Guaranteed Nonlinear Estimation via Interval Analysis

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Introduction

Overview

Interval analysis [Sun58, MY59, KK96, JKDW01]: numerical tool to solve nonlinear problems encountered in engineering

- Computing all global minimizers of a non-convex cost function [Han92],
- Computing all solutions of a set of nonlinear equations [Neu90],
- Characterizing sets defined by nonlinear inequalities [Moo92, JW93a],
- Solving ODEs [BM98, HBM01]
- ...

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- ...

Results provided by interval analysis are guaranteed

- Unlike classical numerical approaches (Monte-Carlo or local methods)
- even when strong nonlinearities and discontinuities appear in the problem.

Overview

Guaranteed:

- properties of results provided using interval analysis may be proved numerically.
- limited precision of representation of numbers on computers taken into account.

Characteristics led to the design of guaranteed parameter and state estimators for non-linear models [Moo92, JW93a, KW06, KW11]

- provide sets containing all values of the parameter or state vector that are consistent with
  - considered model structure
  - bounds on modeling error.

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- Well suited to distributed implementation within networks of wireless sensors [Kie09].
- Guaranteed characterization of asymptotic and non-asymptotic confidence regions [Jau06, KW14].

Overview

Application domains

- robotics,
- chemistry,
- communications,
- celestial mechanics,
- etc.

Outline

1. Introduction - Motivation

2. Estimation problems revisited
   - Parameter bounding
   - Robust estimation
   - Confidence region characterization for Bayesian estimation
   - Non-asymptotic confidence region characterization

3. Main ideas and ressources

Introduction - Motivation

Purpose of this tutorial

- Reformulate various estimation problems as set inversion / constraint satisfaction problems
- Introduce interval methods and constraint propagation techniques
- Show efficient solvers using these tools
- Illustrate with examples taken from biology, robotics, and communication

Particular attention given to robust and distributed estimation problems

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Estimation problems revisited

Classical estimation problems such as

- Parameter estimation
- Robust estimation
- Estimator confidence region characterization

are formulated as set inversion / constraint satisfaction problems

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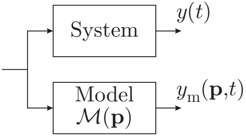
Parameter bounding I

Consider a typical (non-linear) estimation problem



$\mathbf{y} = (y(t_1), \ldots, y(t_n))^{\mathsf{T}}$: vector of experimental data
$\mathbf{p}$: vector of unknown, constant parameters
$\mathbf{y}_{\mathrm{m}}(\mathbf{p}) = (y_{\mathrm{m}}(\mathbf{p}, t_1), \ldots, y_{\mathrm{m}}(\mathbf{p}, t_n))^{\mathsf{T}}$: vector of model outputs

Parameter estimation:

Determine $\widehat{\mathbf{p}}$ from $\mathbf{y}$.

Parameter bounding II

Classical problem formulation:

Minimisation of a cost function

$$\widehat{\mathbf{p}} = \arg\min_{\mathbf{p}} j(\mathbf{p})$$

for example

$$j(\mathbf{p}) = (\mathbf{y} - \mathbf{y}_{\mathrm{m}}(\mathbf{p}))^{\mathrm{T}} (\mathbf{y} - \mathbf{y}_{\mathrm{m}}(\mathbf{p}))$$

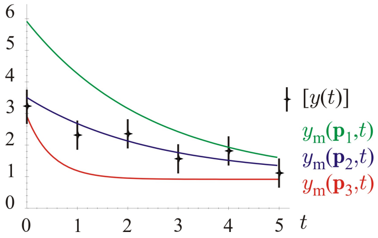
- Local techniques: Gauss-Newton, Levenberg-Marquardt...
- Random search: simulated annealing, genetic algorithms...
- Global guaranteed techniques: Hansen's algorithm

Parameter bounding III

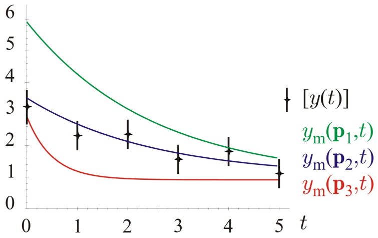
Alternative formulation:

Parameter bounding

$[\varepsilon_i] = [\underline{\varepsilon}_i, \overline{\varepsilon}_i]$, known acceptable errors around $y(t_i)$ at time t_i, $i = 1, \ldots, n$



Parameter bounding IV



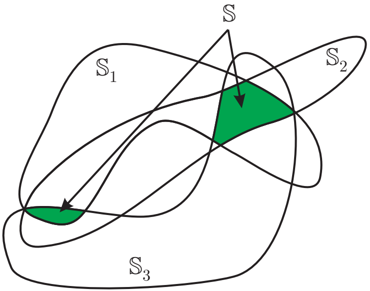
$\mathbf{p} \in \mathscr{P}_0$ deemed acceptable if for all $i = 1, \ldots, n$,

$$\underline{\varepsilon}_i \leqslant y(t_i) - y_m(\mathbf{p}, t_i) \leqslant \overline{\varepsilon}_i.$$

$\Longrightarrow$ Bounded-error parameter estimation:

Characterize $\mathbb{S} = \{\mathbf{p} \in \mathscr{P}_0 \mid y(t_i) - y_m(\mathbf{p}, t_i) \in [\underline{\varepsilon}_i, \overline{\varepsilon}_i],\ i = 1, \ldots, n\}$

Parameter bounding V



$$\mathbb{S} = \bigcap_{\ell=1\ldots,n} \mathbb{S}_\ell,$$

with

$$\begin{aligned}\mathbb{S}_\ell &= \{\mathbf{p} \in \mathscr{P}_0 \mid y_\ell^{\mathrm{m}}(\mathbf{p}) - y_\ell \in [\underline{\varepsilon}_\ell, \overline{\varepsilon}_\ell]\} \\ &= \mathscr{P}_0 \cap (y_\ell^{\mathrm{m}})^{-1}([y_\ell - \overline{\varepsilon}_\ell, y_\ell - \underline{\varepsilon}_\ell])\end{aligned}$$

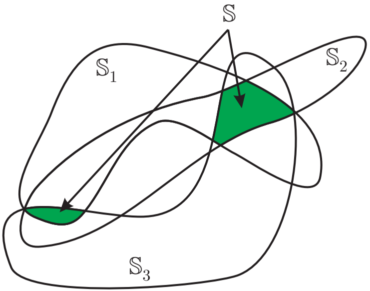
Set-inversion problem

Parameter bounding VI

- When $y_{\mathrm{m}}(\mathbf{p}, t_i)$ is linear in $\mathbf{p}$
  - exact description by polytopes ([WPL88],...)
  - outer approximation by ellipsoids, polytopes, ...([Sch68, FH82],...)

- When $y_{\mathrm{m}}(\mathbf{p}, t_i)$ is non-linear in $\mathbf{p}$
  - outer approximation by polytopes, ellipsoids... ([Nor87, CG90, Cer91, Cer96],...)
  - approximate but guaranteed enclosure of $\mathbb{S}$ by Set Inversion via Interval Analysis ([Moo92, JW93b][Moo92, JW93b],...)

Parameter bounding VII



$$\mathbb{S} = \bigcap_{\ell=1\ldots n} \mathbb{S}_\ell, \text{ with } \mathbb{S}_\ell = \left\{ \mathbf{p} \in \mathscr{P}_0 \mid y_\ell^{\mathrm{m}}(\mathbf{p}) - y_\ell \in [\underline{\varepsilon}_\ell, \overline{\varepsilon}_\ell] \right\}$$

ML estimate of $\mathbf{p}$ assuming independent noise samples bounded in $[\underline{\varepsilon}_\ell, \overline{\varepsilon}_\ell]$.

Interval analysis allows to get

$$\underline{\mathbb{S}} \subset \mathbb{S} \subset \overline{\mathbb{S}}$$

No consistent $\mathbf{p}$ is missed $\Longrightarrow$ guaranteed set estimate.

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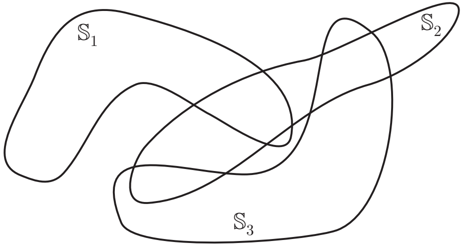
Robust estimation I

Assume now that sensor provides sometimes erroneous measurements

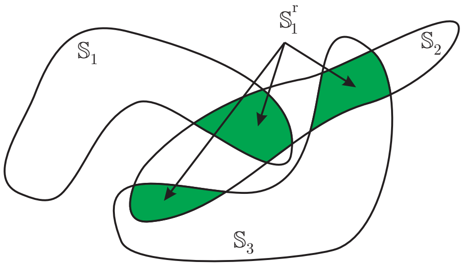


Solution set may be empty

$$\mathbb{S} = \bigcap_{\ell=1\ldots n} \mathbb{S}_\ell = \emptyset.$$

Hypothesis on model or noise violated

Robust estimation II



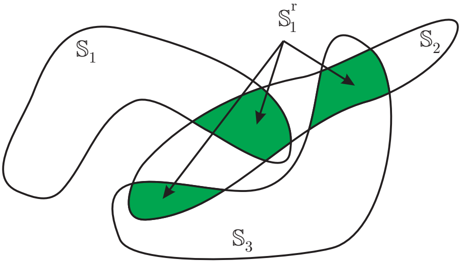
Estimator robust against q outliers

$$\mathbb{S}_q^{\mathrm{r}} = \bigcup_{1 \leqslant \ell_1 < \cdots < \ell_q \leqslant n} \; \bigcap_{\ell \neq \ell_1, \dots, \ell \neq \ell_q} \mathbb{S}_\ell$$

$$= \bigcap^{\{q\}} \mathbb{S}_\ell$$

Union of all intersections of $n - q$ sets among n

Robust estimation III



Non-combinatorial alternative definition

$$\mathbb{S}_q^{\mathrm{r}} = \left\{ \mathbf{p} \in \mathscr{P}_0 \;\middle|\; \sum_{\ell=1}^{q} t_\ell(\mathbf{p}) \geq n - q \right\}$$

with

$$t_\ell(\mathbf{p}) = \left(y_\ell^{\mathrm{m}}(\mathbf{p}) - y_\ell \in [\underline{\varepsilon}_\ell, \overline{\varepsilon}_\ell] \right)$$

Interval analysis: enclosure of $\mathbb{S}_q^{\mathrm{r}}$ evaluated with a complexity of the order of that of $\mathbb{S}$

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Confidence region characterization I

Assume that

$$\mathbf{y} = \mathbf{y}_{\mathrm{m}}(\mathbf{p}) + \mathbf{n},$$

where

- $\mathbf{n}$ is the noise vector with distribution $\pi_n(\mathbf{n})$,
- $\mathbf{y}$ is the data vector
- $\mathbf{p}$ is the parameter vector with prior distribution $\pi_{\mathrm{prior}}(\mathbf{p})$
- $\mathbf{y}_{\mathrm{m}}(\mathbf{p})$ is the model function.

Confidence region characterization II

Using Bayes' rule

$$\pi_{\text{post}}(\mathbf{p}) = \frac{\pi_n(\mathbf{y} - \mathbf{y}_{\text{m}}(\mathbf{p})).\pi_{\text{prior}}(\mathbf{p})}{\int_{\mathbf{p}\in\mathbb{R}^n} \pi_n(\mathbf{y} - \mathbf{y}_{\text{m}}(\mathbf{p})).\pi_{\text{prior}}(\mathbf{p})d\mathbf{p}}.$$

To obtain the maximum *a posteriori* estimate $\widehat{\mathbf{p}}_{\text{MAP}}$, one maximizes

$$f(\mathbf{p}) = \pi_n(\mathbf{y} - \mathbf{y}_{\text{m}}(\mathbf{p})).\pi_{\text{prior}}(\mathbf{p}),$$

since denominator is constant (...and difficult to evaluate).

Confidence region characterization III

But, some difficulties may be encountered

- Parameters of model may not be identifiable uniquely
  $\hookrightarrow$ different values of $\widehat{\mathbf{p}}_{\mathrm{MAP}}$ may yield the same $\mathbf{y}_{\mathrm{m}}\left(\widehat{\mathbf{p}}_{\mathrm{MAP}}\right)$
- Numerical algorithm to compute $\widehat{\mathbf{p}}_{\mathrm{MAP}}$ may get trapped at local minimizer
- Even if single $\widehat{\mathbf{p}}_{\mathrm{MAP}}$ is obtained and if $\mathbf{y} \simeq \mathbf{y}_{\mathrm{m}}\left(\widehat{\mathbf{p}}_{\mathrm{MAP}}\right)$, $\widehat{\mathbf{p}}_{\mathrm{MAP}}$ cannot be considered as final answer to the estimation problem
  $\hookrightarrow$ quality tag is missing.

$$\hat{p}_i = 1.2345 \pm 10^{-4} \text{ is quite different of } \hat{p}_i = 1.2345 \pm 10^{3}.$$

Confidence region characterization IV

Classical approaches are based on

- Level-set [WP97].-
- Monte-Carlo techniques [WP97].
- Evaluation of the density of the estimator [Kay93].
- Bounded-error estimation [MNPLW96, JKDW01].

Confidence region characterization V

The set $\mathbb{S}_\alpha$ defined by

$$\begin{array}{ll}\text{(i)} & \mathbb{S}_\alpha = f^{-1}([s_\alpha, +\infty[), \\ \text{(ii)} & \dfrac{\int_{\mathbb{S}_\alpha} f(\mathbf{p})d\mathbf{p}}{\int_{\mathbb{R}^n} f(\mathbf{p})d\mathbf{p}} = \alpha.\end{array}$$

is the α% confidence region associated with the unnormalized pdf f.
It corresponds to the smallest set which contains $\mathbf{p}$ with a probability equal to α.

Characterizing $\mathbb{S}_\alpha$ is a parametric set-inversion problem
The parameter s_α has to be determined

Confidence region characterization VI

Example:
Consider a random variable p, described by the unnormalized pdf:

$$f(p) = \exp\left(-\frac{p^2}{2}\right).$$

Let us compute its confidence region $\mathbb{S}_{0.95}$. Since,

$$\int_{-\infty}^{\infty} \exp\left(-\frac{p^2}{2}\right) dp = \sqrt{2\pi},$$

we should solve

$$\begin{array}{ll} \text{(i)} & \mathbb{S}_{0.95} = f^{-1}([s_\alpha, +\infty]), \\ \text{(ii)} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}_\alpha} f(\mathbf{p}) d\mathbf{p} = 0.95. \end{array}$$

Confidence region characterization VII

For this example, $\mathbb{S}_{0.95} = [-b, b]$. Thus

$$\begin{array}{ll} \text{(i)} & [-b, b] = f^{-1}([s_\alpha, +\infty]), \\ \text{(ii)} & \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} \exp\left(-\frac{p^2}{2}\right) dp = 0.95. \end{array}$$

Confidence region characterization VIII

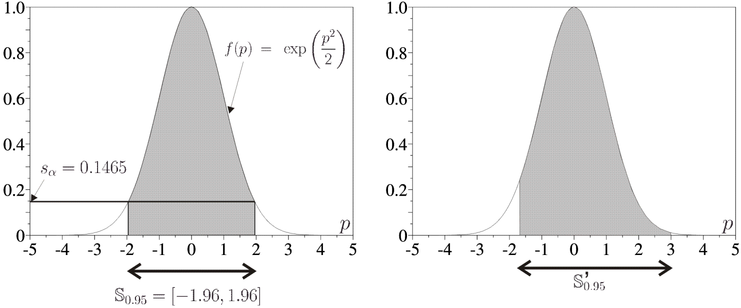
After integration

$$\begin{array}{ll} \text{(i)} & [-b,b] = f^{-1}([s_\alpha, +\infty]), \\ \text{(ii)} & \operatorname{erf}\left(\frac{1}{2} b\sqrt{2}\right) = 0.95. \end{array}$$

We get $b = 1.96$, which corresponds to the well-known result

$$\begin{aligned} \mathbb{S}_{0.95} &= [-1.96, 1.96]. \\ s_\alpha &= f(b) = \exp\left(-\frac{1.96^2}{2}\right) = 0.1465. \end{aligned}$$

Confidence region characterization IX



Interval analysis allows to get
guaranteed inner and outer-approximations of $\mathbb{S}_\alpha$
when $f(\mathbf{p})$ is not that nice (non-gaussian, multimodal...), see also [Jau06].

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Non-asymptotic confidence region characterization I

Classical approaches for confidence region characterization, based on

- Level-set [WP97].
- Monte-Carlo techniques [WP97].
- Evaluation of the density of the estimator [Kay93].
- Bounded-error estimation [MNPLW96, JKDW01].

rely on hypotheses on noise corrupting data

- difficult to check from residuals $\mathbf{y} - \mathbf{y}_{\mathrm{m}}(\hat{\mathbf{p}})$ when n is large,
- impossible to check when only few data points.

Non-asymptotic confidence region characterization II

Campi *et al.* [CW05, DWC07, CCW12] propose two new approaches

- Leave-out Sign-dominant Correlated Regions (LSCR)
- Sign-Perturbed Sums (SPS)

providing

- *exact* characterization of parameter uncertainty
- in *non-asymptotic* conditions.

Hypotheses

1. System generating data must belong to model set (true value $\mathbf{p}^*$ should be meaningful)
2. Noise samples must be independently distributed with distributions symmetric with respect to zero.

Non-asymptotic confidence region characterization III

With LSCR and SPS, obtaining a prescribed level confidence region aims at characterizing

$$\boldsymbol{\Psi}_q = \left\{ \mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i\left(\mathbf{p}\right) \geqslant q \right\},$$

where $\tau_i\left(\mathbf{p}\right)$ is some *indicator* function

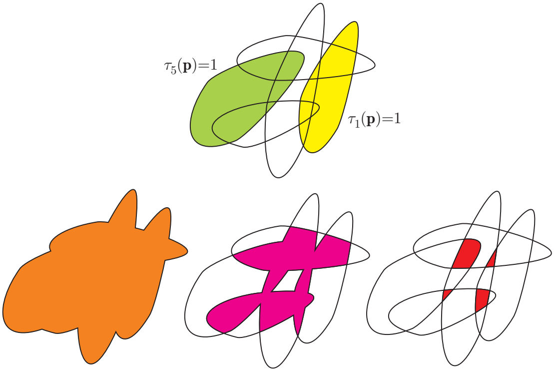
$$\tau_i\left(\mathbf{p}\right) = \begin{cases} 1 & \text{if } f_i\left(\mathbf{p}\right) \geqslant 0, \\ 0 & \text{else,} \end{cases}$$

and where $f_i\left(\mathbf{p}\right)$ depends on the model structure, the measurements, and $\mathbf{p}$.

