Computing positive invariant tubes with interval analysis

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Réunion mixte SDH et MEA.

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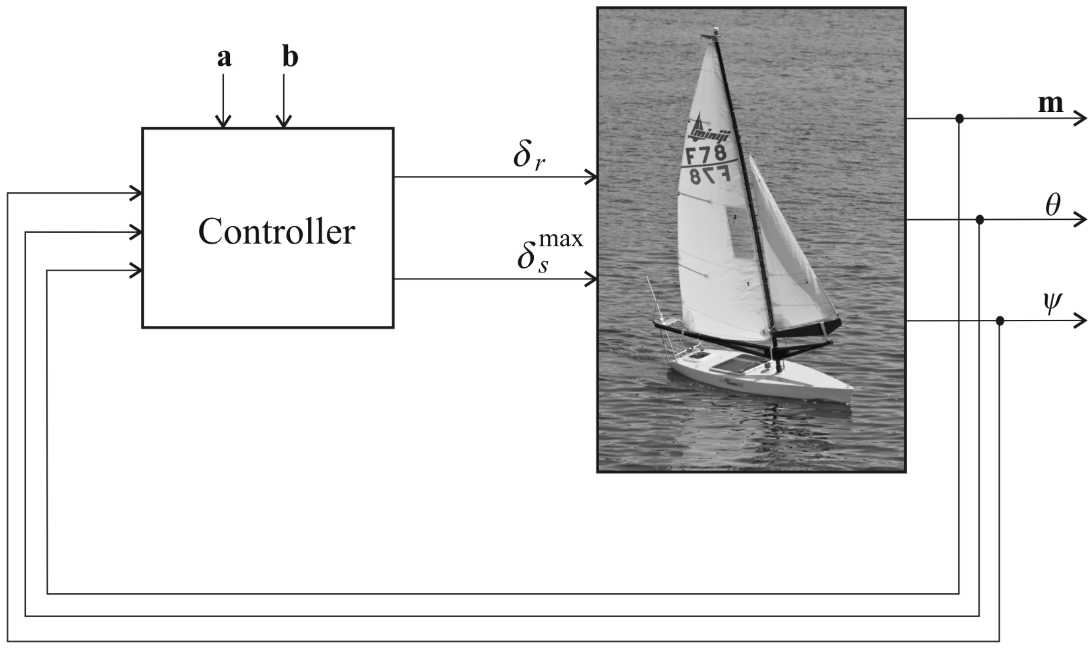
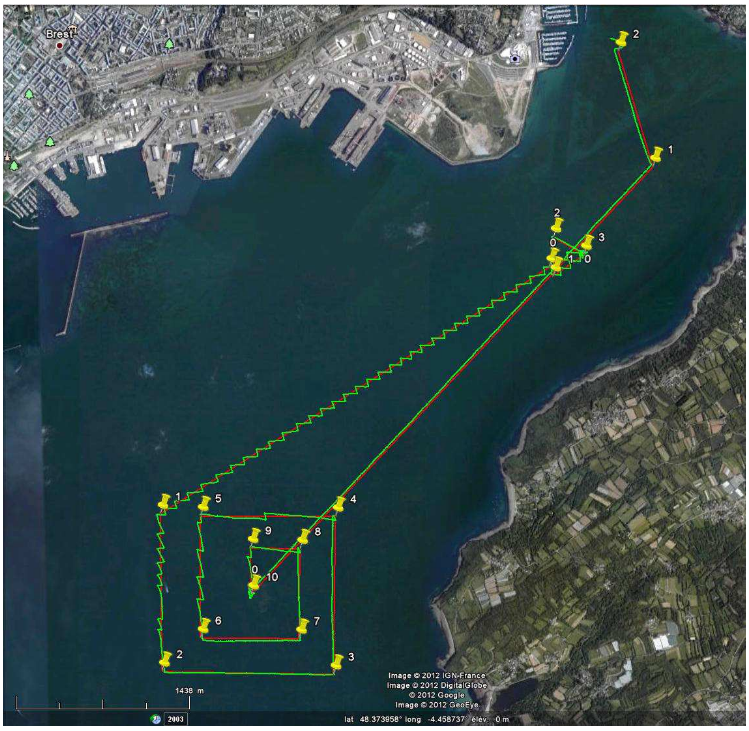
1 V-stability

Vaimos (IFREMER and ENSTA)


a
b
Controller
δ_r
$\delta_s^{\max}$
m
θ
ψ


Brest
2
1
2
0
3
1
0
1
5
4
9
8
0
10
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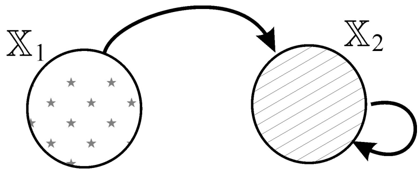
3
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
1438 m
2003
lat 48.373958° long -4.458737° élév. 0 m

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$



$\mathbb{X}_1$: outside the corridor.

