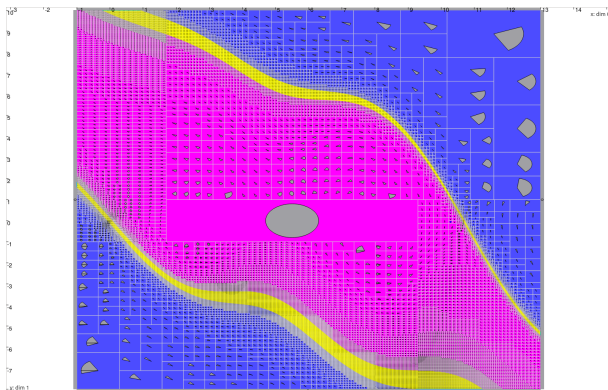
An interpretation can be given only when the fixed point is reached.

# Car on the hill

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 9.81 \sin\left(\frac{11}{24} \cdot \sin x_1 + 0.6 \cdot \sin(1.1 \cdot x_1)\right) - 0.7 \cdot x_2 \end{cases}$$

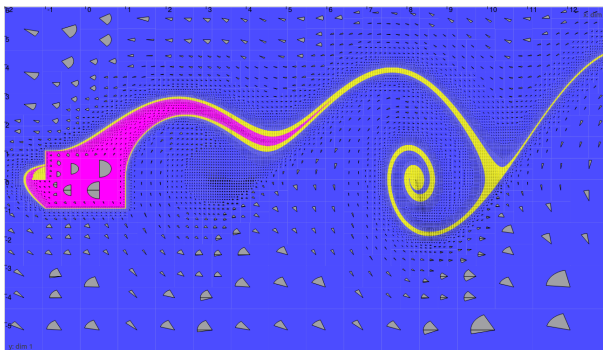






Research box  $\mathbb{X}_0 = [-1, 13] \times [-10, 10]$   
 Blue:  $\mathbb{T}_{out} = \overline{\mathbb{X}_0}$ ; Red:  $\mathbb{T}_{in} = [2, 9] \times [-1, 1]$

Combined with an outer propagation

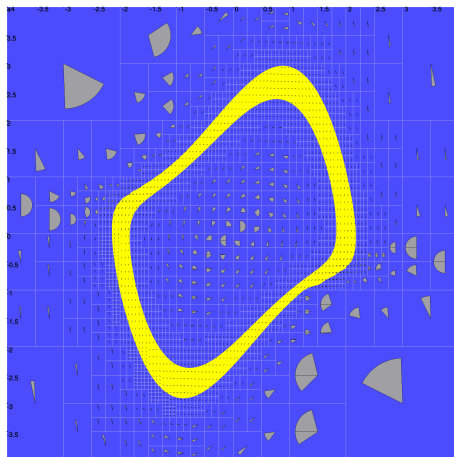


# Van der Pol system

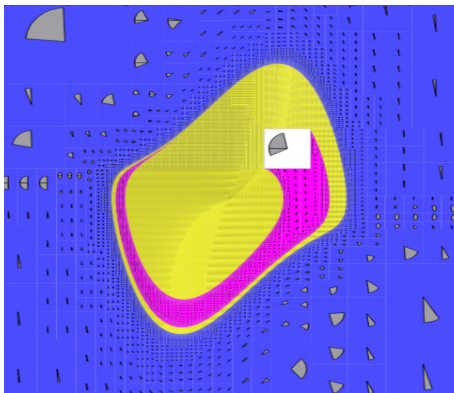
Consider the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2) \cdot x_2 - x_1 \end{cases}$$

and the box  $\mathbb{X}_0 = [-4, 4] \times [-4, 4]$ .

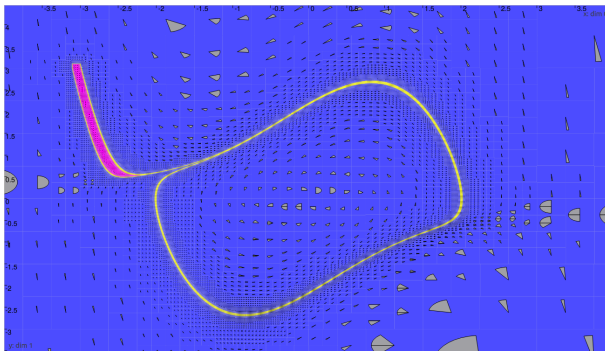


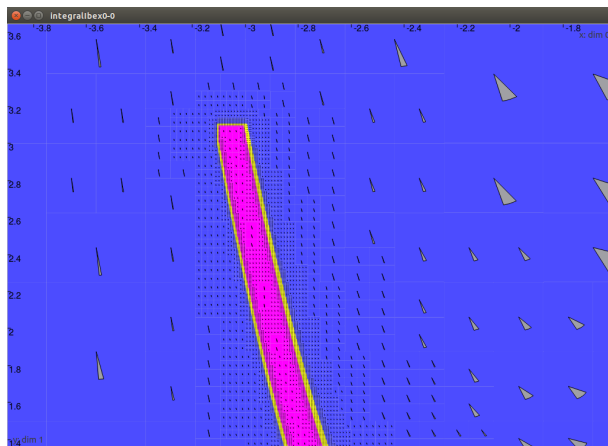
$$f \rightarrow -f ; \mathbb{T} = \overline{\mathbb{X}_0} \cup [-0.1, 0.1]^2.$$



$$\mathbf{f} \rightarrow -\mathbf{f} ; \mathbb{T}_{out} = \overline{\mathbb{X}_0} ; \mathbb{T}_{in} = [0.5, 1]^2.$$

## Combined with an outer propagation





## Reference.

T. Le Mézo, L. Jaulin, and B. Zerr. Inner approximation of a capture basin of a dynamical system. In Abstracts of the 9th Summer Workshop on Interval Methods SWIM'2016. Lyon, France, June 19-22, 2016.