Non-asymptotic confidence region characterization IV

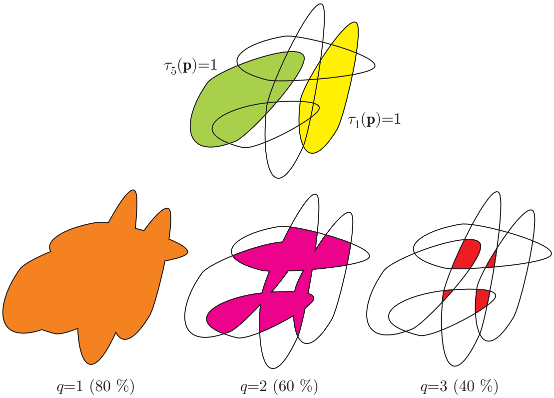
In LSCR (and SPS), one has to characterize

$$\boldsymbol{\Psi}_q = \left\{ \mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i(\mathbf{p}) \geqslant q \right\},$$



q=1 (80 %) q=2 (60 %) q=3 (40 %)

Non-asymptotic confidence region characterization V



Characterization

- approximate using gridding in [CW05, DWC07, CCW12].
- guaranteed using interval analysis here, see also [KW14].

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Main ideas of interval analysis

Interval analysis:

tool to evaluate guaranteed outer-approximation
of the range of a function over intervals

Useful to perform numerical proofs

- function is positive / negative over a given interval
- range of a function is included in some interval
- range of a function does not contain 0.

Results still valid on computers using limited-precision representation of numbers.

Fathers of interval analysis

Introduced by Sunaga [Sun58] in Japan and by Moore [MY59, Moo66] in the USA

- Introduce basics of interval analysis and inclusion functions
- Provides efficient techniques to
  - perform guaranteed deterministic global optimization,
  - evaluate all solutions of a set of nonlinear equations,
  - compute inner and outer approximation of the set of vectors consistent with a set of inequalities,
  - perfom guaranteed numerical integration of ODE...

Limited impact until beginning of the 90s
$\Longrightarrow$ various reasons, among which implementation issues

Ressources

Many books, code libraries, lists

http://www.cs.utep.edu/interval-comp/main.html

Intlab: Matlab Interval analysis toolbox

http://www.ti3.tu-harburg.de/rump/intlab

IBEX: C++ library for constraint processing over real numbers

http://www.ibex-lib.org

Interval analysis: Ongoing standardization process IEEE P1788.

Tools and algorithms

Outline

4. Interval analysis: Basic tools
   - Interval of real numbers
   - Interval arithmetic
   - Inclusion functions
   - Interval vectors or boxes
   - Subpavings
   - Numerical proofs

5. Interval analysis: Basic algorithms
   - Set inversion
   - Simple illustration on bounded-error estimation
   - Contractors
   - Projection of constraints
   - Propagation
   - Contractor algebra
   - Relaxed intersection

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Intervals of real numbers I

Closed and *bounded* subset of $\mathbb{R}$

$$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\} .$$

It is a set $\Longrightarrow$ notions such as

$$=, \in, \subset, \cap$$

are well defined.

When considering $\cup$

$$[x] \cup [y] = \left[\min(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})\right] .$$

Intervals of real numbers II

Width of an interval

$$w([x]) = \overline{x} - \underline{x},$$

Midpoint of an interval

$$m([x]) = \frac{\underline{x} + \overline{x}}{2}.$$

Lower and upper bounds of an interval

$$lb([x]) = \underline{x} \text{ and } ub([x]) = \overline{x}.$$

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Interval arithmetic

Intervals seen as sets of real numbers on which arithmetic operations may be performed

$$[x] \circ [y] = \{x \circ y | x \in [x] \text{ and } y \in [y]\}, \text{ with } \circ \in \{+, -, \times, /\}$$

For example

$$[3, 6] + [4, 7] = ?$$

Interval arithmetic

Intervals seen as sets of real numbers on which arithmetic operations may be performed

$$[x] \circ [y] = \{x \circ y | x \in [x] \text{ and } y \in [y]\}, \text{ with } \circ \in \{+, -, \times, /\}$$

For example

$$\begin{aligned}[3,6]+[4,7] &= [3+4, 6+7] \\ &= [7,13]\end{aligned}$$

$$[3,6]-[2,8] = ?$$

Interval arithmetic

Intervals seen as sets of real numbers on which arithmetic operations may be performed

$$[x] \circ [y] = \{x \circ y | x \in [x] \text{ and } y \in [y]\}, \text{ with } \circ \in \{+, -, \times, /\}$$

For example

$$\begin{aligned} [3, 6] + [4, 7] &= [3 + 4, 6 + 7] \\ &= [7, 13] \end{aligned}$$

$$\begin{aligned} [3, 6] - [2, 8] &= [3 - 8, 6 - 2] \\ &= [-5, 4] \end{aligned}$$

Interval arithmetic

More specifically

$$\begin{cases} [x]+[y]=\left[\underline{x}+\underline{y},\overline{x}+\overline{y}\right], \\ [x]-[y]=\left[\underline{x}-\overline{y},\overline{x}-\underline{y}\right], \\ [x]\times[y]=\left[\min\left(\underline{x}.\underline{y},\overline{x}.\underline{y},\underline{x}.\overline{y},\overline{x}.\overline{y}\right),\max\left(\underline{x}.\underline{y},\overline{x}.\underline{y},\underline{x}.\overline{y},\overline{x}.\overline{y}\right)\right], \\ [x]/[y]=[x]\times\left[1/\overline{y},1/\underline{y}\right], \text{ if } 0\notin[y]. \end{cases}$$

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Inclusion function I

Consider the range of a function over an interval

$$f([x]) = \{f(x) | x \in [x]\}$$

$\Longrightarrow$ difficult to obtain in general
$\Longrightarrow$ sometimes even not an interval.

An inclusion function $[f](.)$ of $f(.)$ satisfies

$$\forall [x] \subset \mathbb{R}, \ f([x]) \subset [f]([x]).$$

Inclusion function is minimal if $\subset$ may be replaced by $=$.

Inclusion function II

Consider a sequence $[x]_1, [x]_2, \dots$ such that

$$[x]_1 \supset [x]_2 \supset [x]_3 \supset \dots$$

and

$$\lim_{n \to \infty} w([x]_n) = 0.$$

- A convergent inclusion function satisfies

$$\lim_{n \to \infty} w([f]([x]_n)) = 0.$$

- An inclusion monotonic inclusion function is such that

$$[x] \subset [y] \implies [f]([x]) \subset [f]([y])$$

Inclusion function III

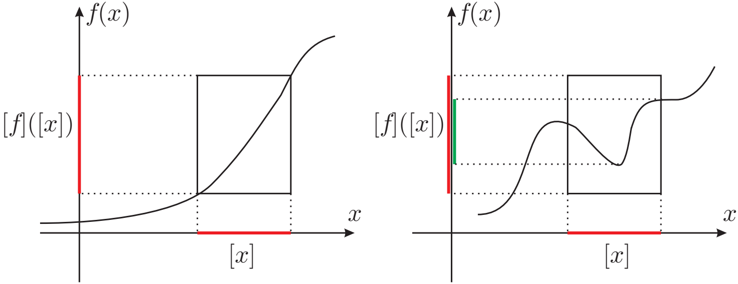
Minimal inclusion functions easy to build for monotone functions

$$\begin{aligned}
\sqrt{[x]} &= \left[\sqrt{\underline{x}}, \sqrt{\overline{x}}\right], \text{ if } \underline{x} \geq 0, \\
\exp([x]) &= [\exp(\underline{x}), \exp(\overline{x})], \\
\tan([x]) &= [\tan(\underline{x}), \tan(\overline{x})], \text{ if } [x] \subseteq [-\pi/2, \pi/2].
\end{aligned}$$

More complicated for other elementary functions
$\Longrightarrow$ algorithm required for $\sin, \cos, \dots$
$\Longrightarrow$ natural inclusion function
$\Longrightarrow$ centred forms...

Natural inclusion function I

Usually, an inclusion function is not minimal



$\Longrightarrow$ some overestimation of the range (pessimism).

Natural inclusion function

$\Downarrow$

Remplace each real variable by its interval counterpart

$$x \longrightarrow [x]$$

Natural inclusion function II

Example:

$$f_1(x) = x(x+1), \quad f_3(x) = x^2 + x,$$
$$f_2(x) = x \times x + x, \quad f_4(x) = (x+\tfrac{1}{2})^2 - \tfrac{1}{4}.$$

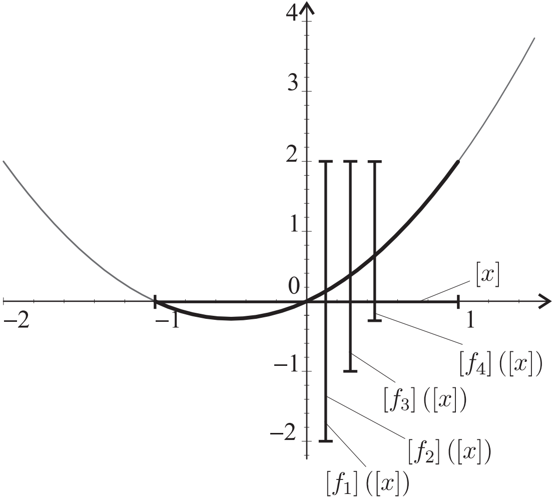
Results for $[x] = [-1, 1]$

$$\begin{aligned}
[f_1]([x]) &= [x]([x]+1) = [-2, 2], \\
[f_2]([x]) &= [x] \times [x] + [x] = [-2, 2], \\
[f_3]([x]) &= [x]^2 + [x] = [-1, 2], \\
[f_4]([x]) &= \left([x] + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = \left[-\tfrac{1}{4}, 2\right].
\end{aligned}$$

Only $[f_4](.)$ is minimal $\Longleftrightarrow$ minimum number of occurences of the interval variable

Natural inclusion function III

Example:



Centred forms

For $f : \mathscr{D} \longrightarrow \mathbb{R}$, differentiable over $[x] \subset \mathscr{D}$, one has $\forall x, m \in [x]$, $\exists \xi \in [x]$ such that

$$f(x) = f(m) + (x - m) f'(\xi).$$

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Then

$$f(x) \in f(m) + (x - m) f'([x]),$$

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Then

$$f(x) \in f(m) + (x - m) f'([x]),$$

and

$$f([x]) \subseteq f(m) + ([x] - m) \left[f'\right]([x]).$$

Centred forms

For $f:\mathscr{D}\longrightarrow\mathbb{R}$, differentiable over $[x]\subset\mathscr{D}$, one has $\forall x,m\in[x]$, $\exists\xi\in[x]$ such that

$$f(x)=f(m)+(x-m)f'(\xi).$$

Then

$$f(x)\in f(m)+(x-m)f'([x]),$$

and

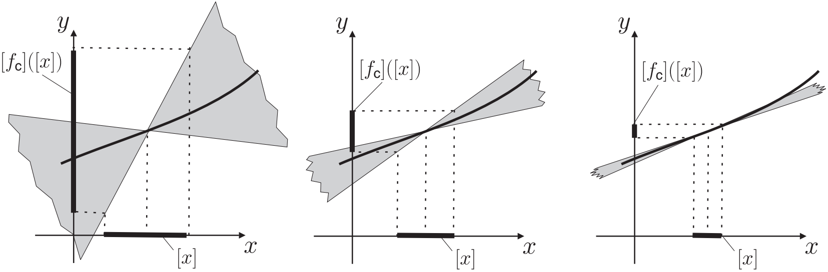
$$f([x])\subseteq f(m)+([x]-m)\left[f'\right]([x]).$$

Centred form: inclusion function defined by

$$[f]_c([x])=f(m)+([x]-m)\left[f'\right]([x])$$

Centred forms I

Interpretation of the centred form



Centred forms II

Example:
Consider

$$f(x) = x^2 \exp(x) - x \exp\left(x^2\right).$$

Compare the natural inclusion fonction and the centred form

$[x]$	$f([x])$	$[f]([x])$	$[f]_c([x])$
$[0.5, 1.5]$	$[-4.148, 0]$	$[-13.82, 9.44]$	$[-25.07, 25.07]$
$[0.9, 1.1]$	$[-0.05380, 0]$	$[-1.697, 1.612]$	$[-0.5050, 0.5050]$
$[0.99, 1.01]$	$[-0.0004192, 0]$	$[-0.1636, 0.1628]$	$[-0.004656, 0.004656]$

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   - Numerical proofs

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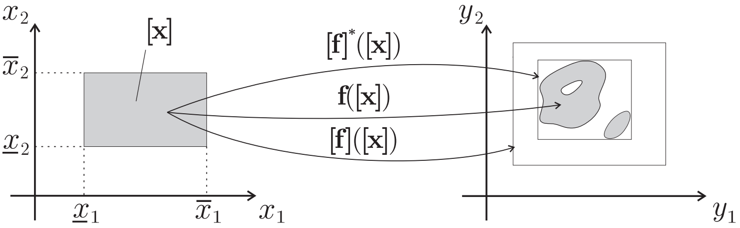
   - Relaxed intersection

Extension to vectors of intervals

Vector of intervals or box

$$[\mathbf{x}] = [x_1] \times \cdots \times [x_n] .$$

Vector inclusion function



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Subpavings I

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.
Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

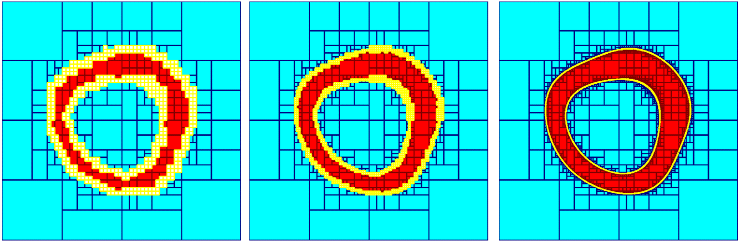
$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Subpavings II

For example, the set

$$\mathbb{X} = \left\{ (x_1, x_2) \,\middle|\, x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9] \right\}$$

can be approximated by the subpavings $\mathbb{X}^-$ and $\mathbb{X}^+$.



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Numerical proofs using interval analysis

Consider an inclusion function $[f](.)$ of $f(.)$. It satisfies

$$\forall [x] \subset \mathbb{R},\ f([x]) \subset [f]([x]).$$

Then

- $0 \notin [f]([x]) \implies 0 \notin f([x])$
- $[f]([x]) \subset [0,\infty[\implies f(x) \geqslant 0, \forall x \in [x]$

Numerical proofs using interval analysis

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- $[f]([x]) \subset [0,\infty[\implies f(x) \geqslant 0, \forall x \in [x]$

But

- $0 \in [f]([x])$ does not necessarily imply $0 \in f([x])$

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Interval analysis: Basic algorithms

Several basic algorithms will be considered

- Set Inversion via Interval Analysis (SIVIA)
- Contractors
- Relaxed contractors

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Set Inversion I

Consider $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$ and define

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

Set Inversion II

Example:
Characterizing the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 x_2 + \sin x_2 \leq 0 \text{ and } x_1 - x_2 = 1\}.$$

is a set inversion problem. Indeed

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

with

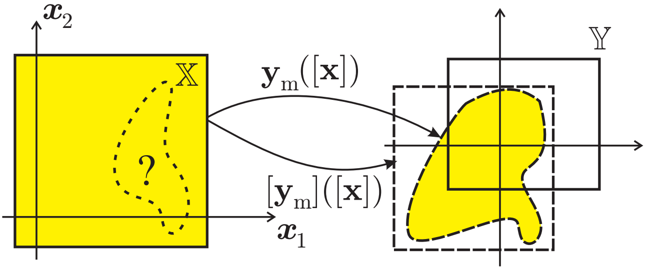
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 x_2 + \sin x_2 \\ x_1 - x_2 \end{pmatrix} \text{ and } \mathbb{Y} = [-\infty, 0] \times \{1\}.$$

Set Inversion III

Basic tests

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.



Set Inversion IV

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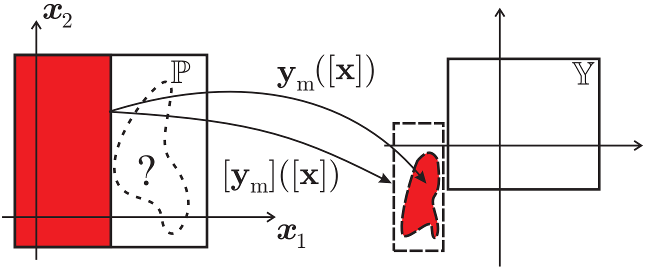
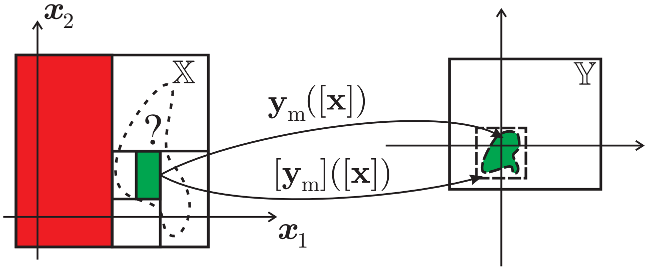


Set Inversion V

Basic tests

$$\begin{array}{llll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.



Set Inversion VI

	Algorithm Sivia(in:$[\mathbf{x}]_0, \mathbf{f}, \mathbb{Y}$)
1	$\mathscr{L} := \{[\mathbf{x}]_0\}$;
2	pull $[\mathbf{x}]$from $\mathscr{L}$;
3	if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$,'red');
4	elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$,'blue');
5	elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$,'yellow')};
6	else bisect $[\mathbf{x}]$and push into $\mathscr{L}$;
7	if $\mathscr{L} \neq \emptyset$, go to 2

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Simple illustration on bounded-error estimation I

Example.
Model:

$$y_{\mathrm{m}}(\mathbf{p}, t) = p_1 e^{-p_2 t}$$

.
Prior feasible box for the parameters:

$$[\mathbf{p}] \subset \mathbb{R}^{n_{\mathbf{p}}}$$

Measurement times:

$$t_1, t_2, \ldots, t_n$$

Data bars:

$$[y_1^-, y_1^+], [y_2^-, y_2^+], \ldots, [y_n^-, y_n^+]$$

$$\mathbb{S} = \{\mathbf{p} \in [\mathbf{p}], y_{\mathrm{m}}(\mathbf{p}, t_1) \in [y_1^-, y_1^+], \ldots, y_{\mathrm{m}}(\mathbf{p}, t_n) \in [y_n^-, y_n^+]\}.$$

Simple illustration on bounded-error estimation II

If

$$\mathbf{y}_{\mathrm{m}}(\mathbf{p}) = \begin{pmatrix} y_{\mathrm{m}}(\mathbf{p}, t_1) \\ \vdots \\ y_{\mathrm{m}}(\mathbf{p}, t_n) \end{pmatrix}$$

and

$$[\mathbf{y}] = [y_1^-, y_1^+] \times \cdots \times [y_n^-, y_n^+]$$

then

$$\mathbb{P} = [\mathbf{p}] \cap \mathbf{y}_{\mathrm{m}}^{-1}([\mathbf{y}]).$$

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Contractors

To characterize $\mathbb{X} \subset \mathbb{R}^n$, bisection algorithms bisect all boxes in all directions and become inefficient when n increase.