$\mathbb{X}_2$: inside the corridor.

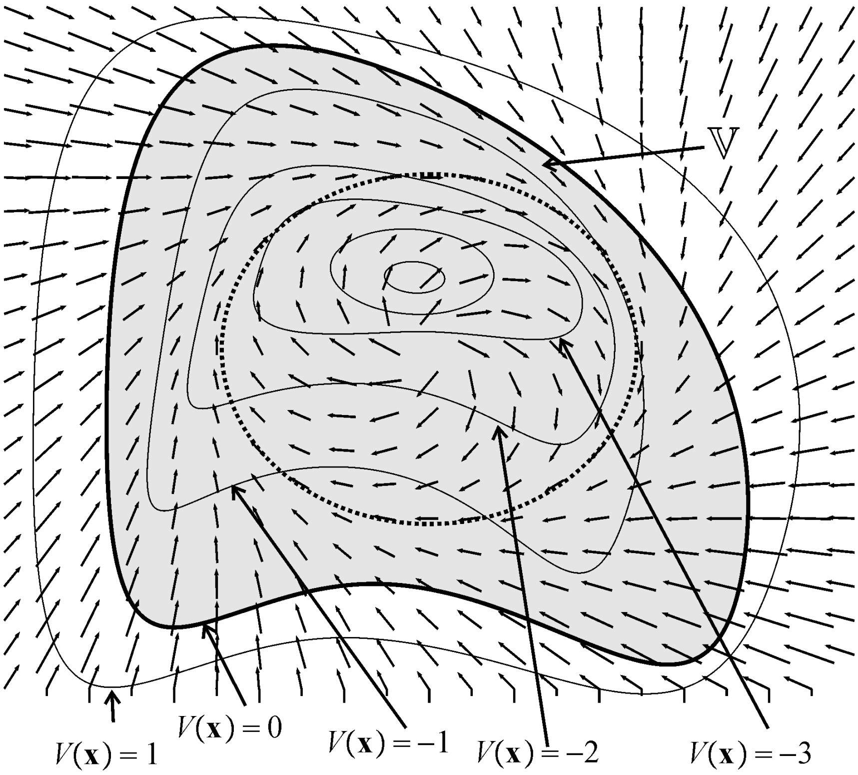
Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$. The system is V-stable if

$$\left(V\left(\mathbf{x}\right) \geq 0 \;\Rightarrow\; \dot{V}\left(\mathbf{x}\right) < 0\right).$$

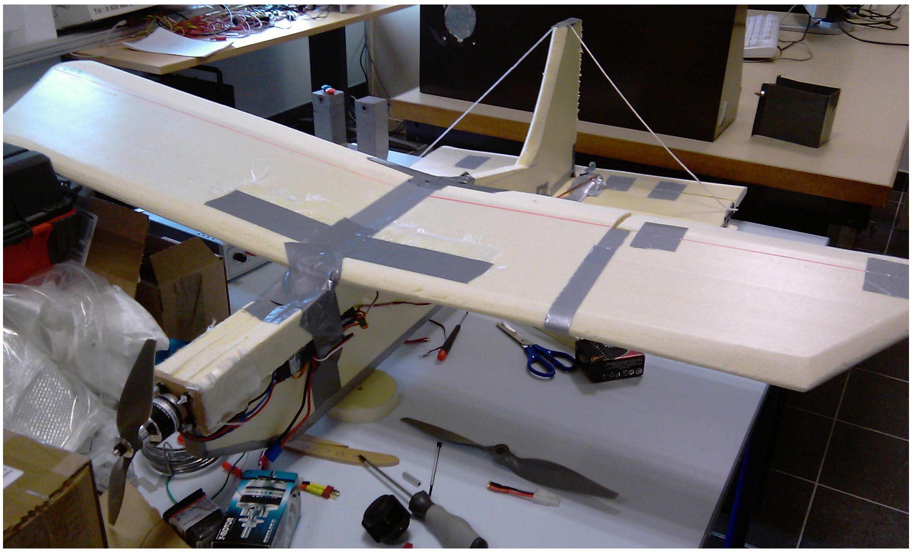
Since

$$\dot{V}\left(\mathbf{x}\right) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left(\mathbf{x}\right) \cdot \mathbf{f}\left(\mathbf{x}\right)$$

Checking the V-stability can be done using using interval analysis.



