Eulerian filter and Eulerian smoother

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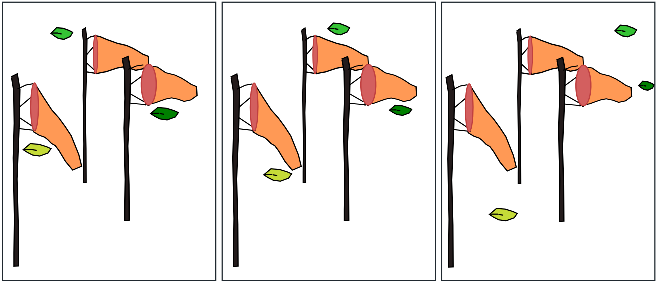
Eulerian state estimation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$$



Leaves: Lagrangian view of the wind; Flags: an Eulerian view [3]

Eulerian state estimation can be formalized as:

$$\begin{array}{lll} (i) & \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t)) & \text{(evolution)} \\ (ii) & \mathbf{x}(t_i)\in\mathbb{X}_i\subset\mathbb{R}^n & \text{(event)} \\ (iii) & \forall(i,j)\in\mathbb{J}, t_i\leq t_j & \text{(precedence)} \end{array}$$

Bracket the set $\mathbb{X}\subset\mathbb{R}^n$ of all feasible $\mathbf{x}(t)$.
The transformation

$$\mathbf{f}(\mathbf{x})\to\mathbf{f}(\rho(\mathbf{x})\cdot\mathbf{x}),\ \rho(\mathbf{x})>0$$

does not change the result.

Invariant sets

Denote by φ the flow map of our system, *i.e.*, with $\mathbf{x}_0 = \mathbf{x}(0)$, the system reaches $\varphi(t, \mathbf{x}_0)$ at time t.

A set $\mathbb{A}$ is *positive invariant* if

$$\mathbf{x} \in \mathbb{A}, t \geq 0 \Longrightarrow \varphi(t, \mathbf{x}) \in \mathbb{A}.$$

The set of all positive invariant sets is a lattice, *i.e.*, the union and the intersection are closed.
Thus, the notion of *largest positive invariant set* contained in $\mathbb{X}$ can be defined.

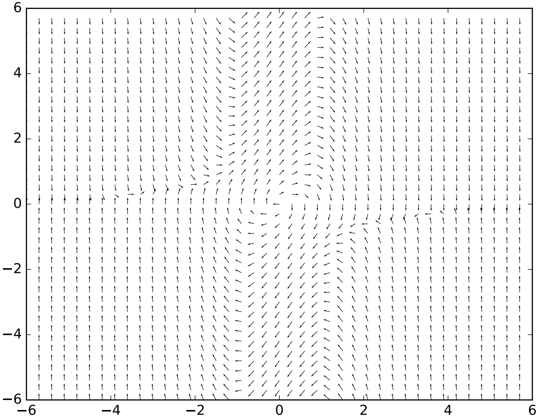
The largest positive invariant set included in $\mathbb{X}$ is:

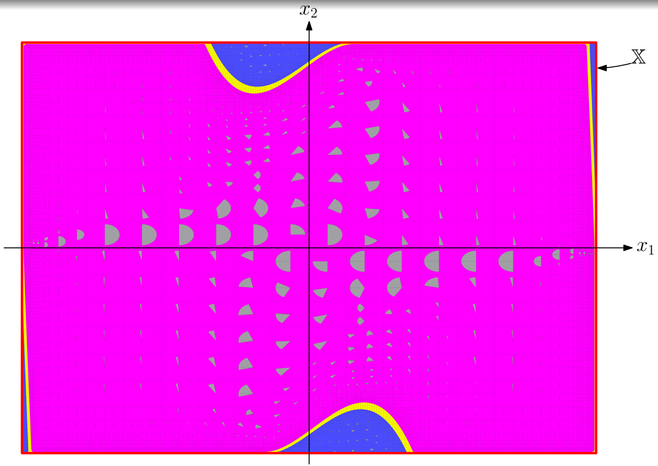
$$Inv^{+}(\mathbf{f},\mathbb{X}) = \{\mathbf{x}_0 \mid \forall t \geq 0, \varphi(t,\mathbf{x}_0) \in \mathbb{X}\}.$$

Mazes allow us to compute an inner and an outer approximation for $Inv^{+}(\mathbf{f},\mathbb{X})$.