Interval methods can still be useful if

- the solution set $\mathbb{X}$ is small (optimization problem, solving equations),
- contraction procedures are used as much as possible,
- bisections are used only as a last resort.

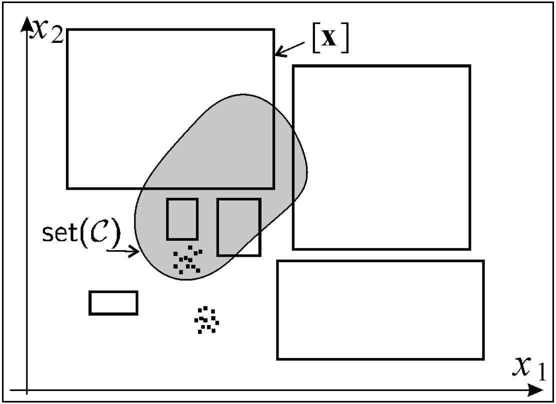
Contractors allow to fight the curse of dimensionality.

Definitions I

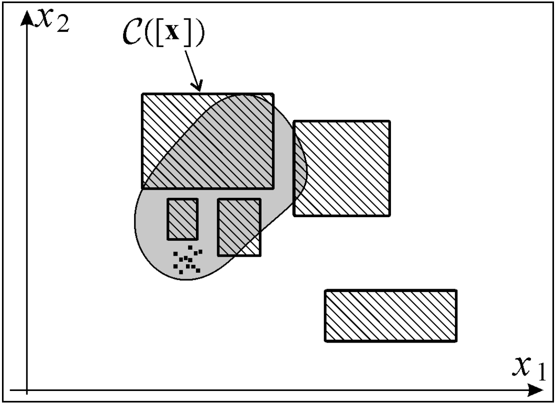
The operator $\mathscr{C}_{\mathbb{X}} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a contractor for the set $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)}, \\ \mathscr{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness)}. \end{cases}$$

Definitions II



Definitions III



Definitions IV

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a contractor for the equation or constraint $f(\mathbf{x}) = 0$, if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) \end{cases}$$

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Projection of constraints I

Let x, y, z be 3 variables such that

$$\begin{aligned} x &\in]-\infty, 5], \\ y &\in]-\infty, 4], \\ z &\in [6, \infty[, \\ z &= x + y. \end{aligned}$$

The values < 2 for x, < 1 for y and > 9 for z are inconsistent.

Projection of constraints II

To project a constraint (here, $z = x + y$), is to compute the smallest intervals which contains all consistent values.

For our example, this amounts to project onto x, y and z the set

$$\mathbb{S} = \{(x,y,z) \in \,]-\infty,5] \times \,]-\infty,4] \times [6,\infty[\;|\; z = x + y\}\,.$$

Numerical method for projection I

Since $x \in]-\infty, 5], y \in]-\infty, 4], z \in [6, \infty[$ and $z = x + y$, we have

$$
\begin{array}{rcl}
z = x + y \Rightarrow & z \in & [6,\infty[\cap (]-\infty,5] +]-\infty,4]) \\
& & = [6,\infty[\cap]-\infty,9] \\
& & = [6,9]. \\
x = z - y \Rightarrow & x \in &]-\infty,5] \cap ([6,\infty[-]-\infty,4]) \\
& & =]-\infty,5] \cap [2,\infty[\\
& & = [2,5]. \\
y = z - x \Rightarrow & y \in &]-\infty,4] \cap ([6,\infty[-]-\infty,5]) \\
& & =]-\infty,4] \cap [1,\infty[\\
& & = [1,4].
\end{array}
$$

Numerical method for projection II

The contractor associated with $z = x + y$ is

	Algorithm pplus(inout: $[z],[x],[y]$)
1	$[z] := [z] \cap ([x] + [y])$;
2	$[x] := [x] \cap ([z] - [y])$;
3	$[y] := [y] \cap ([z] - [x])$.

Numerical method for projection III

Projection procedure developed for plus can be extended to other ternary constraints such as mult: $z = x * y$, or equivalently

$$\text{mult} \triangleq \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = x * y \right\}.$$

Resulting projection procedure becomes

Algorithm pmult(inout: $[z], [x], [y]$)

$[z] := [z] \cap ([x] * [y])$;
$[x] := [x] \cap ([z] * 1/[y])$;
$[y] := [y] \cap ([z] * 1/[x])$.

Numerical method for projection IV

Consider the binary constraint

$$\exp \triangleq \{(x,y) \in \mathbb{R}^n | y = \exp(x)\}.$$

Associated contractor

	Algorithm pexp(inout: $[y],[x]$)
1	$[y] := [y] \cap \exp([x])$;
2	$[x] := [x] \cap \log([y])$.

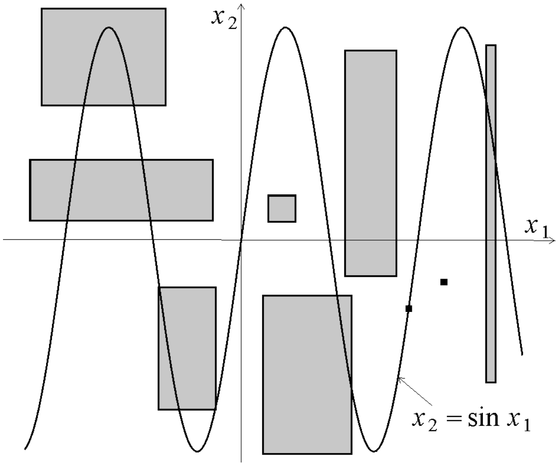
Numerical method for projection V

Any constraint for which such a projection procedure is available is a primitive constraint.

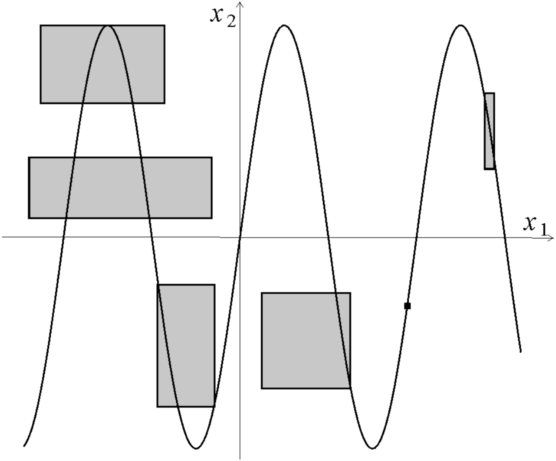
Consider for example the primitive equation:

$$x_2 = \sin x_1.$$

Numerical method for projection VI

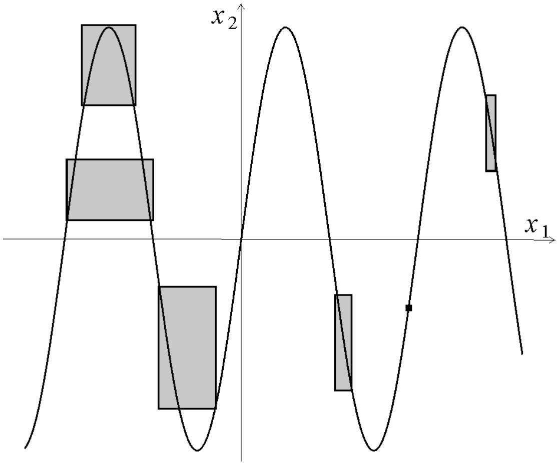


Numerical method for projection VII



Forward contraction

Numerical method for projection VIII



Backward contraction

Decomposition

The constraints

$$\begin{array}{l} x + \sin(xy) \leqslant 0, \\ x \in [-1,1],\, y \in [-1,1] \end{array}$$

Decomposition

The constraints

$$\begin{array}{l} x+\sin(xy) \leqslant 0, \\ x \in [-1,1], y \in [-1,1] \end{array}$$

can be decomposed into

$$\left\{ \begin{array}{l} a = xy \\ b = \sin(a) \\ c = x+b \end{array} \right. , \quad \begin{array}{l} x \in [-1,1] \\ y \in [-1,1] \\ \end{array} \quad \begin{array}{l} a \in]-\infty,\infty[\\ b \in]-\infty,\infty[\\ c \in]-\infty,0] \end{array}$$

Forward-backward contractor (HC4 revise)

For the equation

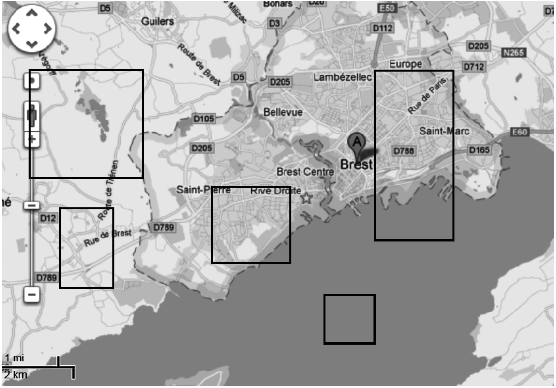
$$(x_1 + x_2) \cdot x_3 \in [1,2],$$

we have the following contractor:

Algorthm $\mathscr{C}$ (inout$[x_1],[x_2],[x_3]$)	
$[a] = [x_1] + [x_2]$	// $a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	// $b = a \cdot x_3$
$[b] = [b] \cap [1,2]$	// $b \in [1,2]$
$[x_3] = [x_3] \cap \frac{[b]}{[a]}$	// $x_3 = \frac{b}{a}$
$[a] = [a] \cap \frac{[b]}{[x_3]}$	// $a = \frac{b}{x_3}$
$[x_1] = [x_1] \cap [a] - [x_2]$	// $x_1 = a - x_2$
$[x_2] = [x_2] \cap [a] - [x_1]$	// $x_2 = a - x_1$

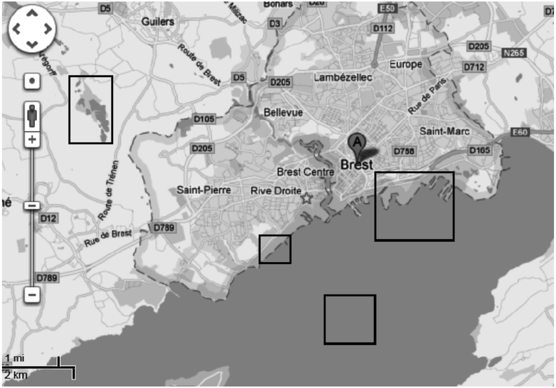
Contractor on images I

The robot with coordinates (x_1, x_2) is in the water.



Contractor on images II

The robot with coordinates (x_1, x_2) is in the water.



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Propagation - first example I

Consider the system of two equations.

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

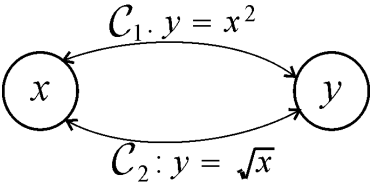
Propagation - first example II

We can build two contractors

$$\mathscr{C}_1 : \left\{ \begin{array}{l}]y] = [y] \cap]x]^2 \\ ]x] = [x] \cap \sqrt{]y]} \end{array} \right. \quad \text{associated to } y = x^2$$

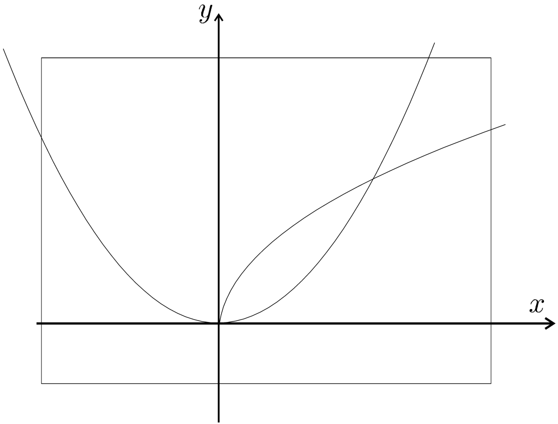
$$\mathscr{C}_2 : \left\{ \begin{array}{l}]y] = [y] \cap \sqrt{]x]} \\ ]x] = [x] \cap]y]^2 \end{array} \right. \quad \text{associated to } y = \sqrt{x}$$

Propagation - first example III

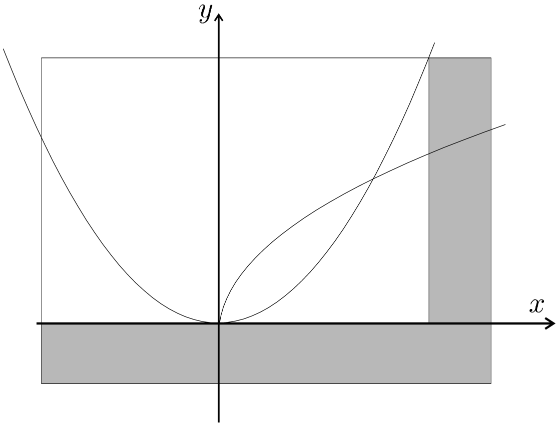


Contractor graph

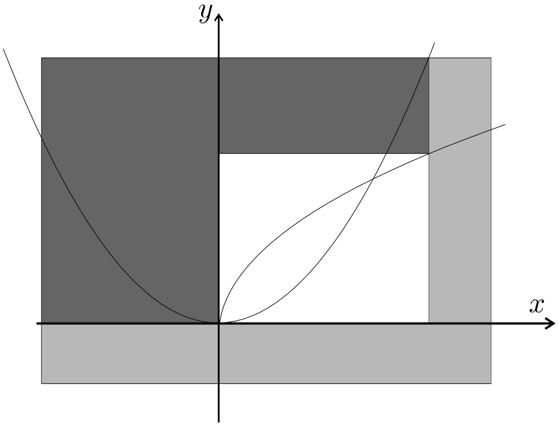
Propagation - first example IV



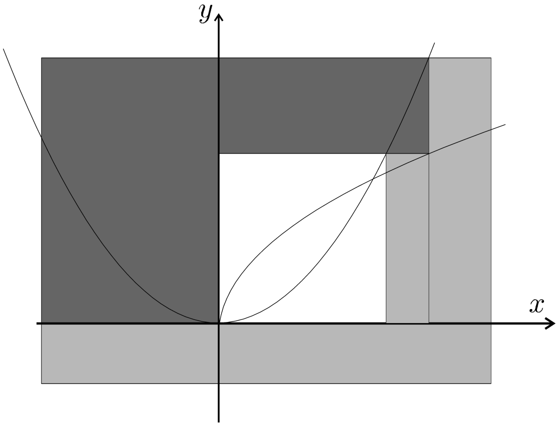
Propagation - first example V



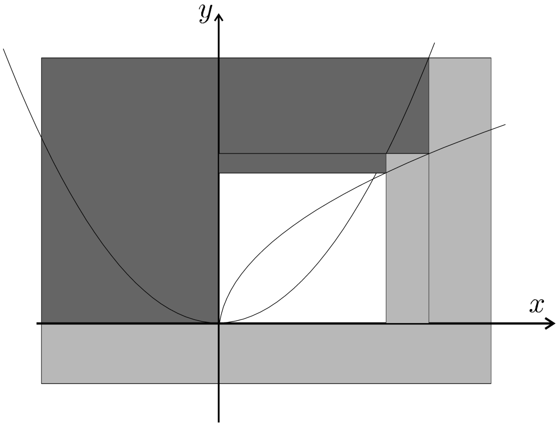
Propagation - first example VI



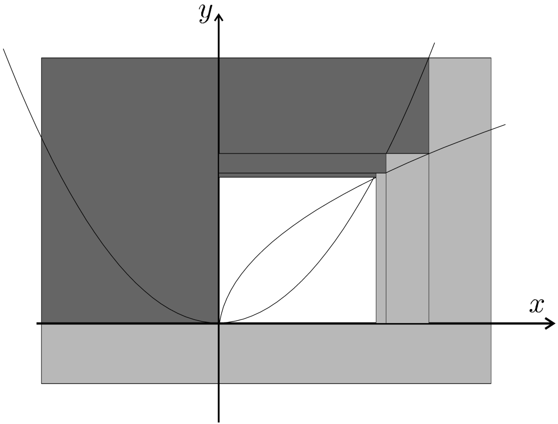
Propagation - first example VII



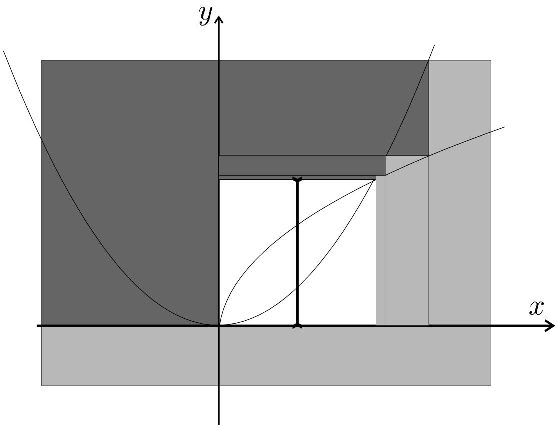
Propagation - first example VIII



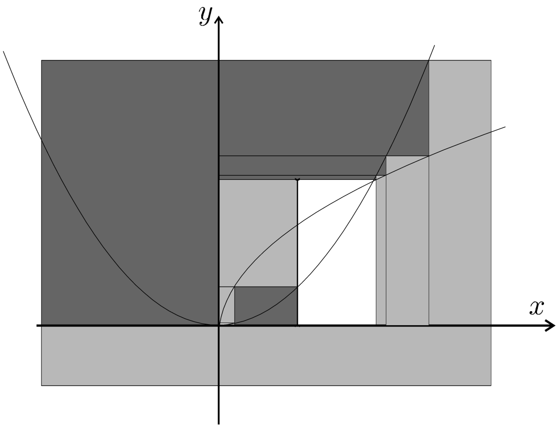
Propagation - first example IX



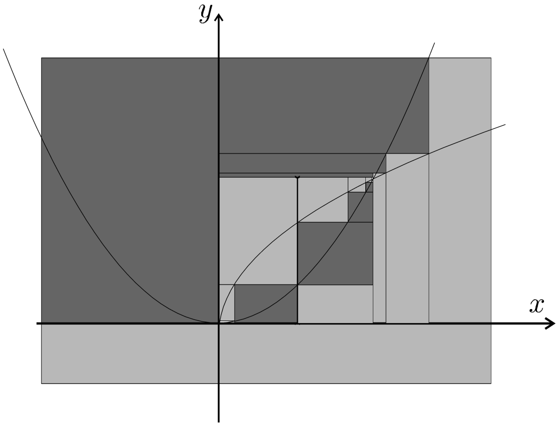
Propagation - first example X



Propagation - first example XI



Propagation - first example XII

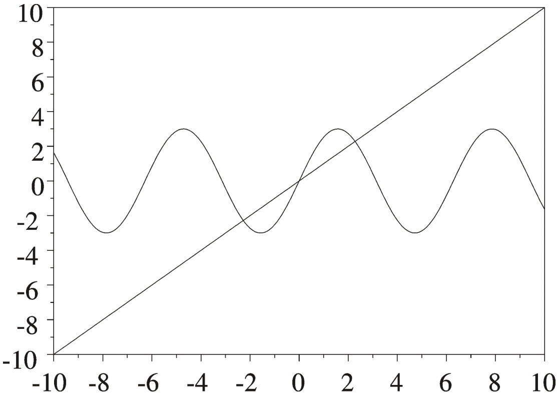


Propagation - second example I

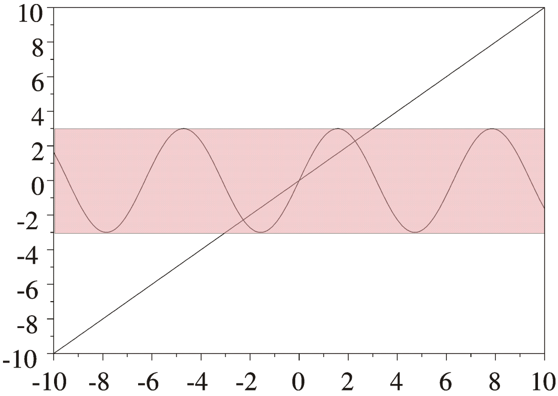
Consider the system

$$\left\{ \begin{array}{lcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \quad x \in \mathbb{R},\ y \in \mathbb{R}.$$