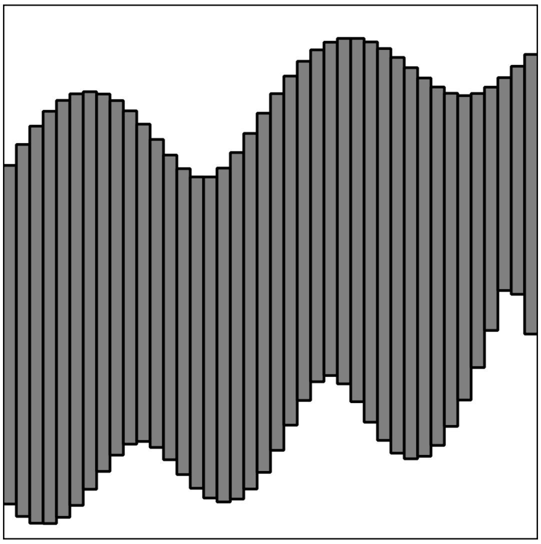
𝕍
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$



Non-holonomic system

2 Tubes

A tube is a function which associates to any $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$.



In the machine a tube can be represented by two stair functions

Example of tubes

$$
\begin{aligned}
[f](t) &= [1,2]\cdot t + \sin([1,3]\cdot t) \\
[g](t) &= [a_0] + [a_1]\,t + [a_2]\,t^2 + [a_3]\,t^3 \\
\int_0^t [g](\tau)\,d\tau &= [a_0]\,t + [a_1]\,\frac{t^2}{2} + [a_2]\,\frac{t^3}{3} + [a_3]\,\frac{t^4}{4}.
\end{aligned}
$$

3 Positibe invariant tubes

Consider the time dependant system

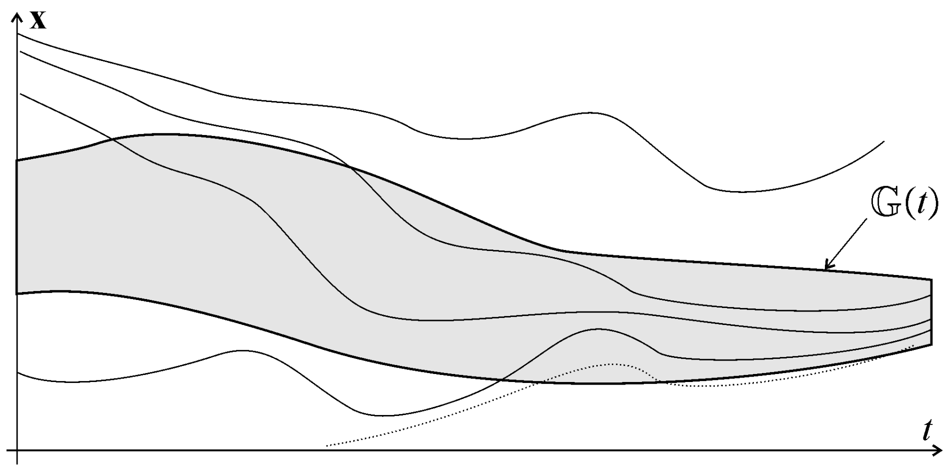
$$\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t\right)$$

and a *tube*

$$\mathbb{G}(t) \subset \mathbb{R}^n, t \in \mathbb{R}.$$

The tube $\mathbb{G}(t)$ is said to be a *positive invariant* if

$$\mathbf{x}(t) \in \mathbb{G}(t), \tau > 0 \Rightarrow \mathbf{x}(t+\tau) \in \mathbb{G}(t+\tau).$$



Theorem. Consider the tube

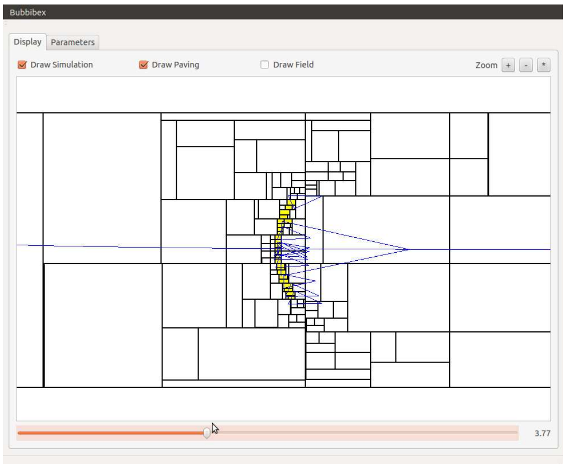
$$\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$$

where $\mathbf{g} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m$. If the *cross out* condition

$$\begin{cases} \underbrace{\frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) . \mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t)}_{\dot{g}_i(\mathbf{x},t)} \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases}$$

is inconsistent for all $(\mathbf{x}, t, i)$, then $\mathbb{G}(t)$ is a capture tube for $\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

A software Bubbibex (using Ibex) made by students from ENSTA Bretagne for MBDA uses interval analysis to prove the inconsistency.