As an illustration, consider the Van der Pol system:

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$


6
4
2
0
−2
−4
−6
−6
−4
−2
0
2
4
6



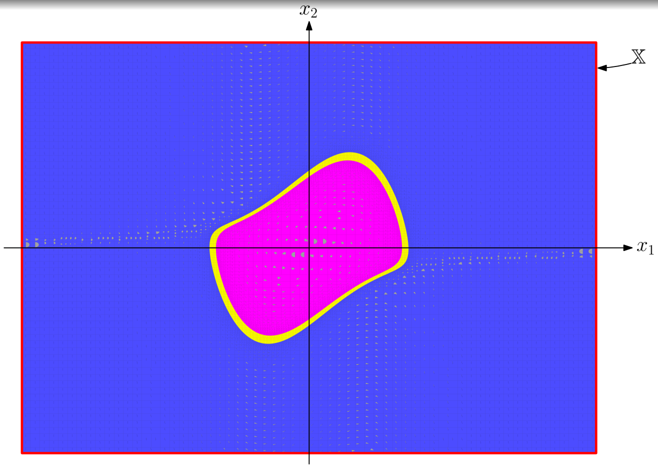
Largest positive invariant set $Inv^+(\mathbf{f}, \mathbb{X})$

Largest negative invariant set.

$$Inv^-(\mathbf{f},\mathbb{X}) = \{\mathbf{x}_0 \mid \forall t \leq 0, \varphi(t,\mathbf{x}_0) \in \mathbb{X}\}.$$

We have

$$Inv^-(\mathbf{f},\mathbb{X}) = Inv^+(-\mathbf{f},\mathbb{X}).$$



Largest negative invariant set $Inv^{-}(\mathbf{f},\mathbb{X})$

Largest invariant set

$$Inv(\mathbf{f},\mathbb{X}) = \{\mathbf{x}_0 \mid \forall t \in \mathbb{R}, \varphi(t,\mathbf{x}_0) \in \mathbb{X}\}.$$

We have

$$Inv(\mathbf{f},\mathbb{X}) \quad = \quad Inv^+(-\mathbf{f},\mathbb{X}) \cap Inv^+(\mathbf{f},\mathbb{X}).$$

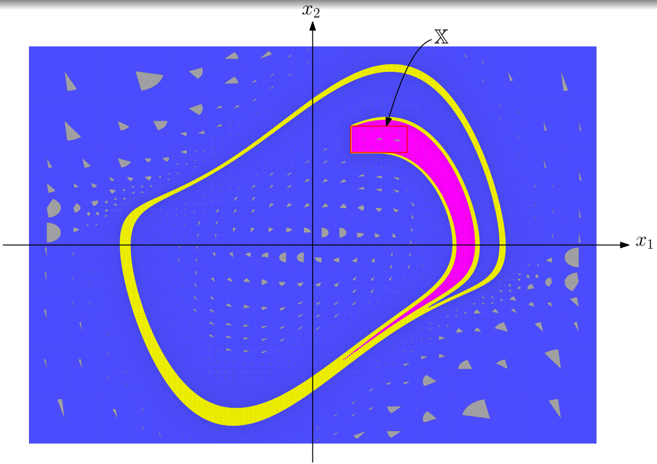
Thus $Inv(\mathbf{f},\mathbb{X})$ can be defined in terms of largest positive invariant sets.

Forward reach set

$$Forw(\mathbf{f},\mathbb{X}) = \{\mathbf{x} \mid \exists t \geq 0, \exists \mathbf{x}_0 \in \mathbb{X}, \varphi(t,\mathbf{x}_0) = \mathbf{x}\}.$$

We have

$$Forw(\mathbf{f},\mathbb{X}) = \overline{Inv^{+}(-\mathbf{f},\overline{\mathbb{X}})}.$$