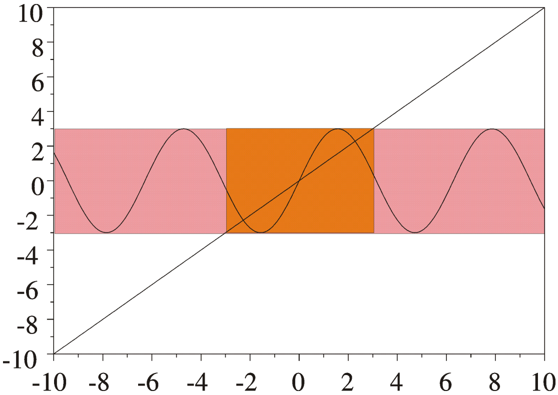
Propagation - second example II



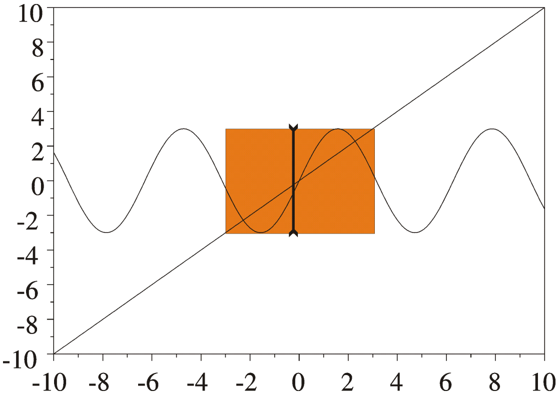
Propagation - second example III



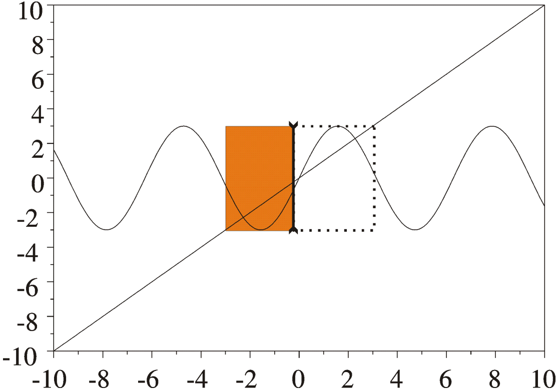
Propagation - second example IV



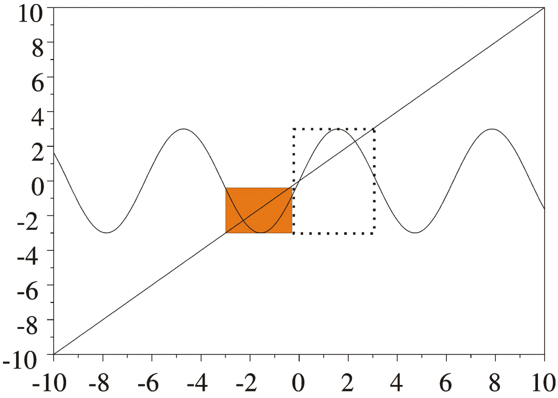
Propagation - second example V



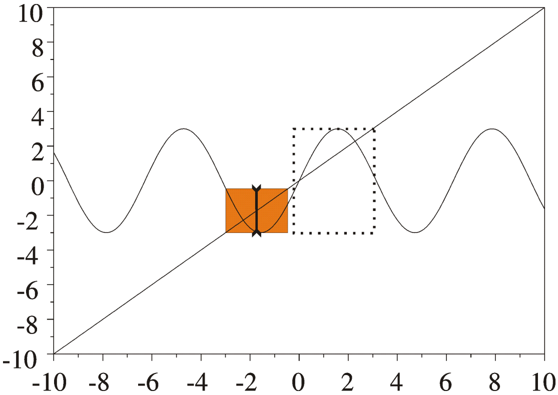
Propagation - second example VI



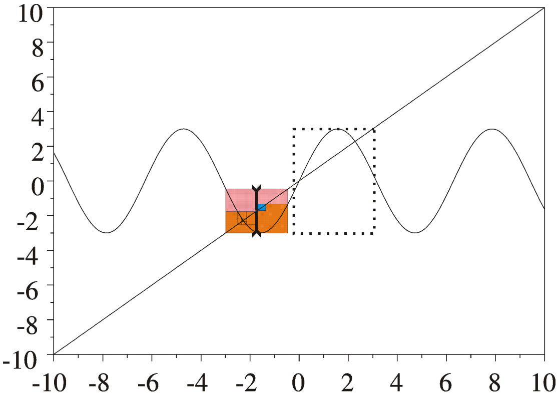
Propagation - second example VII



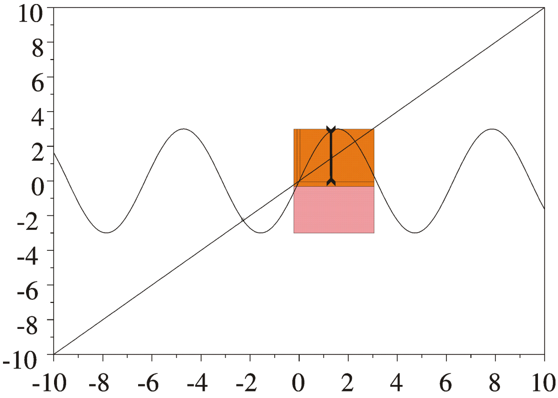
Propagation - second example VIII



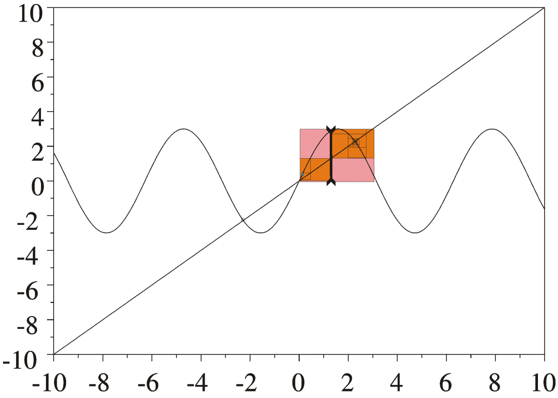
Propagation - second example IX



Propagation - second example X



Propagation - second example XI



Outline

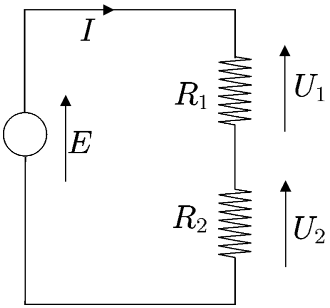
4 Interval analysis: Basic tools
- Interval of real numbers
- Interval arithmetic
- Inclusion functions
- Interval vectors or boxes
- Subpavings
- Numerical proofs

5 Interval analysis: Basic algorithms
- Set inversion
- Simple illustration on bounded-error estimation
- Contractors
- Projection of constraints
- Propagation
- Contractor algebra
- Relaxed intersection

Contractor algebra

intersection	$(\mathscr{C}_1 \cap \mathscr{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathscr{C}_1([\mathbf{x}]) \cap \mathscr{C}_2([\mathbf{x}])$
union	$(\mathscr{C}_1 \cup \mathscr{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathscr{C}_1([\mathbf{x}]) \cup \mathscr{C}_2([\mathbf{x}])]$
composition	$(\mathscr{C}_1 \circ \mathscr{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathscr{C}_1(\mathscr{C}_2([\mathbf{x}]))$
repetition	$\mathscr{C}^\infty \stackrel{\text{def}}{=} \mathscr{C} \circ \mathscr{C} \circ \mathscr{C} \circ \dots$
repeat intersection	$\mathscr{C}_1 \sqcap \mathscr{C}_2 = (\mathscr{C}_1 \cap \mathscr{C}_2)^\infty$
repeat union	$\mathscr{C}_1 \sqcup \mathscr{C}_2 = (\mathscr{C}_1 \cup \mathscr{C}_2)^\infty$

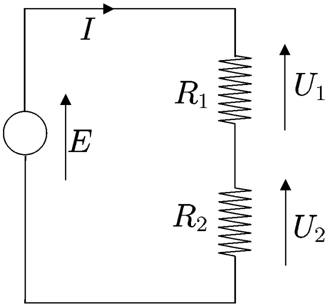
Example I



Domains

$$
\begin{array}{rcl}
E & \in & [23V, 26V]; I \in [4A, 8A]; \\
U_1 & \in & [10V, 11V]; U_2 \in [14V, 17V]; \\
P & \in & [124W, 130W]; R_1 \in [0, \infty[\text{ and } R_2 \in [0, \infty[.
\end{array}
$$

Example II



Constraints

(i) $P = EI$, (ii) $E = (R_1 + R_2)I$, (iii) $U_1 = R_1 I$,
(iv) $U_2 = R_2 I$, (v) $E = U_1 + U_2$.

Example III

Before propagation, we have

$$\begin{array}{rcl} E & \in & [23V, 26V]; I \in [4A, 8A]; \\ U_1 & \in & [10V, 11V]; U_2 \in [14V, 17V]; \\ P & \in & [124W, 130W]; R_1 \in [0, \infty[\text{ and } R_2 \in [0, \infty[. \end{array}$$

After propagation, we get

$$\begin{array}{rcl} E & \in & [24V, 26V]; I \in [4.769A; 5.417A]; \\ U_1 & \in & [10V, 11V]; U_2 \in [14V, 16V]; \\ P & \in & [124W, 130W]; R_1 \in [1.846; 2.307[\text{ and } R_2 \in [2.584; 3.355[. \end{array}$$

Outline

4. Interval analysis: Basic tools
   - Interval of real numbers
   - Interval arithmetic
   - Inclusion functions
   - Interval vectors or boxes
   - Subpavings
   - Numerical proofs

5. Interval analysis: Basic algorithms
   - Set inversion
   - Simple illustration on bounded-error estimation
   - Contractors
   - Projection of constraints
   - Propagation
   - Contractor algebra
   - Relaxed intersection

Relaxed intersection - motivation I

A robot located in a plane at (p_1, p_2) measures its own distance to three marks. The distances and the coordinates of the marks are

beacon	x_i	y_i	$[d_i]$
1	1	3	$[1,2]$
2	3	1	$[2,3]$
3	-1	-1	$[3,4]$

Relaxed intersection - motivation II

The set of all feasible feasible positions is

$$\mathbb{P} = \bigcap_{i\in\{1,2,3\}} \underbrace{\left\{(p_1, p_2) \mid (p_1 - x_i)^2 + (p_2 - y_i)^2 \in \left[d_i^-, d_i^+\right]\right\}}_{\mathbb{P}_i}$$

Relaxed intersection - motivation III

$$
\begin{aligned}
\overline{\mathbb{P}} &= \overline{\bigcap_{i\in\{1,2,3\}} \mathbb{P}_i} = \bigcup_{i\in\{1,2,3\}} \overline{\mathbb{P}_i} \\
&= \bigcup_{i\in\{1,2,3\}} \left\{(p_1,p_2) \mid (p_1-x_i)^2+(p_2-y_i)^2 \in \left[-\infty, d_i^-\right]\right\} \\
&\qquad \cup \left\{(p_1,p_2) \mid (p_1-x_i)^2+(p_2-y_i)^2 \in \left[d_i^+, \infty\right]\right\}
\end{aligned}
$$

Relaxed intersection - dealing with outliers I

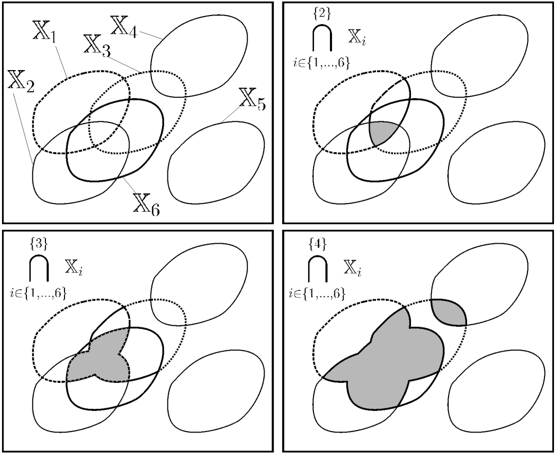
Consider m sets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of $\mathbb{R}^n$.
The q-relaxed intersection

$$\bigcap^{\{q\}} \mathbb{X}_i$$

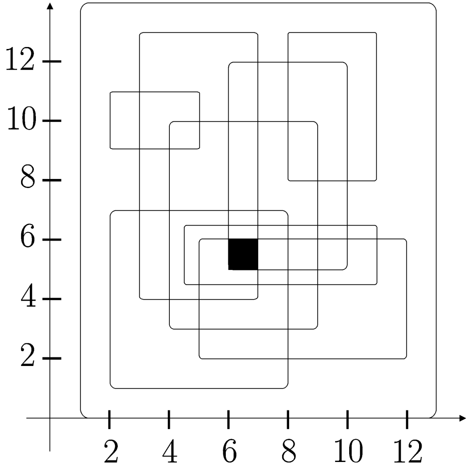
is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most:

$$\mathbf{x} \in \bigcap^{\{q\}} \mathbb{X}_i \Leftrightarrow |\{i | \mathbf{x} \in \mathbb{X}_i\}| \geq m - q$$

Relaxed intersection - dealing with outliers II

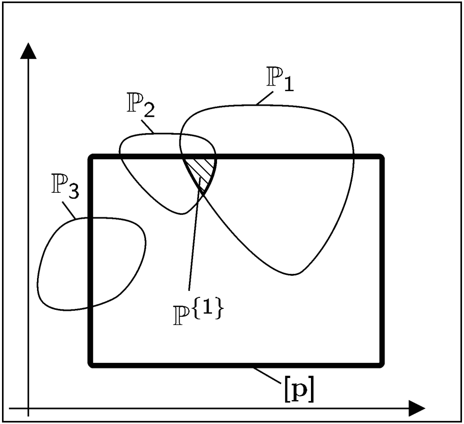


Relaxed intersection - dealing with outliers III

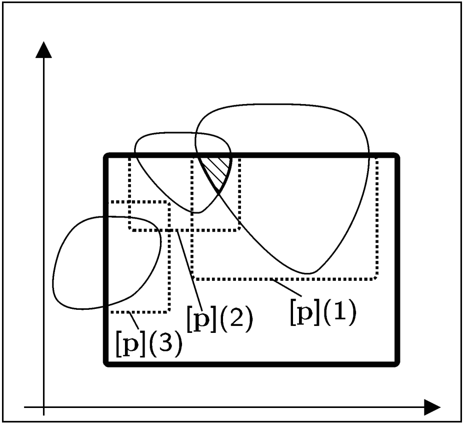


The black box is the 2-intersection of 9 boxes

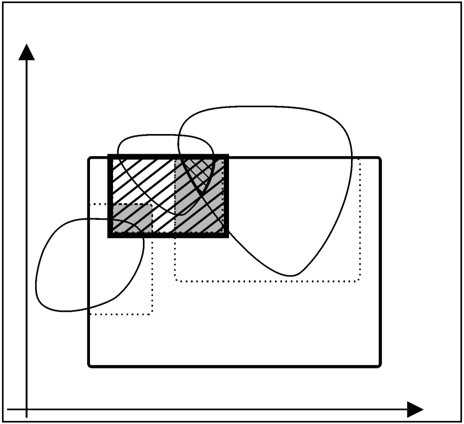
Relaxed intersection - dealing with outliers IV



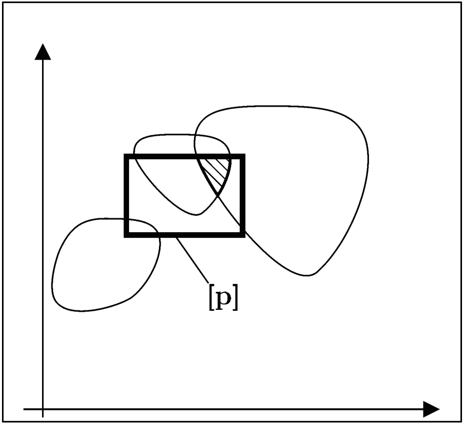
Relaxed intersection - dealing with outliers V



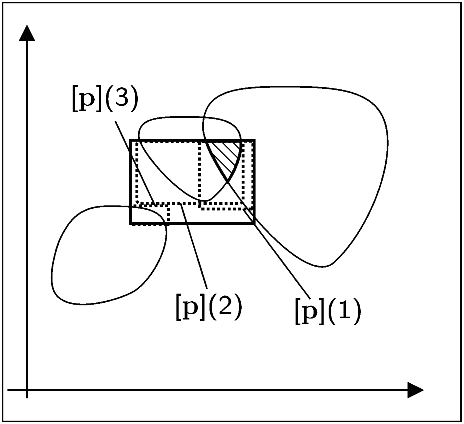
Relaxed intersection - dealing with outliers VI



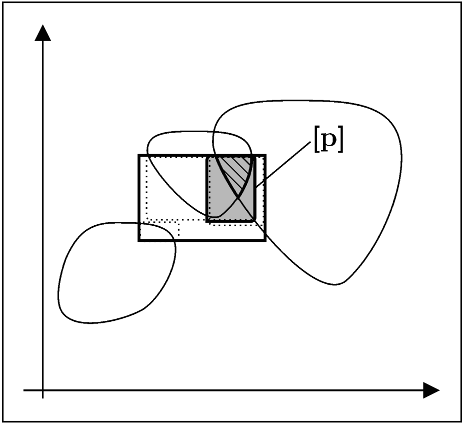
Relaxed intersection - dealing with outliers VII



Relaxed intersection - dealing with outliers VIII

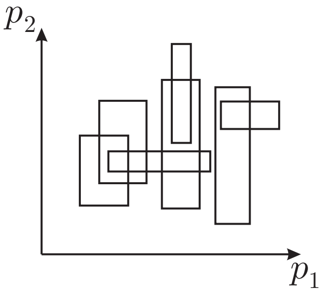


Relaxed intersection - dealing with outliers IX



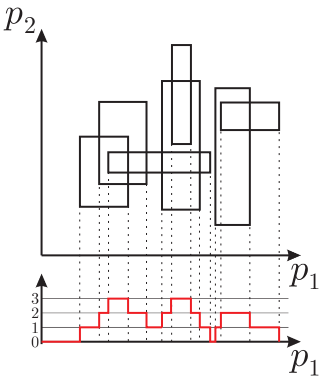
Evaluating the q-relaxed intersection I

Consider a list $\mathscr{L} = \{[p_1], \ldots, [p_m]\}$ of m intervals.