4 Lattice and capture tubes

Consider $\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

If $\mathbb{T}$ is the set of tubes and $\mathbb{T}_c$ is the set of all capture tubes of $\mathcal{S}$ then $(\mathbb{T}_c, \subset)$ is a sublattice of $(\mathbb{T}, \subset)$.

We have indeed

$$\left\{ \begin{array}{lcl} \mathbb{G}_1(t) & \in & \mathbb{T}_c \\ \mathbb{G}_2(t) & \in & \mathbb{T}_c \end{array} \right. \Rightarrow \left\{ \begin{array}{lcl} \mathbb{G}_1(t) \cap \mathbb{G}_2(t) & \in & \mathbb{T}_c \\ \mathbb{G}_1(t) \cup \mathbb{G}_2(t) & \in & \mathbb{T}_c \end{array} \right.$$

5 Computing capture tubes

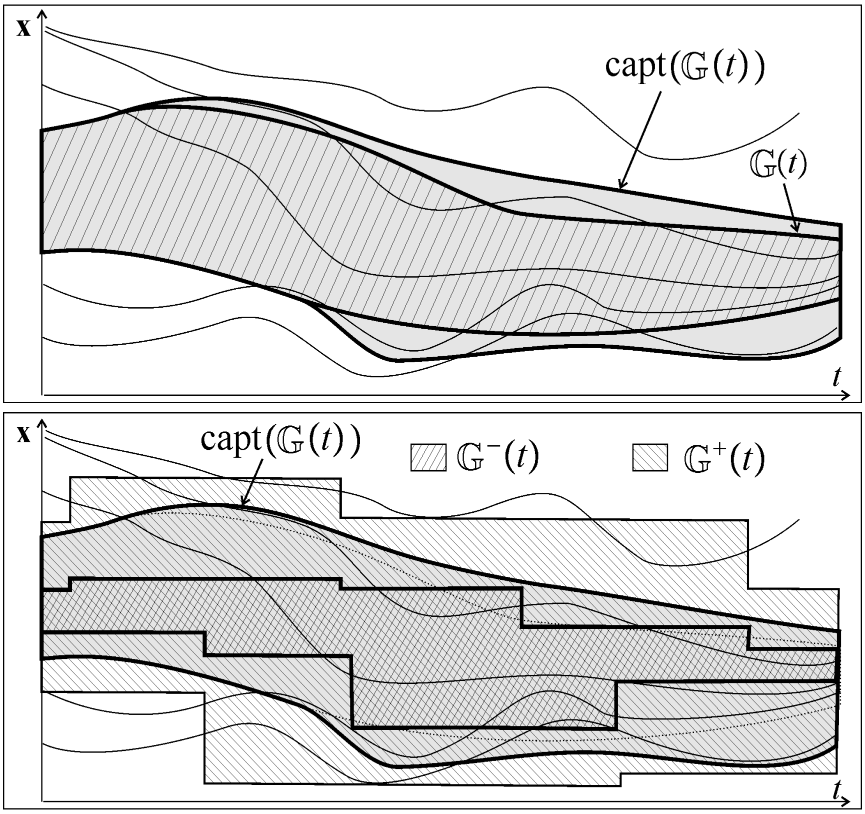
If $\mathbb{G}(t) \in \mathbb{T}$, define

$$\text{capt}\left(\mathbb{G}(t)\right) = \bigcap \left\{ \overline{\mathbb{G}}(t) \in \mathbb{T}_c \mid \mathbb{G}(t) \subset \overline{\mathbb{G}}(t) \right\}.$$

This set is the smallest capture tube enclosing $\mathbb{G}(t)$.