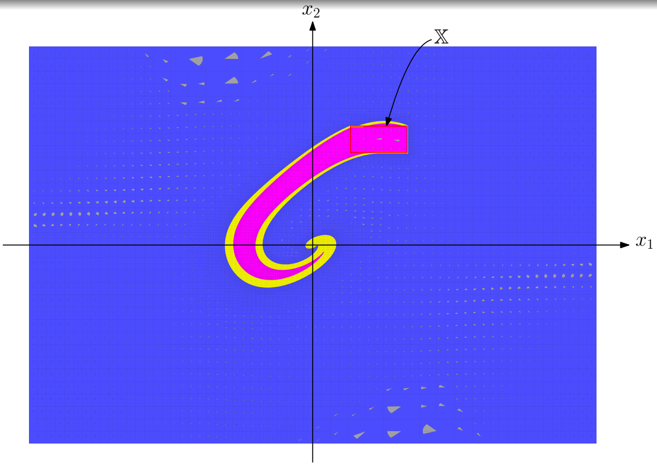
$$Forw(\mathbf{f}, \mathbb{X}) \text{ for } \mathbb{X} = [0.4, 1.0] \times [1.4, 1.8]$$

Backward reach set.

$$Back(\mathbf{f},\mathbb{X})=\{\mathbf{x}_0 \mid \exists t\geq 0, \varphi(t,\mathbf{x}_0)\in\mathbb{X}\}.$$

Since

$$Back(\mathbf{f},\mathbb{X}) \quad = \quad \overline{Inv^{+}(\mathbf{f},\overline{\mathbb{X}})} \ .$$



$Back(\mathbf{f}, \mathbb{X})$ for $\mathbb{X} = [0.4, 1.0] \times [1.4, 1.8]$.

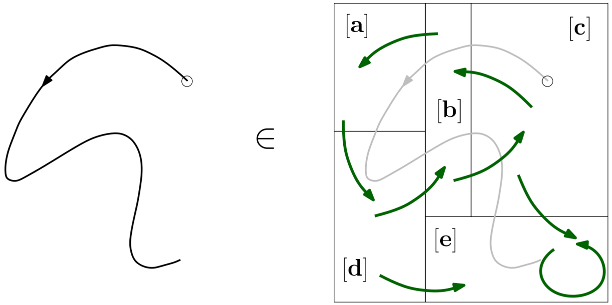
Maze

An ***interval*** is *a domain* which encloses a real number.

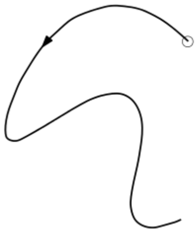
A ***polygon*** is *a domain* which encloses a vector of $\mathbb{R}^n$.

A ***maze*** is *a domain* which encloses a path [2][1].

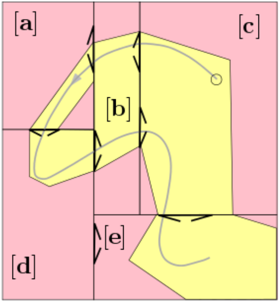
A maze is a set of paths.



Mazes can be made more accurate:



$\in$



Here, a **maze** $\mathscr{L}$ is composed of

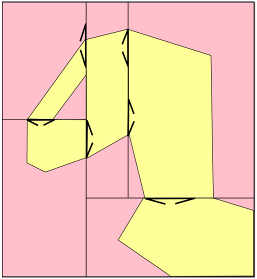
- A paving $\mathscr{P}$
- A polygon for each box of $\mathscr{P}$
- Doors between adjacent boxes

The set of mazes forms a lattice with respect to $\subset$.

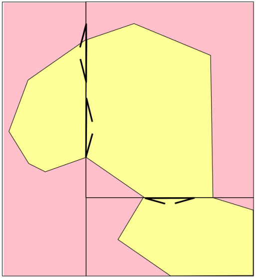
$\mathscr{L}_a \subset \mathscr{L}_b$ means :

- the boxes of $\mathscr{L}_a$ are subboxes of the boxes of $\mathscr{L}_b$.
- The polygons of $\mathscr{L}_a$ are included in those of $\mathscr{L}_b$
- The doors of $\mathscr{L}_a$ are thinner than those of $\mathscr{L}_b$.

The left maze contains less paths than the right maze.



$\subset$

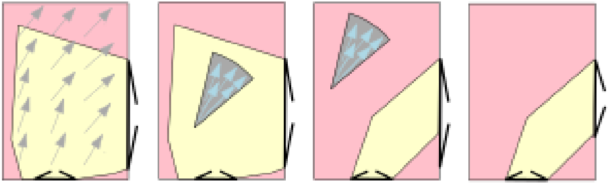


Note that yellow polygons are convex.