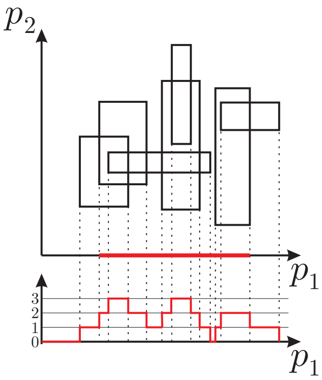
Evaluating the q-relaxed intersection II

Consider a list $\mathscr{L} = \{[p_1], \ldots, [p_m]\}$ of m intervals.



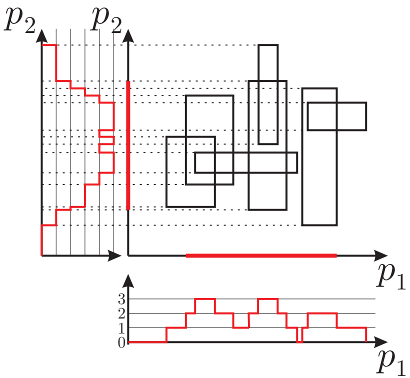
Evaluating the q-relaxed intersection III

Consider a list $\mathscr{L} = \{[p_1], \ldots, [p_m]\}$ of m intervals.



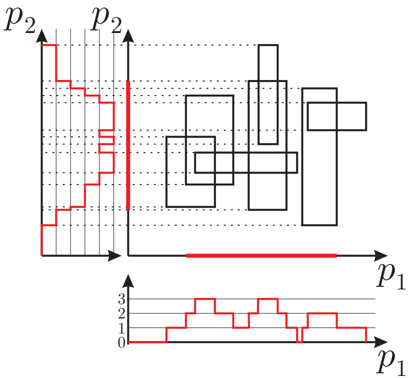
Evaluating the q-relaxed intersection IV

Consider a list $\mathscr{L} = \{[p_1], \dots, [p_m]\}$ of m intervals.



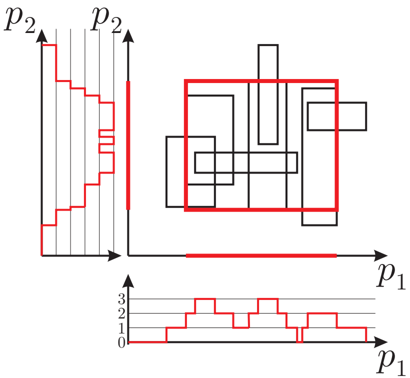
Evaluating the q-relaxed intersection V

Consider a list $\mathscr{L} = \{[p_1], \ldots, [p_m]\}$ of m intervals.



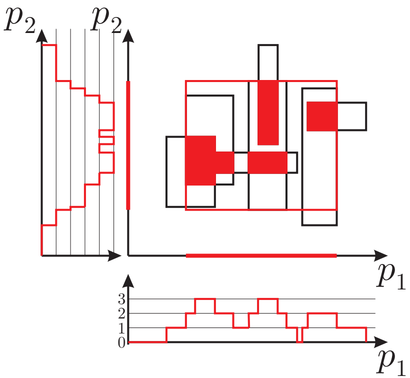
Evaluating the q-relaxed intersection VI

Consider a list $\mathscr{L} = \{[p_1], \ldots, [p_m]\}$ of m intervals.



Evaluating the q-relaxed intersection VII

Consider a list $\mathscr{L} = \{[p_1], \ldots, [p_m]\}$ of m intervals.



Relaxed intersection - bounded-error estimation

$$\mathbb{P}_q^{\mathrm{r}} = \bigcap_i^{\{q\}} \{\mathbf{p} \in \mathbb{R}^n \mid f_i(\mathbf{p}) \in [y_i]\}$$

Relaxed intersection - bounded-error estimation

$$\mathbb{P}_q^{\mathrm{r}} = \bigcap_i^{\{q\}} \{\mathbf{p} \in \mathbb{R}^n \mid f_i(\mathbf{p}) \in [y_i]\}$$

De Morgan's law

$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} = \bigcap^{\{m-q-1\}} \overline{\mathbb{X}}_i$$

Relaxed intersection - bounded-error estimation

$$\mathbb{P}_q^r = \bigcap_i^{\{q\}} \{\mathbf{p} \in \mathbb{R}^n \mid f_i(\mathbf{p}) \in [y_i]\}$$

De Morgan's law

$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} = \bigcap^{\{m-q-1\}} \overline{\mathbb{X}}_i$$

allows to build the following contractors

$$\begin{array}{rcl} \mathscr{C}_i & : & f_i(\mathbf{p}) \in [y_i] \\ \overline{\mathscr{C}_i} & : & f_i(\mathbf{p}) \notin [y_i] \\ \mathscr{C} & = & \bigcap_i^{\{q\}} \mathscr{C}_i \\ \overline{\mathscr{C}} & = & \overline{\bigcap_i^{\{q\}} \mathscr{C}_i} = \bigcap^{\{n-q-1\}} \overline{\mathscr{C}_i} \end{array}$$

Then we call a paver (e.g., IBEX) with $\overline{\mathscr{C}}$ and $\mathscr{C}$.

Relaxed intersection - Illustration I

A robot measures distances to three beacons.

beacon	x_i	y_i	$[d_i]$
1	1	3	$[1,2]$
2	3	1	$[2,3]$
3	-1	-1	$[3,4]$

The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

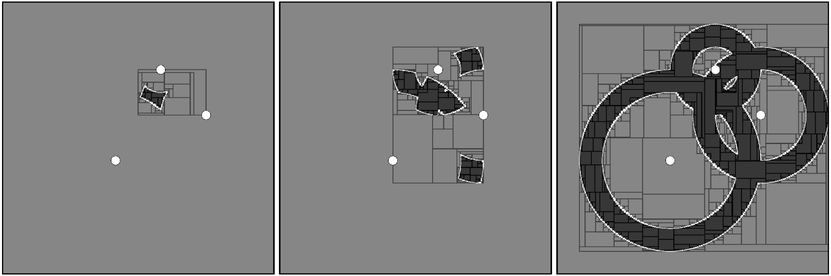
Relaxed intersection - Illustration II

The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5] \right\},$$

where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$.

Relaxed intersection - Illustration III

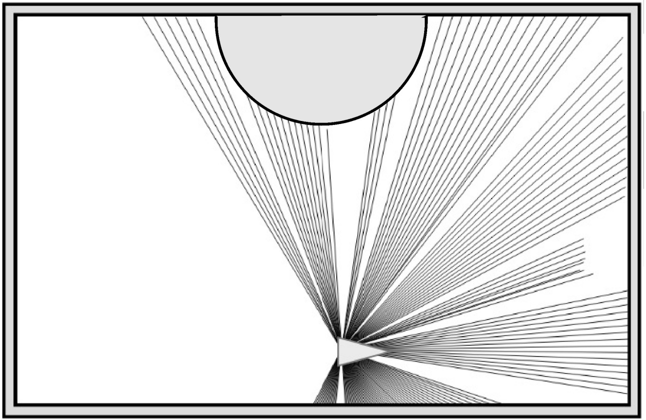


Probabilistic sets $\mathbb{P}^{\{0\}}, \mathbb{P}^{\{1\}}, \mathbb{P}^{\{2\}}$.

Relaxed intersection - Illustration with real data I

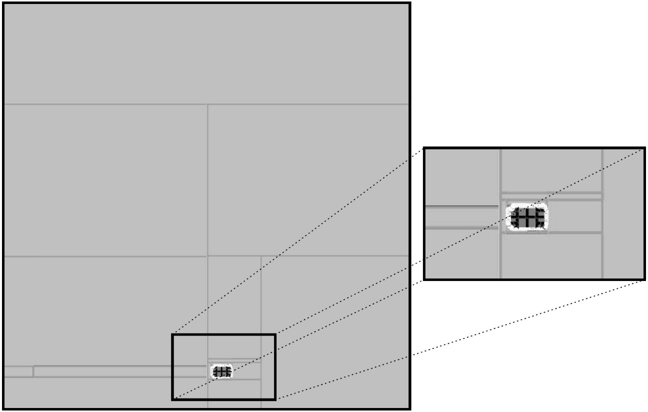
Robot equipped with a laser rangefinder and a compass.

Relaxed intersection - Illustration with real data II



143 distances collected by the rangefinder $\pm 10\,\text{cm}$.

Relaxed intersection - Illustration with real data III



$\mathbb{P}^{\{16\}}$ contains the set all of **p** consistent with all data except 16.

Back to applications

Outline

6. Characterization of non-asymptotic confidence region
   - Approaches proposed by Campi et al.
   - LSCR
   - SPS
   - Guaranteed characterization via interval analysis
   - Example (LSCR)
   - Example (SPS)
7. Bounded-error estimation
   - Interval trajectories
   - Tube arithmetics
   - Tube contractors
   - Time-space estimation
   - Mass spring problem
   - Swarm localization
8. Characterization of confidence regions in a Bayesian context
   - Lattices and intervals
   - Interval subpavings - some additional concepts
   - Interval staircase functions
   - Algorithm
   - Application to Bayesian estimation

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LSCR and SPS

Campi *et al.* [CW05, DWC07, CCW12] propose two new approaches named LSCR and SPS

- *exact characterization* of parameter uncertainty
- in *non-asymptotic* conditions.

Hypotheses

1. System generating data must belong to model set (true value $\mathbf{p}^*$ should be meaningful)
2. Noise samples must be independently distributed with distributions symmetric with respect to zero.

Outline

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   - Approaches proposed by Campi et al.
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LSCR

Introduction - main idea

LSCR [CW05]: *leave-out sign-dominant correlated regions*

LSCR

Introduction - main idea

LSCR [CW05]: *leave-out sign-dominant correlated regions*

- Independent estimates of the correlation of the prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

  should have random signs.

LSCR

Introduction - main idea

LSCR [CW05]: *leave-out sign-dominant correlated regions*

- Independent estimates of the correlation of the prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

  should have random signs.

- Leave out subset of parameter space where sign does not appear random (*i.e.* is sign dominant)

LSCR

Introduction - main idea

LSCR [CW05]: *leave-out sign-dominant correlated regions*

- Independent estimates of the correlation of the prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

  should have random signs.

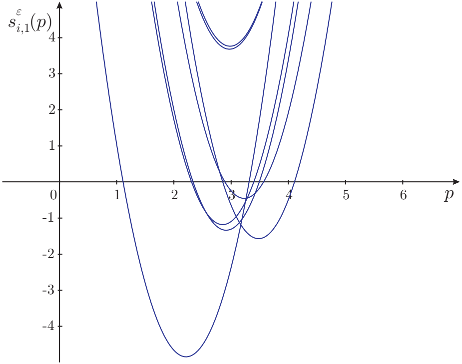
- Leave out subset of parameter space where sign does not appear random (*i.e.* is sign dominant)

Defines, without any approximation,

region $\boldsymbol{\Theta}$ to which $\mathbf{p}^*$ belongs with specified probability.

LSCR
Example

Model $y_t^{\mathrm{m}}(p) = p$, with 8 noisy data generated with $p^* = 3$.



7 different empirical correlations of the prediction errors

LSCR

Description

Consider prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

such that $\varepsilon_t(\mathbf{p}^*)$ is realization of noise corrupting data at time t.

1. Select two integers $r \geqslant 0$ and $q \geqslant 0$.

LSCR

Description

Consider prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

such that $\varepsilon_t(\mathbf{p}^*)$ is realization of noise corrupting data at time t.

1. Select two integers $r \geqslant 0$ and $q \geqslant 0$.

2. For $t = 1 + r, \dots, k + r = n$, compute

$$c^{\varepsilon}_{t-r,r}(\mathbf{p}) = \varepsilon_{t-r}(\mathbf{p})\,\varepsilon_t(\mathbf{p})\,.$$

LSCR

Description

Consider prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

such that $\varepsilon_t(\mathbf{p}^*)$ is realization of noise corrupting data at time t.

1. Select two integers $r \geqslant 0$ and $q \geqslant 0$.
2. For $t = 1 + r, \ldots, k + r = n$, compute

$$c^{\varepsilon}_{t-r,r}(\mathbf{p}) = \varepsilon_{t-r}(\mathbf{p})\,\varepsilon_t(\mathbf{p})\,.$$

3. Compute

$$s^{\varepsilon}_{i,r}(\mathbf{p}) = \sum_{k \in \mathbb{I}_i} c^{\varepsilon}_{k,r}(\mathbf{p})\,,\ \ i = 1, ..., m.$$

where $\mathbb{I}_i \subset \mathbb{I}$, set of indexes. Collection $\mathbb{G}$ of subsets $\mathbb{I}_i$, $i = 1, ..., m$, forms a group under the symmetric difference operation, *i.e.*, $(\mathbb{I}_i \cup \mathbb{I}_j) - (\mathbb{I}_i \cap \mathbb{I}_j) \in \mathbb{G}$.

LSCR

Description

Consider prediction error

$$\varepsilon_t(\mathbf{p}) = y_t - y_t^{\mathrm{m}}(\mathbf{p})$$

such that $\varepsilon_t(\mathbf{p}^*)$ is realization of noise corrupting data at time t.

1. Select two integers $r \geqslant 0$ and $q \geqslant 0$.
2. For $t = 1 + r, \ldots, k + r = n$, compute
$$c^{\varepsilon}_{t-r,r}(\mathbf{p}) = \varepsilon_{t-r}(\mathbf{p})\,\varepsilon_t(\mathbf{p}).$$
3. Compute
$$s^{\varepsilon}_{i,r}(\mathbf{p}) = \sum_{k \in \mathbb{I}_i} c^{\varepsilon}_{k,r}(\mathbf{p}),\ \ i = 1, \ldots, m.$$
where $\mathbb{I}_i \subset \mathbb{I}$, set of indexes. Collection $\mathbb{G}$ of subsets $\mathbb{I}_i$, $i = 1, \ldots, m$, forms a group under the symmetric difference operation, *i.e.*, $(\mathbb{I}_i \cup \mathbb{I}_j) - (\mathbb{I}_i \cap \mathbb{I}_j) \in \mathbb{G}$.
4. Find $\mathbf{\Theta}^{\varepsilon}_{r,q}$ such that at least q of functions $s^{\varepsilon}_{i,r}(\mathbf{p})$ are larger than 0 and at least q are smaller than 0.

LSCR

Description

Exemple of $\mathbb{G}$ st $\forall \mathbb{I}_i \in \mathbb{G}, \forall \mathbb{I}_j \in \mathbb{G}$ one has $(\mathbb{I}_i \cup \mathbb{I}_j) - (\mathbb{I}_i \cap \mathbb{I}_j) \in \mathbb{G}$

	1	2	3	4	5	6	7
$\mathbb{I}_1$	•	•		•	•		
$\mathbb{I}_2$	•		•	•		•	
$\mathbb{I}_3$		•	•		•	•	
$\mathbb{I}_4$	•	•				•	•
$\mathbb{I}_5$	•		•		•		•
$\mathbb{I}_6$		•	•	•			•
$\mathbb{I}_7$				•	•	•	•
$\mathbb{I}_8$							

$$s^{\varepsilon}_{1,r=1}(\mathbf{p}) = \varepsilon_1(\mathbf{p})\varepsilon_2(\mathbf{p}) + \varepsilon_2(\mathbf{p})\varepsilon_3(\mathbf{p}) + \varepsilon_4(\mathbf{p})\varepsilon_5(\mathbf{p}) + \varepsilon_5(\mathbf{p})\varepsilon_6(\mathbf{p})$$
$$s^{\varepsilon}_{2,r=1}(\mathbf{p}) = \varepsilon_1(\mathbf{p})\varepsilon_2(\mathbf{p}) + \varepsilon_3(\mathbf{p})\varepsilon_4(\mathbf{p}) + \varepsilon_4(\mathbf{p})\varepsilon_5(\mathbf{p}) + \varepsilon_6(\mathbf{p})\varepsilon_7(\mathbf{p})$$
$$s^{\varepsilon}_{3,r=1}(\mathbf{p}) = \dots$$

LSCR

Properties

The set $\mathbf{\Theta}^{\varepsilon}_{r,q}$ is such that [CW05]

$$\Pr\left(\mathbf{p}^{*} \in \mathbf{\Theta}^{\varepsilon}_{r,q}\right) = 1 - 2q/m.$$

Shape and size of $\mathbf{\Theta}^{\varepsilon}_{r,q}$ depend on

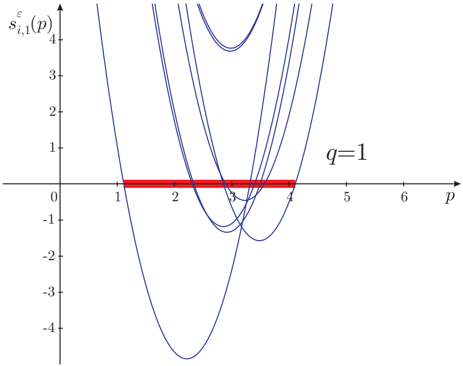
- values given to q and r
- group $\mathbb{G}$ and its number of elements m.

A procedure for generating $\mathbb{G}$ of appropriate size suggested in [Gor74].

LSCR

Example (continued)

Model $y_t^{\mathrm{m}}(p) = p$, with 8 noisy data generated with $p^* = 3$.



7 empirical correlations, and 71% confidence region

LSCR

More formal definition

The set $\boldsymbol{\Theta}_{r,q}^{\varepsilon}$ may be defined more formally as

$$\boldsymbol{\Theta}_{r,q}^{\varepsilon} = \boldsymbol{\Theta}_{r,q}^{\varepsilon,-} \cap \boldsymbol{\Theta}_{r,q}^{\varepsilon,+},$$

with

$$\boldsymbol{\Theta}_{r,q}^{\varepsilon,-} = \left\{ \mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i^{\varepsilon,-}(\mathbf{p}) \geqslant q \right\},$$

$$\boldsymbol{\Theta}_{r,q}^{\varepsilon,+} = \left\{ \mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i^{\varepsilon,+}(\mathbf{p}) \geqslant q \right\},$$

where $\mathbb{P}$ is prior domain for $\mathbf{p}$.