Problem. Given $\mathbb{G}(t) \in \mathbb{T}$, compute an interval $\left[\mathbb{G}^{-}(t), \mathbb{G}^{+}(t)\right] \in \mathbb{IT}$ such that

$$\text{capt}\left(\mathbb{G}(t)\right) \in \left[\mathbb{G}^{-}(t), \mathbb{G}^{+}(t)\right].$$


x
capt(𝔾(t))
𝔾(t)
t
x
capt(𝔾(t))
𝔾⁻(t)
𝔾⁺(t)
t

Flow. The flow associated with $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is a function $\phi_{t_0,t_1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \Rightarrow \phi_{t_0,t_1}(\mathbf{x}(t_0)) = \mathbf{x}(t_1).$$

Proposition. For the system $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ and the tube $\mathbb{G}(t)$, we have

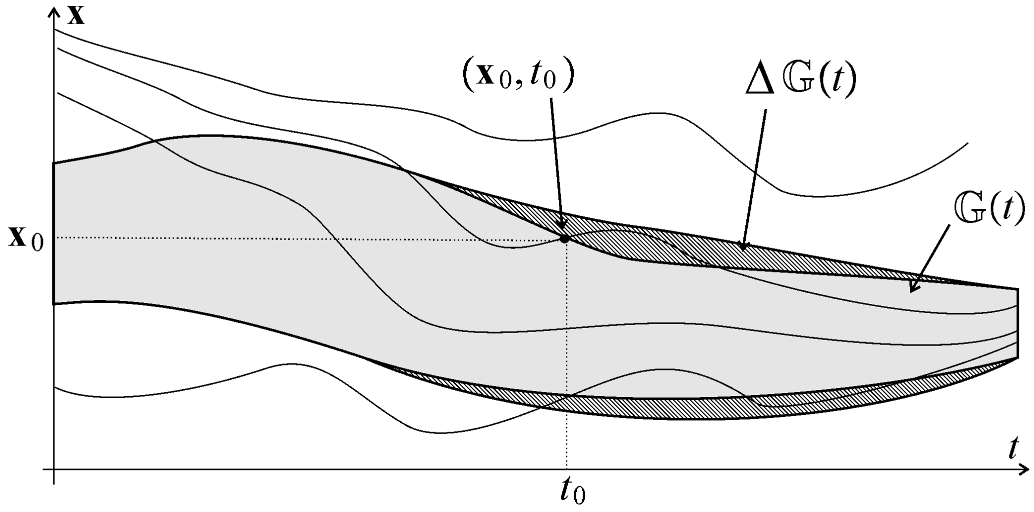
$$\text{capt}\left(\mathbb{G}(t)\right) = \mathbb{G}(t) \cup \mathbf{\Delta}\mathbb{G}(t),$$

with

$$\mathbf{\Delta}\mathbb{G}(t) = \left\{ (\mathbf{x}, t) \;\middle|\; \begin{array}{l} \exists\, (\mathbf{x}_0, t_0) \text{ satisfying the cross out condition} \\ t \geq t_0,\ \phi_{t_0,t}(\mathbf{x}_0) \notin \mathbb{G}(t) \end{array} \right\}$$

Recall the cross out condition:

$$\begin{cases} \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) \,.\mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases}$$

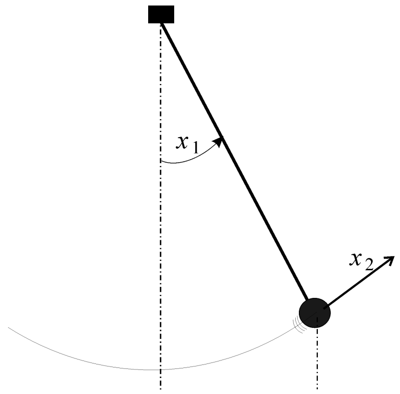


x
$(\mathbf{x}_0, t_0)$
$\Delta\mathbb{G}(t)$
$\mathbb{G}(t)$
$\mathbf{x}_0$
t
t_0

6 Pendulum

Pendulum:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 - 0.15 \cdot x_2 \end{cases}$$



The energy

$$E(\mathbf{x}) = \frac{1}{2}\dot{x}_1^2 - \cos x_1 + 1 = \frac{1}{2}x_2^2 - \cos x_1 + 1$$

allows us to find candidate for the positive invariant tube:

$$g(\mathbf{x}, t) = E(\mathbf{x}) - 1 = \frac{1}{2}x_2^2 - \cos x_1.$$

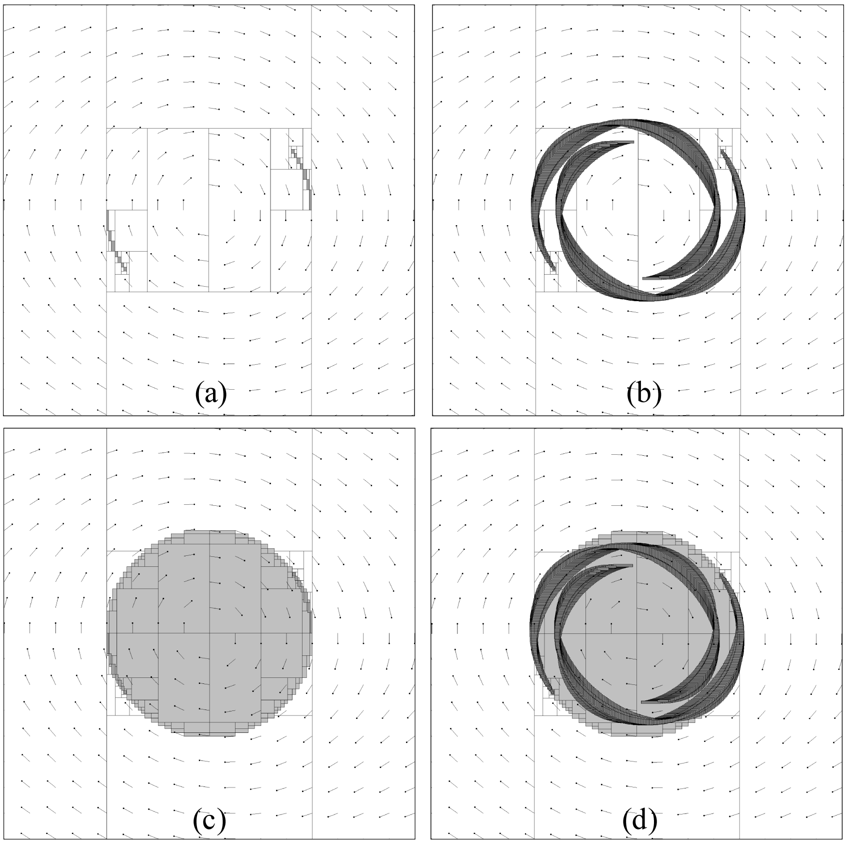
The cross-out conditions

$$\begin{cases} \text{(i)} & \begin{pmatrix} \sin x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -\sin x_1 - 0.15 \cdot x_2 \end{pmatrix} = -0.15 \cdot x_2^2 \ \geq 0, \\ \text{(ii)} & \frac{1}{2} x_2^2 - \cos x_1 = 0. \end{cases}$$

has two solutions: $\mathbf{x} = \left(\pm\frac{\pi}{2}, 0\right)$.

Without considering the energy, we consider, as an candidate tube:

$$g(\mathbf{x}, t) = x_1^2 + x_2^2 - 1.$$



(a) boxes which enclose the points satisfying the cross-out condition
guaranteed integration $\Delta\mathbb{G}$ of these boxes; (c) inner approximation $\mathbb{C}$
Capt($\mathbb{G}$); (d) outer approximation $\mathbb{C}^{+}$ of Capt($\mathbb{G}$)

7 Dubin's car

Consider the Dubin's car

$$\begin{cases} \dot{x} &= \cos\theta \\ \dot{y} &= \sin\theta \\ \dot{\theta} &= u \end{cases}$$

where $u \in [-2, 2]$.

To move toward the target (x_d, y_d), we take the controller:

$$\left\{\begin{array}{lll} \mathbf{n} & = & \frac{1}{\sqrt{(x_d-x)^2+(y_d-y)^2}}\left(\begin{array}{c} x_d - x \\ y_d - y \end{array}\right) + \frac{2}{\sqrt{\dot{x}_d^2+\dot{y}_d^2}}\left(\begin{array}{c} \dot{x}_d \\ \dot{y}_d \end{array}\right) \\ \theta_d & = & \text{atan2}\,(\mathbf{n}) \\ u & = & -2 \cdot \sin\left(\theta - \theta_d\right). \end{array}\right.$$

Target

$$\begin{cases} x_d(t) &= \rho_x \cos t \\ y_d(t) &= \rho_y \sin t. \end{cases}$$

For the derivative, we get

$$\begin{cases} \dot{x}_d(t) &= -\rho_x \sin t \\ \dot{y}_d(t) &= \rho_y \cos t. \end{cases}$$