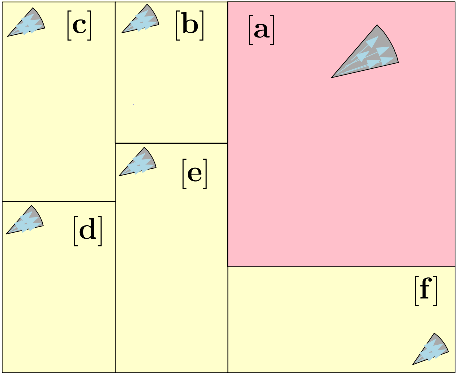
Inner approximation

Target contractor. If a box [x] of $\mathscr{P}$ is outside $\mathbb{X}$ (it is outside $Inv^{+}(\mathbb{X})$) then remove [x] and close all doors entering in [x].

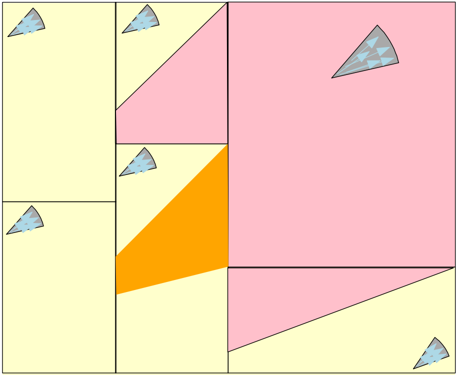
Flow contractor. For each box $[\mathbf{x}]$ of $\mathscr{P}$, we contract the polygon using the constraint $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.



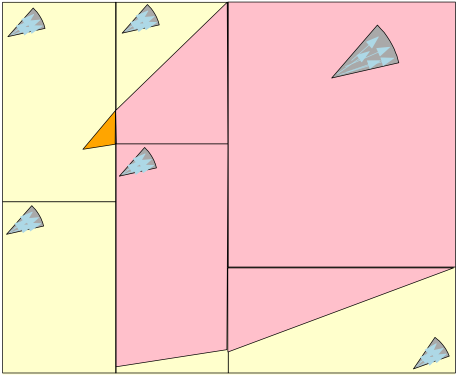
Propagation



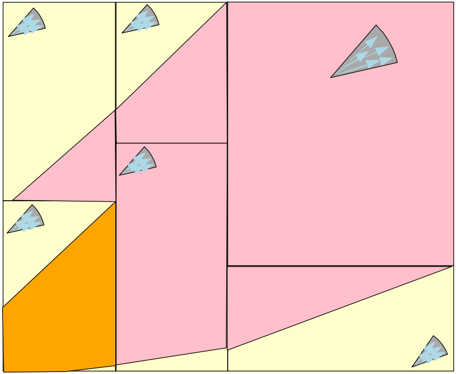
Yellow area: $\mathbb{X}$



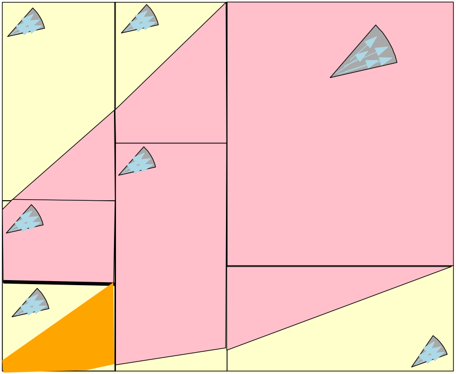
The red parts have been deleted



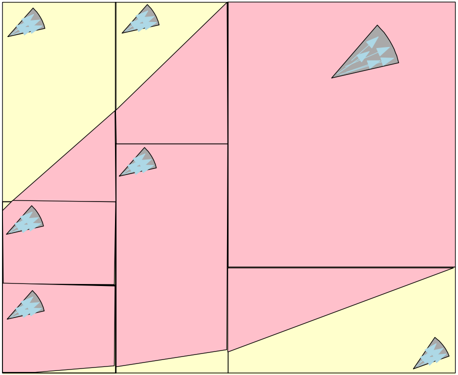
The yellow area is contracted



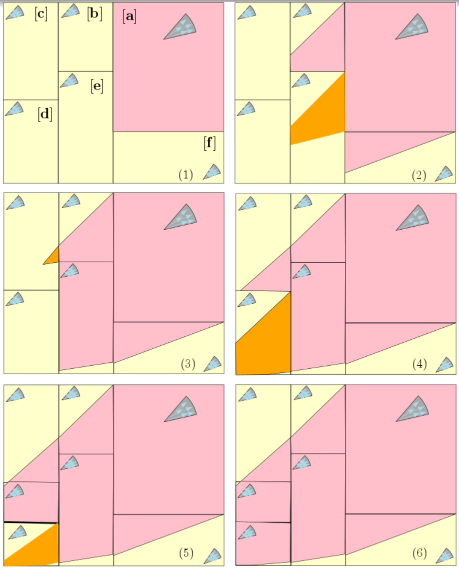
At each step, the yellow area encloses $Inv^+(\mathbb{X})$