LSCR

More formal definition

Moreover

$$\tau_i^{\varepsilon,-}(\mathbf{p}) = \begin{cases} 1 & \text{if } -s_{i,r}^{\varepsilon}(\mathbf{p}) \geqslant 0, \\ 0 & \text{else,} \end{cases}$$

and

$$\tau_i^{\varepsilon,+}(\mathbf{p}) = \begin{cases} 1 & \text{if } s_{i,r}^{\varepsilon}(\mathbf{p}) \geqslant 0, \\ 0 & \text{else.} \end{cases}$$

$\boldsymbol{\Theta}_{r,q}^{\varepsilon,-}$ contains all $\mathbf{p} \in \mathbb{P}$ such that at least q of the functions $s_{i,r}^{\varepsilon}(\mathbf{p})$ are smaller than 0,
$\boldsymbol{\Theta}_{r,q}^{\varepsilon,+}$ contains all $\mathbf{p} \in \mathbb{P}$ such that at least q of the functions $s_{i,r}^{\varepsilon}(\mathbf{p})$ are larger than 0.

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SPS

Introduction

SPS [CCW12]: sign-perturbed sums.

SPS is designed for linear regression, where

$$y_t = \varphi_t^\top \mathbf{p}^* + w_t, t = 1, \ldots, n,$$

with φ_t known regression vector.

SPS computes an exact confidence region for $\mathbf{p}^*$ around least-squares estimate $\hat{\mathbf{p}}$, which is solution to *normal equations*

$$\sum_{t=1}^{n} \varphi_t \left(y_t - \varphi_t^\top \hat{\mathbf{p}} \right) = \mathbf{0}.$$

SPS

Description

For a generic **p**, define

$$\mathbf{s}_0(\mathbf{p}) = \sum_{t=1}^{n} \varphi_t \left(y_t - \varphi_t^\top \mathbf{p} \right),$$

and the sign-perturbed sums

$$\mathbf{s}_i(\mathbf{p}) = \sum_{t=1}^{n} \alpha_{i,t} \varphi_t \left(y_t - \varphi_t^\top \mathbf{p} \right),$$

where $i = 1, \ldots, m-1$ and $\alpha_{i,t} = \pm 1$ with equal probability, and

$$z_i(\mathbf{p}) = \|\mathbf{s}_i(\mathbf{p})\|_2^2, i = 0, \ldots, m-1.$$

SPS
Description

Confidence region $\mathbf{\Sigma}_q$ is set of all $\mathbf{p}$ such that $z_0(\mathbf{p})$ is *not* among the q largest values of $(z_i(\mathbf{p}))_{i=0}^{m-1}$.

[CCW12] shows that $\mathbf{p}^* \in \mathbf{\Sigma}_q$ with exact probability $1 - q/m$.

SPS

More formal definition

$\boldsymbol{\Sigma}_q$ may be defined more formally as

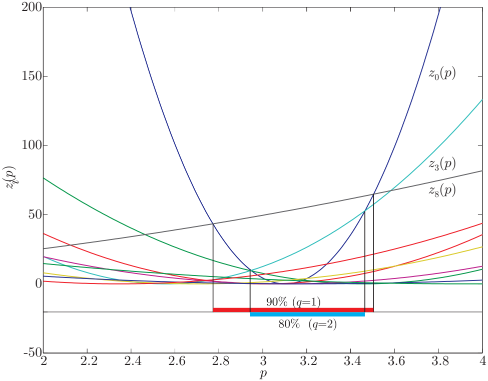
$$\boldsymbol{\Sigma}_q = \left\{ \mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m-1} \tau_i\left(\mathbf{p}\right) \geqslant q \right\}$$

where

$$\tau_i\left(\mathbf{p}\right) = \begin{cases} 1 & \text{if } z_i\left(\mathbf{p}\right) - z_0\left(\mathbf{p}\right) > 0, \\ 0 & \text{else.} \end{cases}$$

SPS
Illustration

Model $y_t^{\mathrm{m}}(p) = p$, with 20 noisy data generated for $p^* = 3$. We choose $m = 10$.



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Guaranteed characterization

In LSCR (and SPS), one has to characterize

$$\boldsymbol{\Psi}_q = \left\{ \mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i(\mathbf{p}) \geqslant q \right\},$$

where $\tau_i(\mathbf{p})$ is some *indicator* function

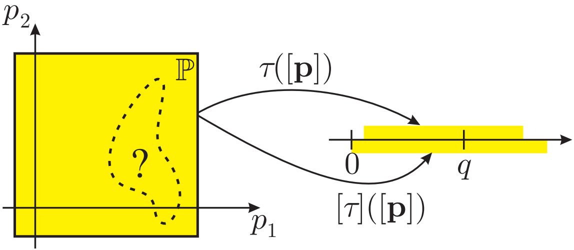
$$\tau_i(\mathbf{p}) = \begin{cases} 1 & \text{if } f_i(\mathbf{p}) \geqslant 0, \\ 0 & \text{else,} \end{cases}$$

and where $f_i(\mathbf{p})$ depends on the model structure, the measurements, and $\mathbf{p}$.

Guaranteed characterization using SIVIA

To characterize $\boldsymbol{\Psi}_q = \left\{\mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i(\mathbf{p}) \geqslant q\right\}$, one uses SIVIA and an inclusion function $[\tau]([\mathbf{p}])$ of

$$\tau(\mathbf{p}) = \sum_{i=1}^{m} \tau_i(\mathbf{p}).$$



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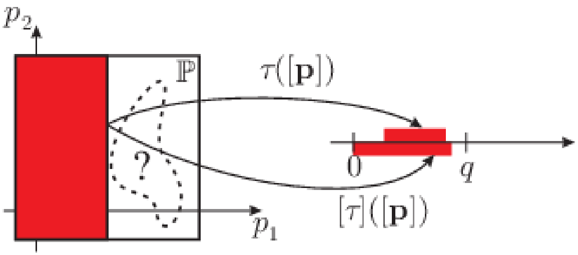

Guaranteed characterization using SIVIA

To characterize $\boldsymbol{\Psi}_q = \left\{\mathbf{p} \in \mathbb{P} \text{ such that } \sum_{i=1}^{m} \tau_i(\mathbf{p}) \geqslant q\right\}$, one uses an inclusion function $[\tau]([\mathbf{p}])$ of

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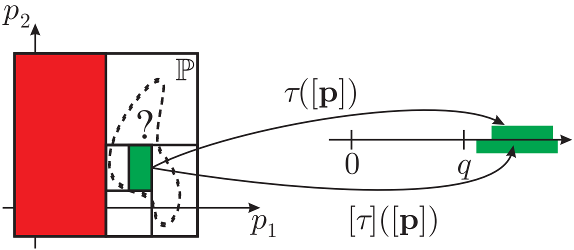

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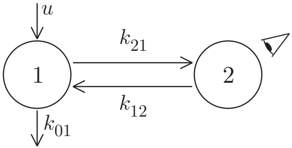
System and Model

Consider the two-compartment model



System and Model

System output obtained from

$$y_t = \alpha(\mathbf{p}^*)\left(\exp\left(\lambda_1(\mathbf{p}^*)\,t\right) - \exp\left(\lambda_2(\mathbf{p}^*)\,t\right)\right) + w_t,$$

where $\mathbf{p} = (k_{01}, k_{12}, k_{21})^{\mathsf{T}}$,

$$\alpha(\mathbf{p}) = \frac{k_{21}}{\sqrt{(k_{01} - k_{12} + k_{21})^2 + 4k_{12}k_{21}}},$$

$$\lambda_{1,2}(\mathbf{p}) = -\frac{1}{2}\left((k_{01} + k_{12} + k_{21}) \pm \left((k_{01} - k_{12} + k_{21})^2 + 4k_{12}k_{21}\right)^{-1/2}\right)$$

and w_t's are realizations of iid $\mathcal{N}\left(0, \sigma^2\right)$ variables, for $t = 0, T, \ldots, (n-1)\,T$.

System and Model

Data generated with $\mathbf{p}^* = (1, 0.25, 0.5)^\mathsf{T}$, $\sigma^2 = 10^{-4}$.

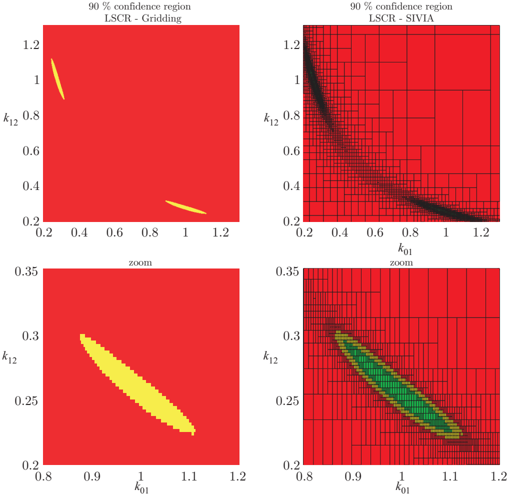
Sampling period is $T = 0.02$ s and $n = 64$.

Only k_{01} et k_{12} are estimated, value k_{21}^* of k_{21} assumed known.

Measurement noise is additive, LSCR method applies directly.

Confidence region obtained by LSCR

$\mathbb{P} = [0,5] \times [0,5]$ and $\varepsilon = 0.0025$.



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System and Model

Consider the system

$$y_t = y_t^{\mathrm{m}}(\mathbf{p}) + w_t,$$

with the FIR model

$$y_t^{\mathrm{m}}(\mathbf{p}) = \sum_{i=0}^{n_{\mathbf{a}}-1} a_i u_{t-i},$$

where $\mathbf{p} = (a_0, \ldots, a_{n_{\mathbf{a}}-1})^{\mathrm{T}}$ and $u_n = 0$ for $n \leqslant 0$.

For $t = 1, \ldots, n$, the w_ts are iid noise samples.

In linear regression form, one has

$$y_t = \varphi_t^{\mathrm{T}} \mathbf{p}^* + w_t$$

with $\varphi_t = (u_t, \ldots, u_{t-n_{\mathbf{a}}+1})^{\mathrm{T}}$ and $\mathbf{p}^* = \left(a_0^*, \ldots, a_{n_{\mathbf{a}}-1}^*\right)^{\mathrm{T}}$.

Inclusion function (low-dim model)

When the dimension of $\mathbf{p}$ is small, $\mathbf{\Sigma}_q$ may be characterized using SIVIA and inclusion functions for τ_i

$$[\tau_i]([\mathbf{p}]) = \begin{cases} 1 & \text{if } \inf([z_i - z_0]([\mathbf{p}])) \geqslant 0, \\ 0 & \text{if } \sup([z_i - z_0]([\mathbf{p}])) < 0, \\ [0,1] & \text{else,} \end{cases}$$

where $[z_i - z_0]([\mathbf{p}])$ is an inclusion function for the difference between $z_i(\mathbf{p})$ and $z_0(\mathbf{p})$.

Simulation conditions (low-dim model)

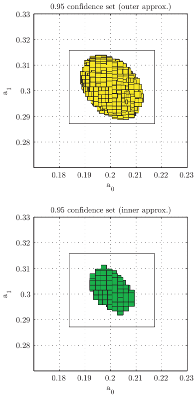
Data are generated for $a_0^* = 0.2$, $a_1^* = 0.3$, and $a_2^* = 0.4$ considering:

1. a filtered Gaussian input $u_t = \alpha u_{t-1} + v_t$, with $\alpha = 0.2$ and $v_t \sim \mathcal{N}(0, 0.65)$
2. a random iid sequence of ± 1 (D-optimal input when input has to remain in $[-1, 1]$.

w_t zero-mean Laplacian (SNR=15 dB).

We choose $n = 1024$, $m = 255$, and $q = 13$ (95 % confidence region), $\varepsilon = 2.5 \times 10^{-3}$.

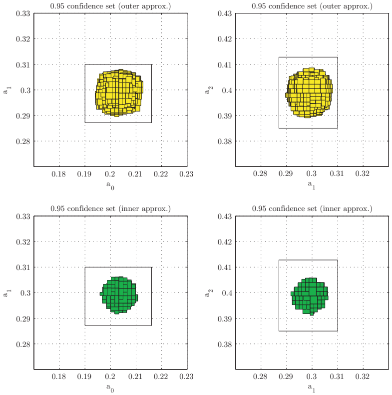
Results (low-dim model)



Projections of the obtained outer and inner-approximations

Gaussian input

Results (low-dim model)



Projections of the obtained outer and inner-approximations

D-optimal input

Simulation conditions (high-dim model)

FIR models with $n_a = 20$ random parameters in $[-2, 2]^{n_a}$ are generated, then

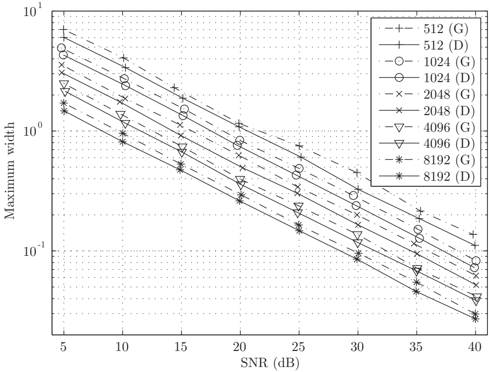
- $n = 512$, 1024, 2048, 4096, and 8192 noise-free data points are generated
- white Laplacian noise is added to the data.

Standard deviation of noise set up to get an SNR of 5 dB to 40 dB.

We choose $n = 1024$, $m = 255$, and $q = 13$ (95 % confidence). Only outer approximations may be obtained.

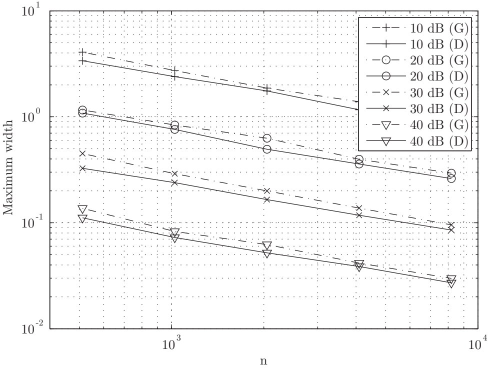
Initial search box $\mathbb{P} = \left[-10^4, 10^4\right]^{20}$.

Results (high-dim model)



Maximum width as a function of the SNR

Results (high-dim model)



Maximum width as a function of n

Slope is about $-1/2$, consistent with ML estimation with additive Gaussian noise

Discussion

- Interval analysis provides guaranteed outer- and inner-approximations of non-asymptotic confidence regions defined by LSCR and SPS.
- Illustrations provided for FIR and non-linear models.
- Accurate inclusion functions are particularly difficult to obtain for the functions involved in SPS,
- Symbolic manipulations of the involved expression to reduce the number of occurrences of the parameters are particularly useful to
  - improve the efficiency of SIVIA
  - to design better contractors

Code available at `http://www.l2s.supelec.fr/perso/kieffer-0`

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Bounded-error estimation

More sophisticated examples are considered here

- State estimation
- Delayed measurements
- ...

Some additional notions are required

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Interval trajectories

A trajectory is a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$. For instance

$$\mathbf{f}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

is a trajectory.

Interval trajectories

A trajectory is a function $\mathbf{f}:\mathbb{R}\to\mathbb{R}^n$. For instance

$$\mathbf{f}(t)=\left(\begin{array}{c}\cos t\\ \sin t\end{array}\right)$$

is a trajectory.

Order relation

$$\mathbf{f}\leq\mathbf{g}\Leftrightarrow\forall t,\forall i,f_i(t)\leq g_i(t).$$

Interval trajectories

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$$\mathbf{f}(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

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Order relation

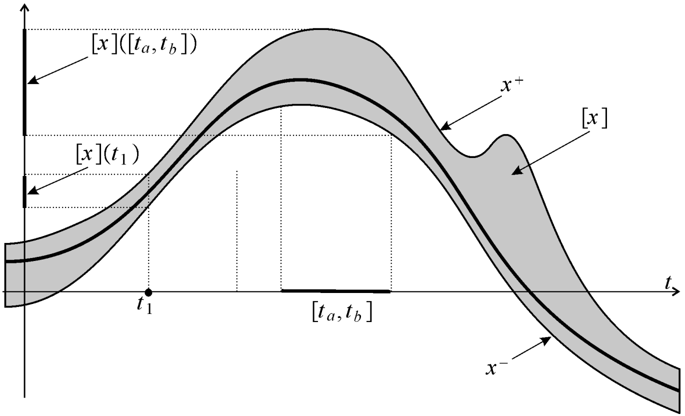
$$\mathbf{f}\leq\mathbf{g}\Leftrightarrow\forall t,\forall i,f_i(t)\leq g_i(t).$$

We have

$$\begin{aligned}\mathbf{h}&=\mathbf{f}\wedge\mathbf{g}\Leftrightarrow\forall t,\forall i,h_i(t)=\min(f_i(t),g_i(t)),\\ \mathbf{h}&=\mathbf{f}\vee\mathbf{g}\Leftrightarrow\forall t,\forall i,h_i(t)=\max(f_i(t),g_i(t)).\end{aligned}$$

Interval trajectories I

The set of trajectories is a lattice. Interval of trajectories (tubes) can be defined.



Interval trajectories II

Example.

$$[\mathbf{f}](t) = \begin{pmatrix} \cos t + [0, t^2] \\ \sin t + [-1, 1] \end{pmatrix}$$

is an interval trajectory (or tube).

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Tube arithmetics

If $[x]$ and $[y]$ are two scalar tubes, we have

$$\begin{array}{ll}
[z]=[x]+[y]\Rightarrow[z](t)=[x](t)+[y](t) & \text{(sum)}\\
[z]=\text{shift}_a([x])\Rightarrow[z](t)=[x](t+a) & \text{(shift)}\\
[z]=[x]\circ[y]\Rightarrow[z](t)=[x]([y](t)) & \text{(composition)}\\
[z]=\int[x]\Rightarrow[z](t)=\left[\int_0^t x^-(\tau)\,d\tau,\int_0^t x^+(\tau)\,d\tau\right] & \text{(integral)}
\end{array}$$

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Tube contractors I

Tube arithmetic allows us to build contractors.