Target tube. We want the robot to stay inside the set

$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\},$$

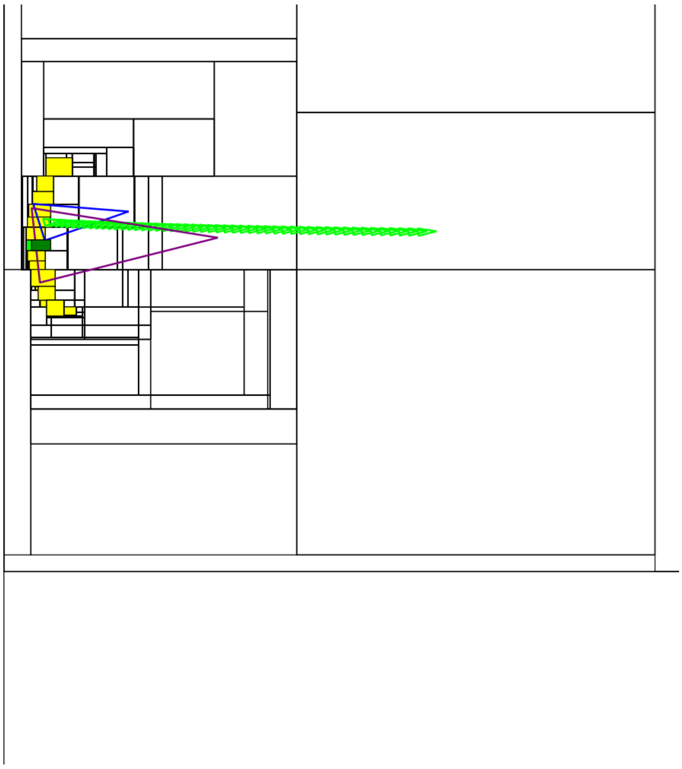
with

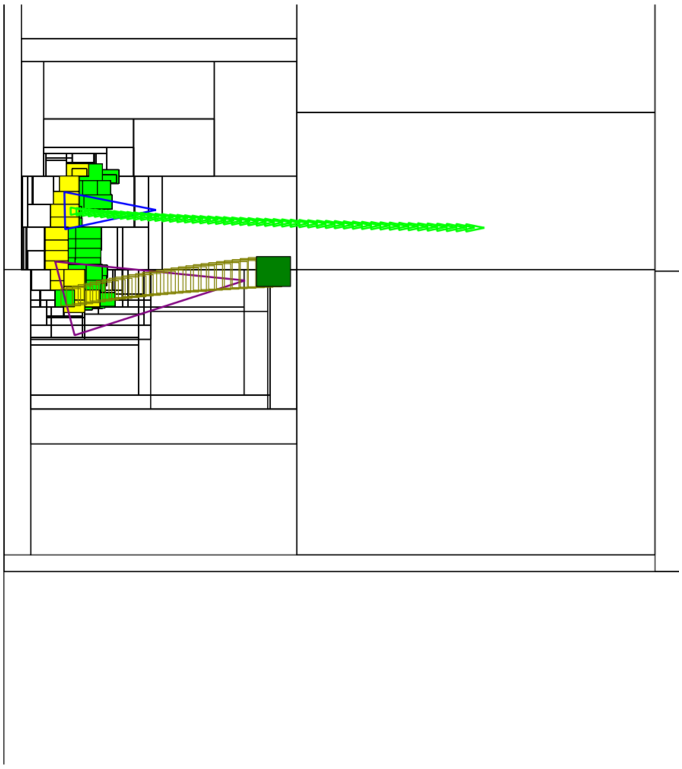
$$\begin{cases} g_1(\mathbf{x}, t) &= (x - x_d)^2 + (y - y_d)^2 - \rho^2 \\ g_2(\mathbf{x}, t) &= \left(\cos\theta - \frac{n_x}{\|\mathbf{n}\|}\right)^2 + \left(\sin\theta - \frac{n_y}{\|\mathbf{n}\|}\right)^2 - \alpha^2. \end{cases}$$

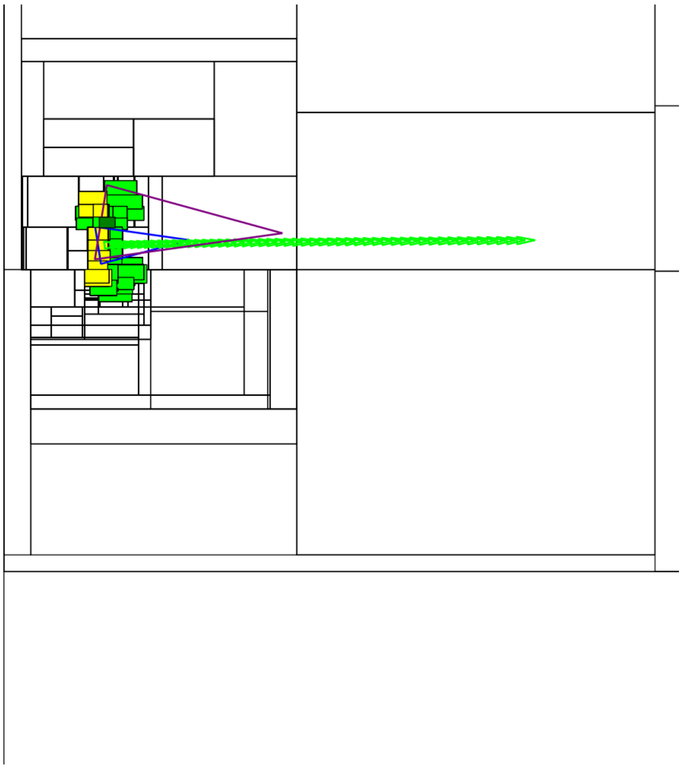
Resolution. We used the solver Bubbibex.