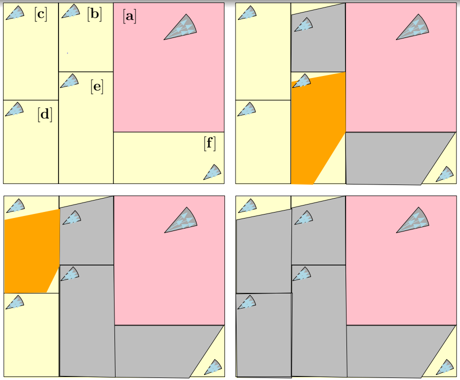
At each step, the red area is outside $Inv^+(\mathbb{X})$



The yellow area encloses $Inv^{+}(\mathbb{X})$


[c]
[b]
[a]
[e]
[d]
[f]
(1)
(2)
(3)
(4)
(5)
(6)

Inflation propagation



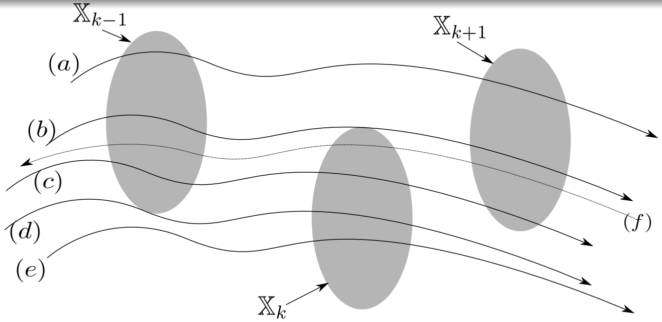
An interpretation can be given only when the fixed point is reached.
The yellow area is an inner approximation of $Inv^{+}(\mathbb{X})$

Eulerian filter

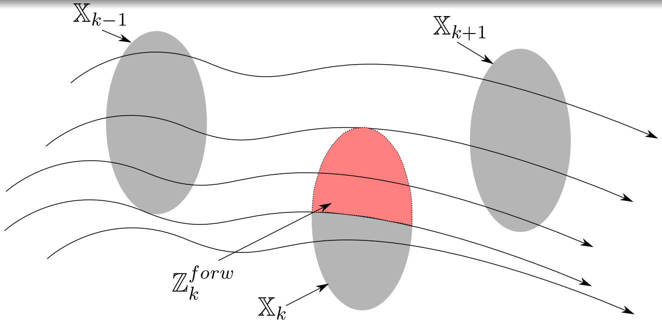
Define ℓ sets $\mathbb{X}_0, \mathbb{X}_1, \dots, \mathbb{X}_\ell$ of the state space. Define $\mathbb{Z}_k^{forw}$ the set of all state vectors $\mathbf{x}(t)$ inside $\mathbb{X}_k$ that have visited $\mathbb{X}_0, \mathbb{X}_1, \dots, \mathbb{X}_{k-1}$. We have

$$\mathbb{Z}_{k+1}^{forw} = Forw\left(\mathbb{Z}_k^{forw}\right) \cap \mathbb{X}_{k+1}$$

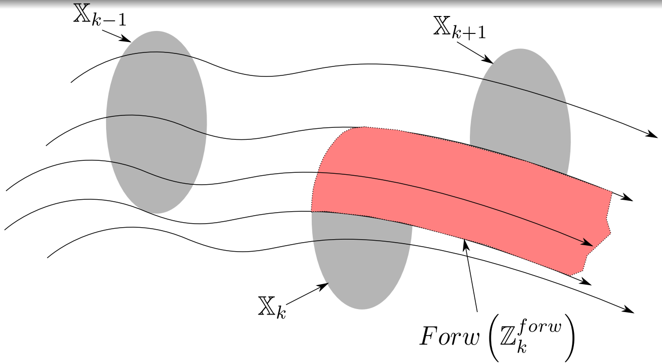
with $\mathbb{Z}_0^{forw} = \mathbb{X}_0$.