Consider for instance the differential constraint

$$\begin{array}{rcl} \dot{x}(t) & = & x(t+1)\cdot u(t), \\ x(t) & \in & [x](t), \dot{x}(t)\in[\dot{x}](t), u(t)\in[u](t) \end{array}$$

We decompose as follows

$$\left\{ \begin{array}{rcl} x(t) & = & x(0)+\int_0^t y(\tau)\,d\tau \\ y(t) & = & a(t)\cdot u(t). \\ a(t) & = & x(t+1) \end{array} \right.$$

Tube contractors II

Possible contractors are

$$\left\{\begin{array}{lll} [x](t) & = & [x](t)\cap\left([x](0)+\int_0^t [y](\tau)\,d\tau\right) \\ [y](t) & = & [y](t)\cap [a](t)\cdot[u](t) \\ [u](t) & = & [u](t)\cap \frac{[y](t)}{[a](t)} \\ [a](t) & = & [a](t)\cap \frac{[y](t)}{[u](t)} \\ [a](t) & = & [a](t)\cap [x](t+1) \\ [x](t) & = & [x](t)\cap [a](t-1) \end{array}\right.$$

Tube contractors III

Example.
Consider $x(t) \in [x](t)$ with the constraint

$$\forall t,\ x(t) = x(t+1)$$

Contract the tube $[x](t)$.

Tube contractors IV

We first decompose into primitive trajectory constraints

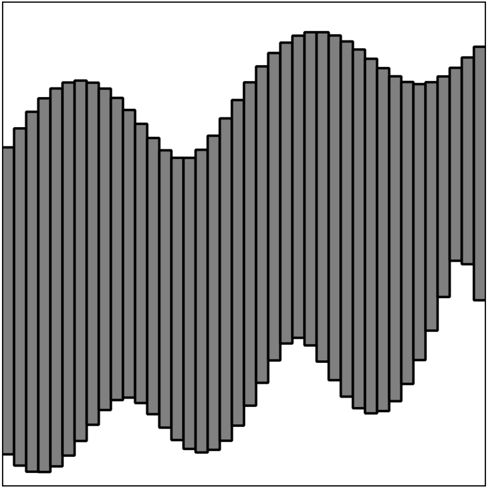
$$\begin{aligned} x(t) &= a(t+1) \\ x(t) &= a(t). \end{aligned}$$

Tube contractors V

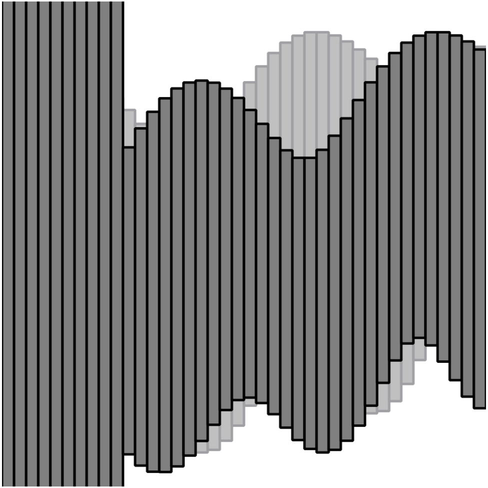
Contractors

$$
\begin{aligned}
[x](t) \quad &: \quad = [x](t) \cap [a](t+1) \\
[a](t) \quad &: \quad = [a](t) \cap [x](t-1) \\
[x](t) \quad &: \quad = [x](t) \cap [a](t) \\
[a](t) \quad &: \quad = [a](t) \cap [x](t)
\end{aligned}
$$

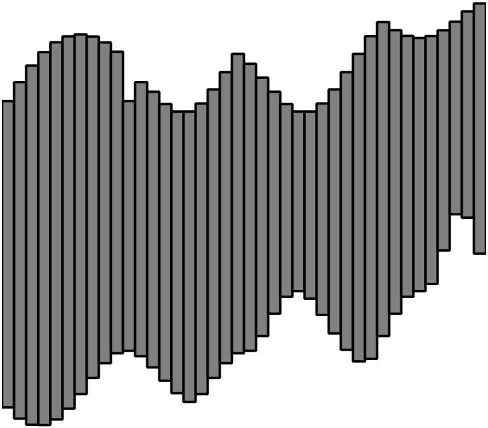
Tube contractors VI



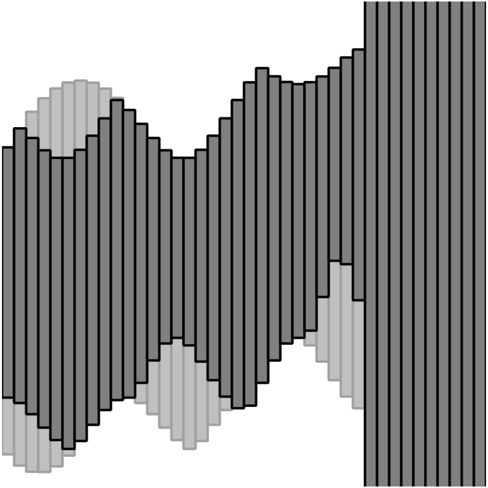
Tube contractors VII



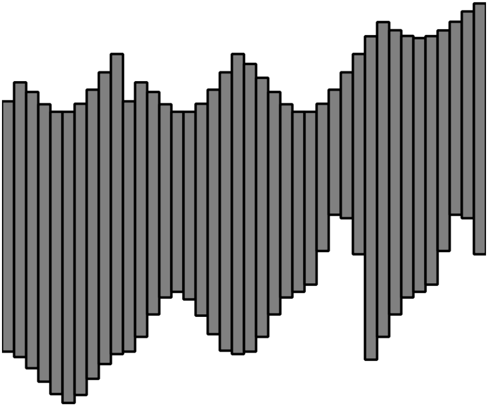
Tube contractors VIII



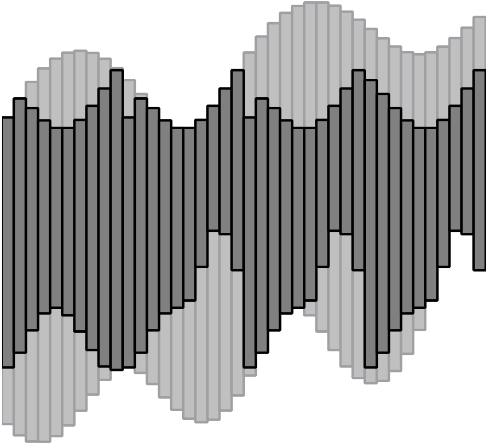
Tube contractors IX



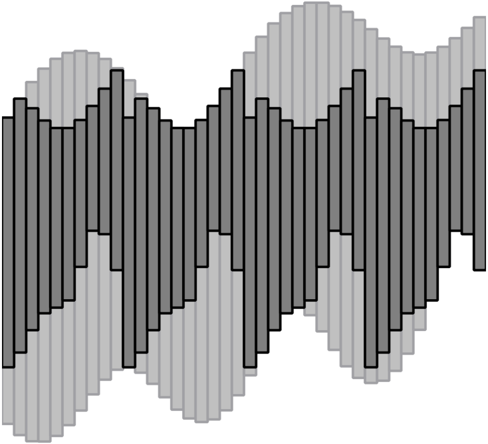
Tube contractors X



Tube contractors XI



Tube contractors XII



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Time-space estimation I

Classical state estimation

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & \quad t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t),t) & \quad t \in \mathbb{T} \subset \mathbb{R}. \end{cases}$$

Space constraint $\mathbf{g}(\mathbf{x}(t),t) = 0$.

Time-space estimation II

Example.

$$\begin{cases} \dot{x}_1 = x_3 \cos x_4 \\ \dot{x}_2 = x_3 \cos x_4 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = u_2 \\ (x_1(5) - 1)^2 + (x_2(5) - 2)^2 - 4 = 0 \\ (x_1(7) - 1)^2 + (x_2(7) - 2)^2 - 9 = 0 \end{cases}$$

Time-space estimation III

With time-space constraints

$$\begin{cases} \underline{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t), \mathbf{x}(t'), t, t') & (t, t') \in \mathbb{T} \subset \mathbb{R} \times \mathbb{R}. \end{cases}$$

Time-space estimation IV

Example.
An ultrasonic underwater robot with state

$$\mathbf{x} = (x_1, x_2, \dots) = (x, y, \theta, v, \dots)$$

At time t the robot emits an onmidirectional sound. At time t' it receives it

$$\left(x_1 - x_1^{'}\right)^2 + \left(x_2 - x_2^{'}\right)^2 - c\,(t - t')^2 = 0.$$

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Mass spring problem I

The mass spring satisfies

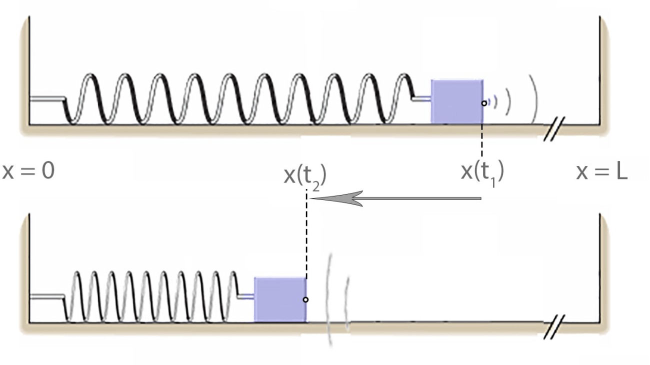
$$\ddot{x} + \dot{x} + x - x^3 = 0$$

i.e.

$$\left\{ \begin{array}{lll} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_2 - x_1 + x_1^3 \end{array} \right.$$

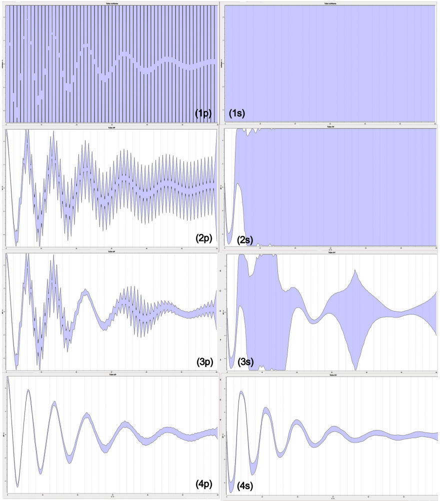
The initial state is unknown.

Mass spring problem II



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - x_1 + x_1^3 \\ L - x_1(t_1) + L - x_1(t_2) = c(t_2 - t_1). \end{cases}$$

Mass spring problem III



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Swarm localization

Consider n robots $\mathscr{R}_1, \dots, \mathscr{R}_n$ described by

$$\underline{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i].$$

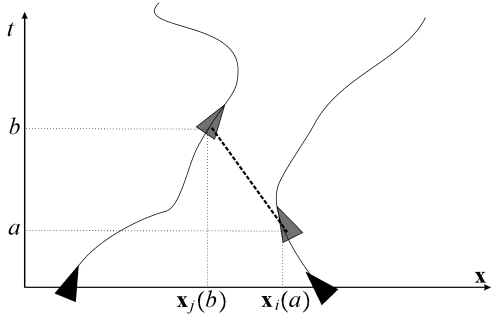
Swarm localization

Consider n robots $\mathscr{R}_1, \ldots, \mathscr{R}_n$ described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i].$$

Omnidirectional sounds are emitted and received.

A *ping* is a 4-uple (a, b, i, j) where a is the emission time, b is the reception time, i is the emitting robot and j the receiver.



Bounded-error swarm localization I

With the time-space constraints

$$\begin{array}{l}\underline{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\ g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0\end{array}$$

where

$$g(\mathbf{x}_i, \mathbf{x}_j, a, b) = \|x_1 - x_2\| - c(b - a).$$

Bounded-error swarm localization II

Clocks are uncertain. We only have measurements $\tilde{a}(k), \tilde{b}(k)$ of $a(k), b(k)$ thanks to clocks h_i. Thus

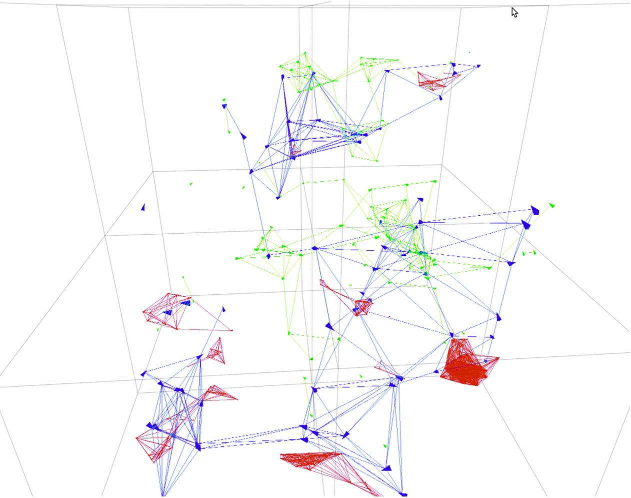
$$\begin{array}{l}
\underline{\dot{\mathbf{x}}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k))
\end{array}$$

Bounded-error swarm localization III

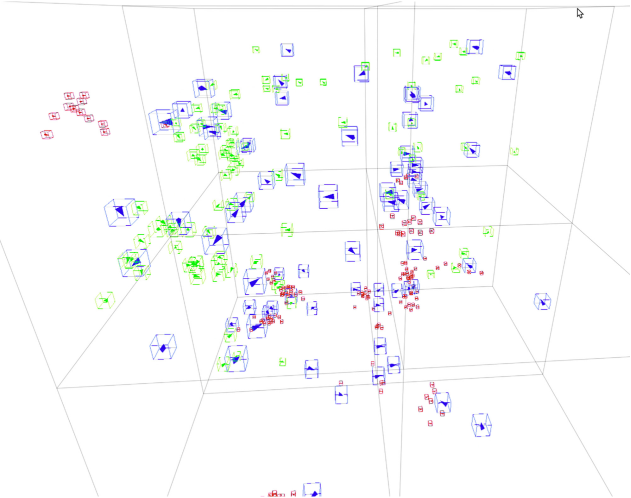
The drift of the clocks is bounded

$$
\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}\left(\mathbf{x}_i, \mathbf{u}_i\right), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}\left(a(k)\right), \mathbf{x}_{j(k)}\left(b(k)\right), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}\left(a(k)\right) \\
\tilde{b}(k) = h_{j(k)}\left(b(k)\right) \\
\dot{h}_i = 1 + n_h, \;\; n_h \in [n_h]
\end{array}
$$

Bounded-error swarm localization IV



Bounded-error swarm localization V



Discussion

Interval techniques

- useful in bounded-error state estimation
- robustness to outliers
- account for the drift in measurements

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Confidence region characterization

Remind that one has to characterize the set $\mathbb{S}_\alpha$ defined by

$$\begin{array}{ll} \text{(i)} & \mathbb{S}_\alpha = f^{-1}([s_\alpha, +\infty[), \\ \text{(ii)} & \dfrac{\int_{\mathbb{S}_\alpha} f(\mathbf{p}) d\mathbf{p}}{\int_{\mathbb{R}^{\boldsymbol{n}}} f(\mathbf{p}) d\mathbf{p}} = \alpha. \end{array}$$

corresponding to the smallest set which contains $\mathbf{p}$ with a probability equal to α.
Again, additional interval notions have to be introduced.

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Lattices and intervals I

A *lattice* $(\mathscr{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds.

- The least upper bound (*join*) of x and y is written $x \vee y$.
- The greatest lower bound (*meet*) is written $x \wedge y$.

A lattice $\mathscr{E}$ is *complete* if for all subsets $\mathscr{A}$ of $\mathscr{E}$, $\vee\mathscr{A}$ and $\wedge\mathscr{A}$ belong to $\mathscr{E}$.

Lattices and intervals II

Example 1 : The set $\mathbb{R}$ is not a complete lattice whereas $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is.
Example 2 : The set $\mathbb{R}^n$ is a lattice with respect to the partial order relation given by

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow \forall i \in \{1, \ldots, n\}, x_i \leq y_i.$$

We have

$$\begin{aligned} \mathbf{x} \wedge \mathbf{y} &= (\min(x_1, y_1), \ldots, \min(x_n, y_n)) \text{ and} \\ \mathbf{x} \vee \mathbf{y} &= (\max(x_1, y_1), \ldots, \max(x_n, y_n)). \end{aligned}$$

Lattices and intervals III

An *interval* $[x]$ of a complete lattice $\mathscr{E}$ is a subset of $\mathscr{E}$ which satisfies

$$[x] = \{x \in \mathscr{E} \mid \wedge [x] \leq x \leq \vee [x]\}.$$

Both $\emptyset$ and $\mathscr{E}$ are intervals of $\mathscr{E}$.

Examples

- The sets $[0,1]_{\bar{\mathbb{R}}}$ and $[0,\infty]_{\bar{\mathbb{R}}}$ are intervals of $\bar{\mathbb{R}}$.
- The set $\{2,3,4,5\} = [2,5]_{\bar{\mathbb{N}}}$ is an interval of $\bar{\mathbb{N}}$.
- The set $\{4,6,8,10\} = [4,10]_{2\bar{\mathbb{N}}}$ is an interval of $2\bar{\mathbb{N}}$.
- The set $[1,2] \times [3,4]) = [(1,3),(2,4)]_{\bar{\mathbb{R}}^2}$ is an interval of $\bar{\mathbb{R}}^2$.

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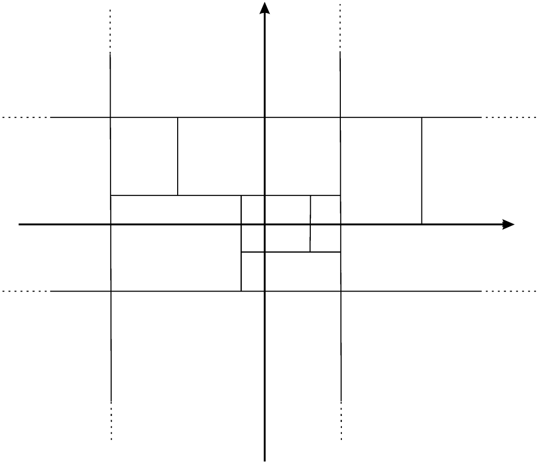
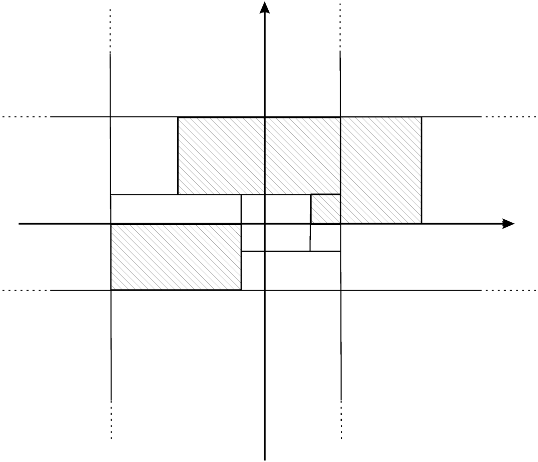
Interval subpavings - some additional concepts I

A paving $\mathcal{Q}$ of $\mathbb{R}^n$ is a set of nonoverlapping boxes covering $\mathbb{R}^n$.