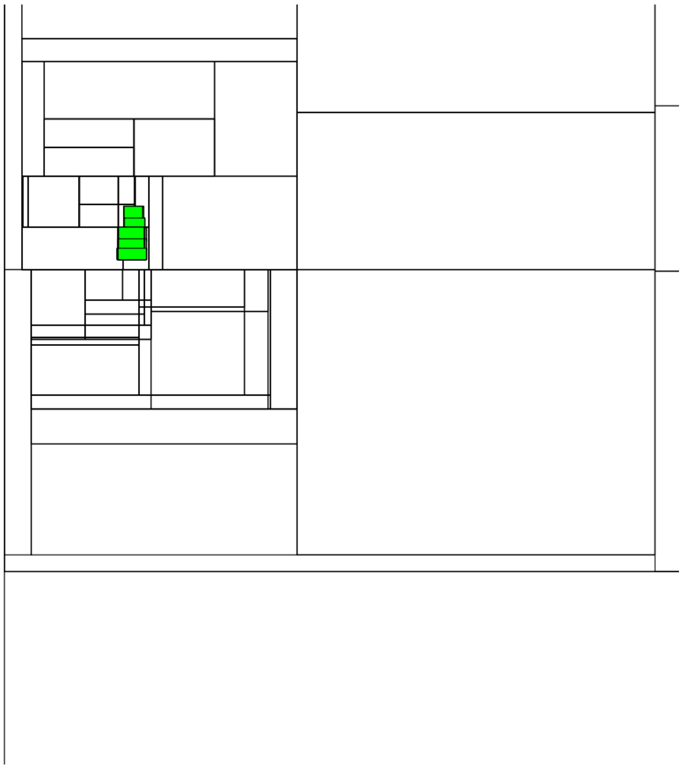
The tube is proved to be unsafe.

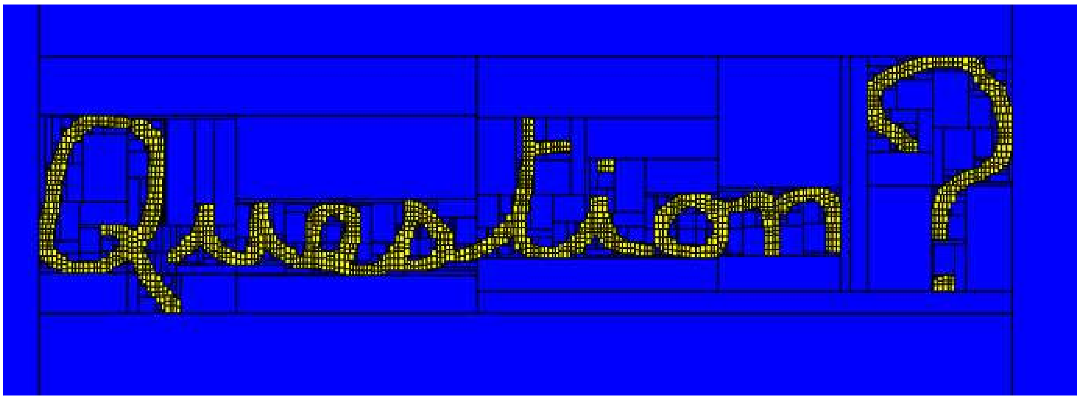
Bubbibex is able to compute the margin (*i.e.*, $\text{width}\left(\left[\mathbb{G}^-(t), \mathbb{G}^+(t)\right]\right)$).










Question?

8 References

L. Jaulin and F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE Transaction on Robotics, Volume 27, Issue 5.

L. Jaulin, D. Lopez, Le Doze, S. Le Menec, J. Ninin, G. Chabert, M. S. Ibnseddik, A. Stancu (2015), Computing capture tubes, Reliable Computing.