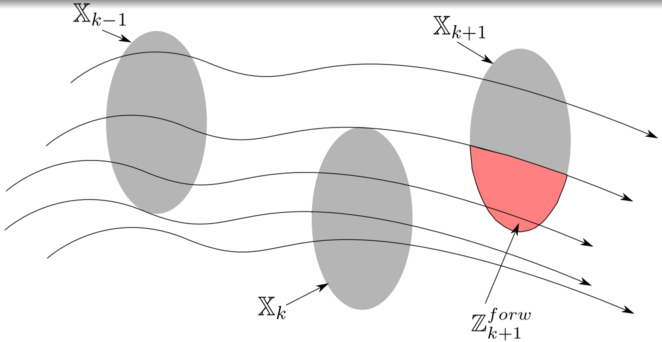
The trajectories (b),(c) are consistent with the sets $\mathbb{X}_{k-1}, \mathbb{X}_k, \mathbb{X}_{k+1}$



Set $\mathbb{Z}_k^{forw}$ of all $\mathbf{x}(t)$ in $\mathbb{X}_k$ that have already visited $\mathbb{X}_0, \mathbb{X}_1, \dots, \mathbb{X}_{k-1}$



$Forw\left(\mathbb{Z}_k^{forw}\right)$ corresponds to all states $\mathbf{x}(t)$ that have visited $\mathbb{X}_0, \mathbb{X}_1, \dots, \mathbb{X}_k$



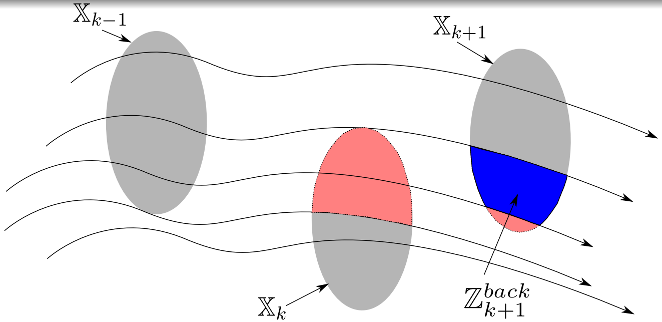
Set $\mathbb{Z}_{k+1}^{forw}$ of all states $\mathbf{x}(t)$ in $\mathbb{X}_{k+1}$ that have already visited $\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_k$

Eulerian smoother

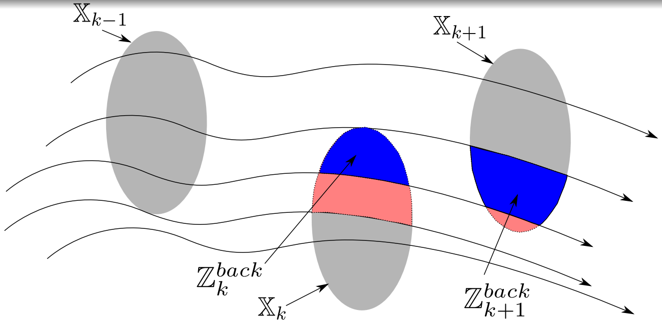
Define the set $\mathbb{Z}_k^{back}$ of all states $\mathbf{x}(t)$ inside $\mathbb{X}_k$ that have visited $\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_{k-1}$ in the past and will visit $\mathbb{X}_{k+1}, \ldots, \mathbb{X}_\ell$ in the future. We have

$$\mathbb{Z}_k^{back} = Back\left(\mathbb{Z}_{k+1}^{back}\right) \cap \mathbb{Z}_k^{forw}$$

with $\mathbb{Z}_\ell^{back} = \mathbb{Z}_\ell^{forw}$. The will be called the *Eulerian smoother.*



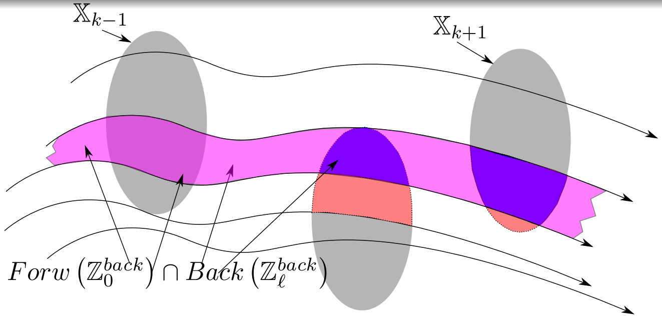
Set $\mathbb{Z}_{k+1}^{back}$ of all states $\mathbf{x}(t)$ inside $\mathbb{Z}_{k+1}^{forw}$ that will visit $\mathbb{X}_{k+2},\dots,\mathbb{X}_{\ell}$