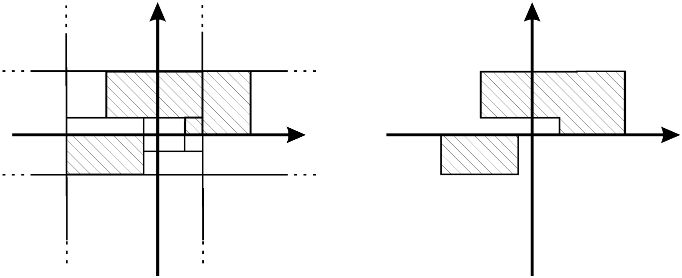
Interval subpavings - some additional concepts II

A subpaving of $\mathcal{Q}$ is a subset of $\mathcal{Q}$.



Interval subpavings - some additional concepts III

The support $\{\mathscr{K}\} \subset \mathbb{R}^n$ of a subpaving $\mathscr{K}$ is the union of all boxes of $\mathscr{K}$.



Interval subpavings - some additional concepts IV

If $\mathscr{P}(Q)$ denotes the set of all subpavings of $\mathscr{Q}$ then $(\mathscr{P}(\mathscr{Q}), \subset)$ is a complete lattice.

- The least upper bound (*join*) is the union

$$\mathscr{K}_1 \vee \mathscr{K}_2 = \mathscr{K}_1 \cup \mathscr{K}_2.$$

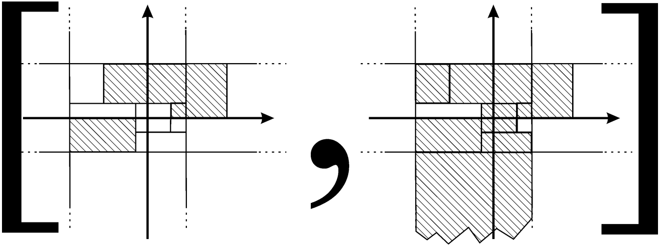
- The greatest lower bound (*meet*) is the intersection

$$\mathscr{K}_1 \wedge \mathscr{K}_2 = \mathscr{K}_1 \cap \mathscr{K}_2.$$

As a consequence intervals of $(\mathscr{P}(\mathscr{Q}), \subset)$ can be defined.

Interval subpavings - some additional concepts V

An interval subpaving $[\mathscr{K}^-, \mathscr{K}^+]$ of $\mathscr{Q}$ can be represented by pair of subpavings of $\mathscr{Q}$ such that $\mathscr{K}^- \subset \mathscr{K}^+$.



One has

$$\mathbb{S} \in [\mathscr{K}^-, \mathscr{K}^+] \Leftrightarrow \{\mathscr{K}^-\} \subset \mathbb{S} \subset \{\mathscr{K}^+\}.$$

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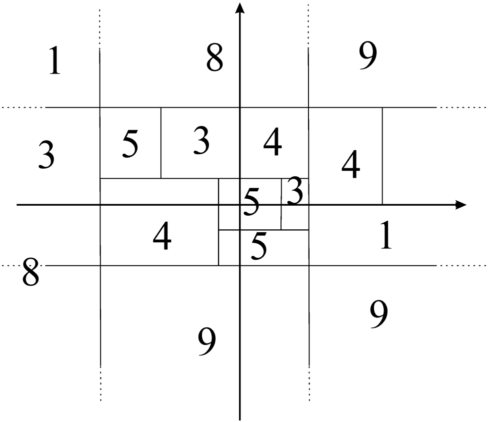
Interval staircase functions I

A staircase function $\hat{f}$ associated with a paving $\mathscr{Q}$ is a function from $\mathscr{Q}$ to $\bar{\mathbb{R}}$

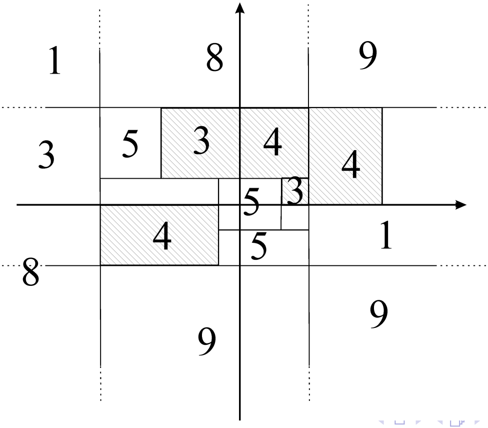


Interval staircase functions II

If $[s] = [s^-, s^+] \in \mathbb{IR}$, the reciprocal image of $[s]$ by $\hat{f}$ is the subpaving of $\mathscr{Q}$ defined by

$$\hat{f}^{-1}([s]) \triangleq \left\{ [\mathbf{p}] \in \mathscr{Q} \mid \hat{f}([\mathbf{p}]) \in [s^-, s^+] \right\}.$$

For instance, $\hat{f}^{-1}([2,4])$ is represented as



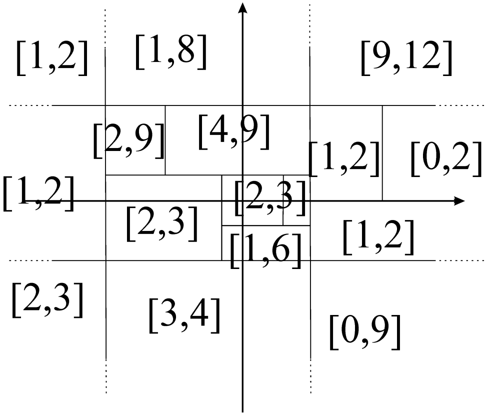
Interval staircase functions III

The set of all staircase functions $(\hat{\mathscr{F}}, \leq)$ is a complete lattice.
Interval staircase functions can thus be defined

An interval staircase function $[\hat{f}] = [\hat{f}^-, \hat{f}^+]$ can be represented a pair of staircase functions such that

$$\forall [\mathbf{p}] \in \mathscr{Q}\ ,\ \hat{f}^-([\mathbf{p}]) \leq \hat{f}^+([\mathbf{p}]).$$

Interval staircase functions IV



Interval staircase functions V

A function f from $\mathbb{R}^n \to \mathbb{R}$ is said to belong to the interval staircase function $[\hat{f}]$ if

$$\forall [\mathbf{p}] \in \mathscr{Q}, \forall \mathbf{p} \in [\mathbf{p}], f(\mathbf{p}) \in \left[\hat{f}^{-}([\mathbf{p}]), \hat{f}^{+}([\mathbf{p}])\right].$$

An interval staircase function for a function $f : \mathbb{R}^n \to \mathbb{R}$ can easily be obtained by using interval techniques.

Interval staircase functions VI

The reciprocal image of the interval $[s^-, s^+] \in \mathbb{IR}$ by the interval staircase function $[\hat{f}] = [\hat{f}^-, \hat{f}^+]$ is the interval subpaving of $\mathscr{Q}$ defined by

$$[\hat{f}]^{-1}([s^-, s^+]) \triangleq \left[\left\{[\mathbf{p}] \in \mathscr{Q} \mid [\hat{f}]([\mathbf{p}]) \subset [s^-, s^+]\right\}, \left\{[\mathbf{p}] \in \mathscr{Q} \mid [\hat{f}]([\mathbf{p}]) \cap [s^-, s^+] \neq \emptyset\right\}\right]$$

Interval staircase functions VII

Theorem
If f belongs to $[\hat{f}]$, then for all $[s^-, s^+] \in \mathbb{IR}$,

$$f^{-1}([s^-, s^+]) \in [\hat{f}]^{-1}([s^-, s^+]).$$

Interval staircase functions VIII

Example

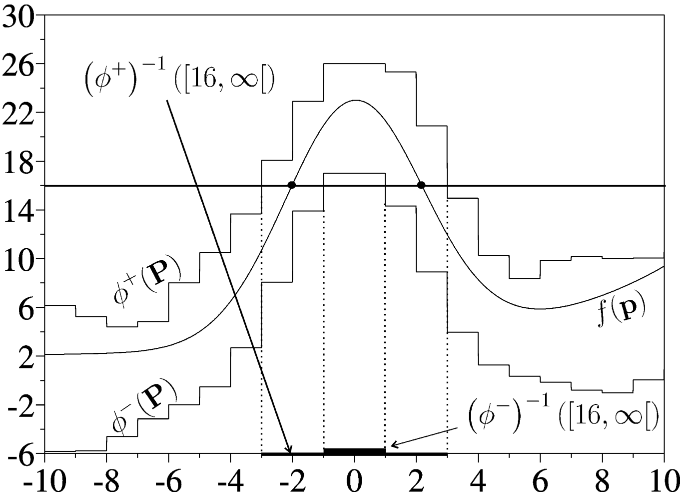
If $[s^-, s^+] = [16, \infty]$ and $\mathscr{Q} = \{[i, i+1],\ i \in \mathbb{N}\}$, then

$$\begin{aligned}
&\{[\mathbf{p}] \in \mathscr{Q} \mid [\hat{f}]([\mathbf{p}]) \subset [s^-, s^+]\ \} \\
&\quad = \{[-1,0],[0,1]\} \equiv [-1,1], \\
&\{[\mathbf{p}] \in \mathscr{Q} \mid [\hat{f}]([\mathbf{p}]) \cap [s^-, s^+] \neq \emptyset\} \\
&\quad = \{[-3,-2],[-2,-1],[-1,0],[0,1],[1,2],[2,3]\} \\
&\quad \equiv [-3,3].
\end{aligned}$$

We have

$$[-1,1] \subset f^{-1}([16,\infty]) \subset [-3,3].$$

Interval staircase functions IX



Interval staircase functions X

If $\left[\mathscr{K}^-,\mathscr{K}^+\right]$ is an interval subpaving of $\mathscr{Q}$ and if $[\hat{f}]$ is a positive interval staircase function, the *integral* of $[\hat{f}]$ over $\left[\mathscr{K}^-,\mathscr{K}^+\right]$ is

$$\int_{[\mathscr{K}^-,\mathscr{K}^+]}[\hat{f}](\mathbf{p})d\mathbf{p} \triangleq \left[\sum_{[\mathbf{p}]\in\mathscr{K}^-}\hat{f}^-([\mathbf{p}]).\text{volume}([\mathbf{p}])\ .\right.$$
$$\left.\sum_{[\mathbf{p}]\in\mathscr{K}^+}\hat{f}^+([\mathbf{p}]).\text{volume}([\mathbf{p}])\right]$$

Interval staircase functions XI

Theorem
If $f \in [\hat{f}]$ and if $\mathbb{S} \in [\mathscr{K}^-, \mathscr{K}^+]$, then

$$\int_{\mathbb{S}} f(\mathbf{p})d\mathbf{p} \in \int_{[\mathscr{K}^-, \mathscr{K}^+]} [\hat{f}](\mathbf{p})d\mathbf{p}.$$

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Algorithm I

Equation in s_α to be solved

$$\alpha = h(s_\alpha) \triangleq \frac{\int_{f^{-1}([s_\alpha,\infty[)} f(\mathbf{p})d\mathbf{p}}{\int_{\mathbb{R}^n} f(\mathbf{p})d\mathbf{p}}$$

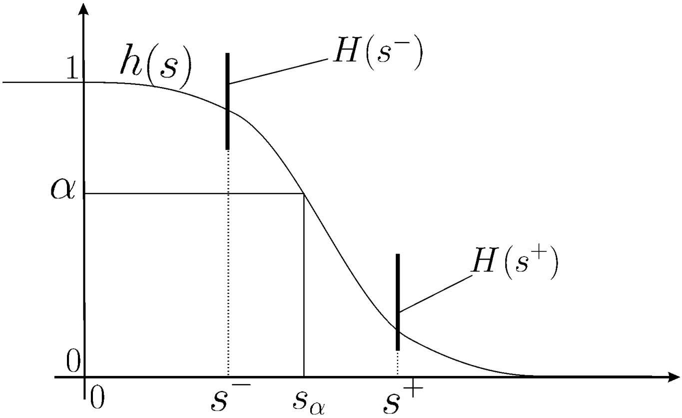
The function $h(s)$ is decreasing. Moreover,

$$h(s) \in [h](s) \triangleq \frac{\int_{[\hat{f}]^{-1}([s,\infty[)} [f](\mathbf{p})d\mathbf{p}}{\int_{\mathbb{R}^n} [\hat{f}](\mathbf{p})d\mathbf{p}}.$$

Algorithm II

Thus

$$\begin{array}{llll} (a) & \alpha < lb([h](s^-)) & \Rightarrow s^- < s_\alpha \\ (b) & \alpha > ub([h](s^+)) & \Rightarrow s^+ > s_\alpha \end{array}$$



Algorithm III

1. Take a paving $\mathscr{Q}$ of $\mathbb{R}^n$; $s^- := +\infty$; $s^+ := 0$;
2. Compute an interval staircase function $[\hat{f}]$ enclosing f;
3. Decrease s^- until $\alpha < lb([h](s^-))$
4. Increase s^+ until $\alpha > ub([h](s^+))$;
5. $\left[\mathscr{K}_\alpha^-, \mathscr{K}_\alpha^+\right] := ([\hat{f}] - [s^-, s^+])^{-1}([0, \infty[)$.

Algorithm IV

Theorem : After completion of this algorithm, we have

$$\mathbb{S}_\alpha \in \left[\mathscr{K}_\alpha^-, \mathscr{K}_\alpha^+\right] \text{ and } s_\alpha \in [s^-, s^+].$$

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   - Swarm localization
8. Characterization of confidence regions in a Bayesian context
   - Lattices and intervals
   - Interval subpavings - some additional concepts
   - Interval staircase functions
   - Algorithm
   - Application to Bayesian estimation

Application to Bayesian estimation I

Model:

$$y(t) = p_1 \sin(p_2 t) + n(t)$$

where $n(t)$ is a white normal random noise with

$$\pi_n(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{n^2}{2\sigma^2}\right),$$

where the standard deviation is $\sigma = \frac{1}{2}$.

Application to Bayesian estimation II

Sampling times and data:

$$\begin{aligned} \mathbf{t} &= (1, 2, 3), \\ \mathbf{y} &= (0.8,\ 1.0,\ 0.2)^{\mathsf{T}}. \end{aligned}$$

Therefore

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} p_1 \sin(p_2) \\ p_1 \sin(2p_2) \\ p_1 \sin(3p_2) \end{pmatrix}}_{\mathbf{y_m}(\mathbf{p})} + \underbrace{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}}_{\mathbf{n}}$$

Application to Bayesian estimation III

Since $n(t)$ is white,

$$\begin{aligned}\pi_n(\mathbf{n}) &= \pi_n(n_1).\pi_n(n_2).\pi_n(n_3) \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^3}\exp(-2n_1^2)\exp(-2n_2^2)\exp(-2n_3^2).\end{aligned}$$

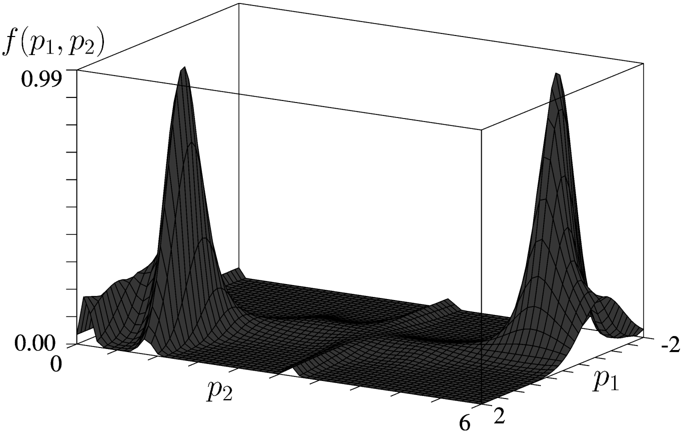
Considering

$$\pi_{\text{prior}}(\mathbf{p}) = \frac{\text{door}_{[-2,2]}(p_1).\text{door}_{[0,6]}(p_2)}{24},$$

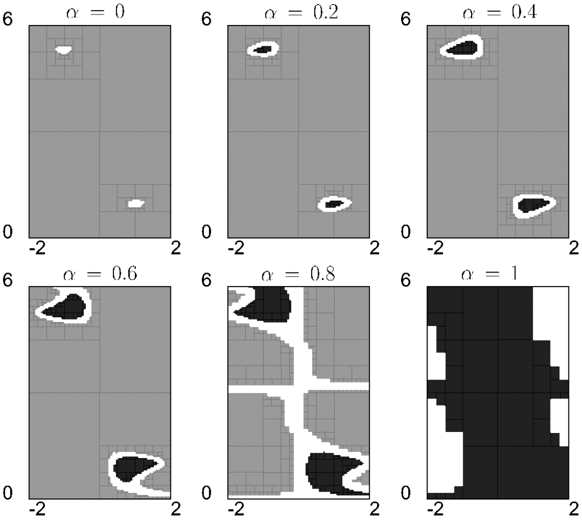
the posterior unnormalized pdf for $\mathbf{p}$:

$$\begin{aligned}f(\mathbf{p}) &= \left(\prod_{k=1}^{3}\exp(-2\,(y_k - p_1\sin(kp_2))^2)\right) \\ &\quad .\text{door}_{[-2,2]}(p_1).\text{door}_{[0,6]}(p_2).\end{aligned}$$

Application to Bayesian estimation IV

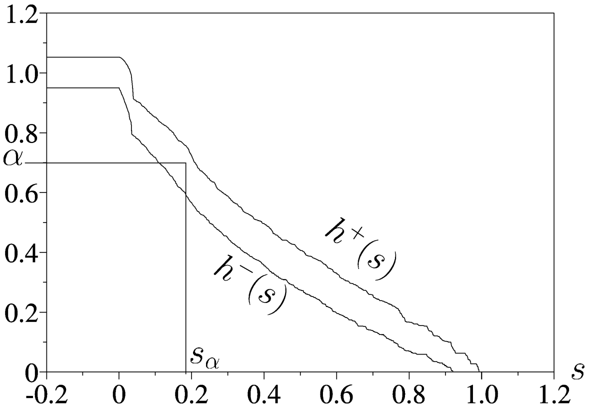


Application to Bayesian estimation V



$\left[\mathscr{K}_\alpha^-, \mathscr{K}_\alpha^+\right]$ for $\alpha \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$;

Application to Bayesian estimation VI



Conclusions

Outline

9. Conclusions and open research directions

Summary

Interval analysis: numerical tool to solve in a guaranteed way nonlinear problems encountered in engineering

- Characterizing sets defined by nonlinear inequalities
- Computing all global minimizers of a non-convex cost function,
- Computing all solutions of a set of nonlinear equations,

Summary

Interval analysis: numerical tool to solve in a guaranteed way nonlinear problems encountered in engineering

- Characterizing sets defined by nonlinear inequalities
- Computing all global minimizers of a non-convex cost function,
- Computing all solutions of a set of nonlinear equations,

Allows development of original solution in estimation problems

- Bounded-error estimation.
- Robust estimation.
- Distributed estimation.
- Guaranteed characterization of asymptotic and non-asymptotic confidence regions.

Open research directions

- Getting efficient contractors is always the key to success.
- Estimation using interval analysis
  - very efficient when explicit expression of model output is available
  - efficient when model described by system of ODEs
  - not efficient when considering PDEs
- Continue to develop tools to faciltate the use of such techniques.

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