Set $\mathbb{Z}_k^{back}$ of all states $\mathbf{x}(t)$ inside $\mathbb{Z}_k^{forw}$ that will visit $\mathbb{X}_{k+1},\dots,\mathbb{X}_\ell$

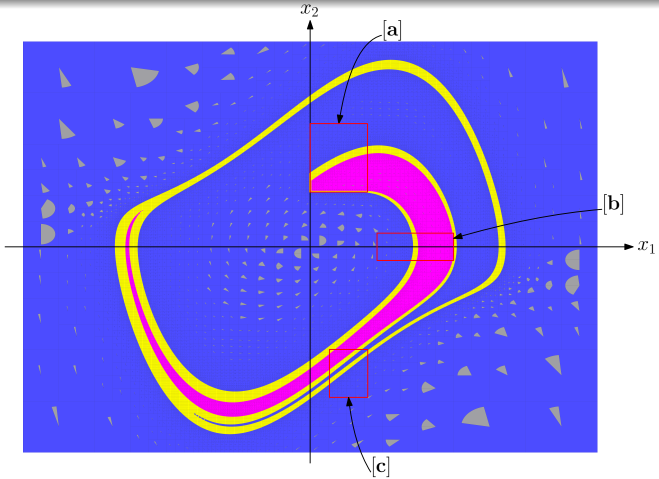
The set of trajectories that started inside $\mathbb{X}_0$ and visited the sets $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_{\ell-1}$sequentially, and that ended in $\mathbb{X}_\ell$ can thus be enclosed by

$$Forw\left(\mathbb{Z}_0^{back}\right) \cap Back\left(\mathbb{Z}_\ell^{back}\right).$$



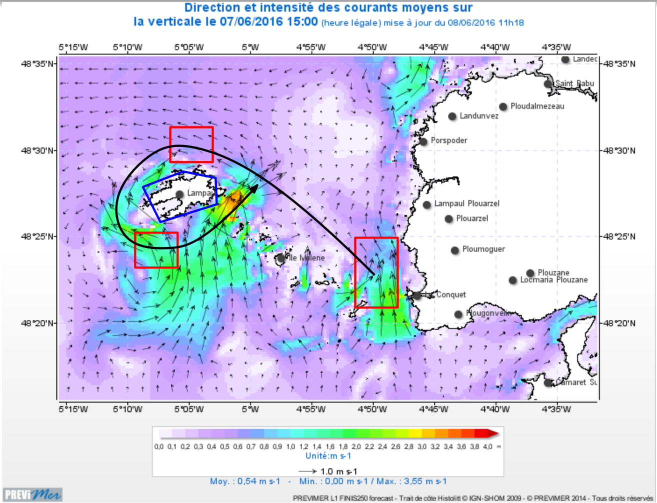
Set $Forw\left(\mathbb{Z}_0^{back}\right) \cap Back\left(\mathbb{Z}_\ell^{back}\right)$ enclosing the trajectory consistent with the past and future visits

Example. Take the Van der Pol system with
$\mathbb{X}_0 = [\mathbf{a}] = [0, 0.6] \times [0.8, 1.8]$, $\mathbb{X}_1 = [\mathbf{b}] = [0.7, 1.5] \times [-0.2, 0.2]$ and
$\mathbb{X}_2 = [\mathbf{c}] = [0.2, 0.6] \times [-2.2, -1.5]$.



Feasible states associated to the Eulerian state estimation problem

An application of Eulerian state estimation moving taking advantage of ocean currents.



Visiting the three red boxes using a buoy that follows the currents is an Eulerian state estimation problem

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Inner approximation of a capture basin of a dynamical system.
In *Abstracts of the 9th Summer Workshop on Interval Methods*. Lyon, France, June 19-22, 2016.

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An interval approach to compute invariant sets.
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Validating a Hamilton-Jacobi Approximation to Hybrid System Reachable Sets.
In Maria Domenica Di Benedetto and Alberto Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control*, number 2034 in Lecture Notes in Computer Science, pages 418–432. Springer Berlin Heidelberg, 2001.