Computing positive invariant sets with intervals

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Constraint network

A Constraint Network is composed of

1) a set of variables $\mathscr{V} = \{x_1 \in \mathbb{X}_1, \ldots, x_n \in \mathbb{X}_n\}$,
2) a set of constraints $\mathscr{C} = \{c_1, \ldots, c_m\}$ and
3) a set of domains $\{[x_1], \ldots, [x_n]\}$.

Example

1) a set of variables $\mathscr{V} = \{x \in \mathbb{R}, y \in \mathbb{R}\}$,
2) a set of constraints $\mathscr{C} = \left\{y = x^2, y = \sqrt{x}\right\}$ and
3) a set of domains $\{[-1,2],[-1,2]\}$.

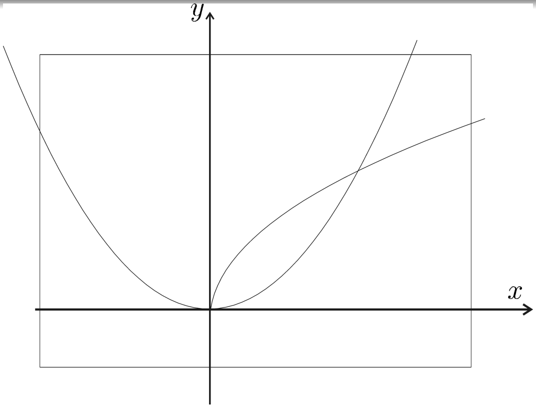
We have a system of two equations.

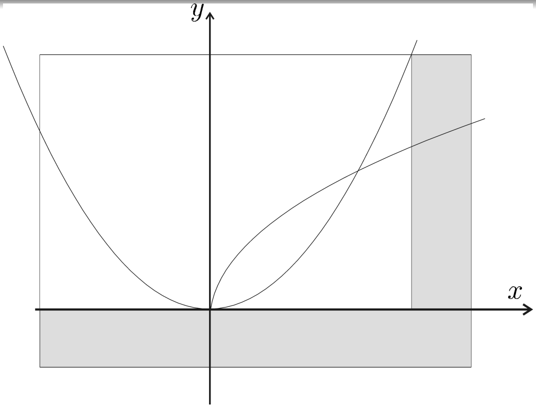
$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

We can build two contractors

$$\mathscr{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

$$\mathscr{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$


y
x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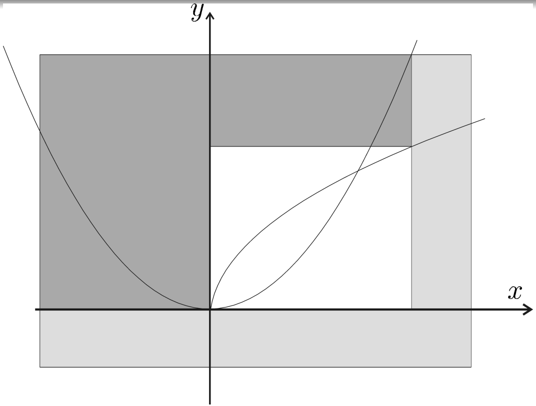
y
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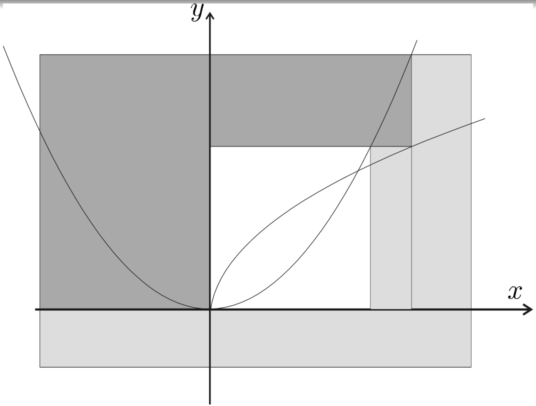
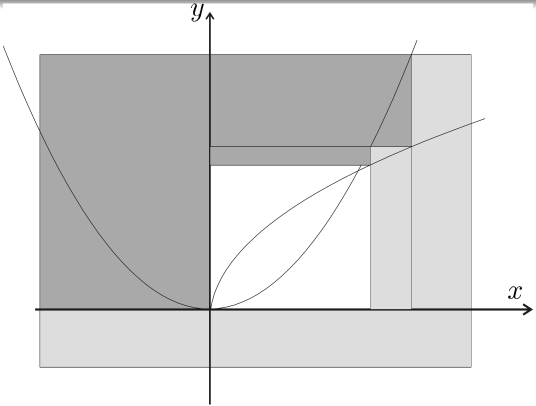

y
x


y
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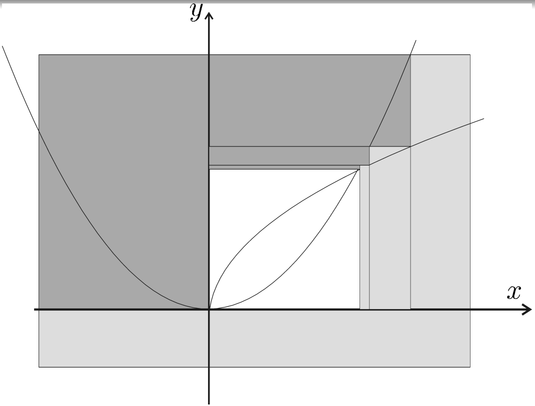
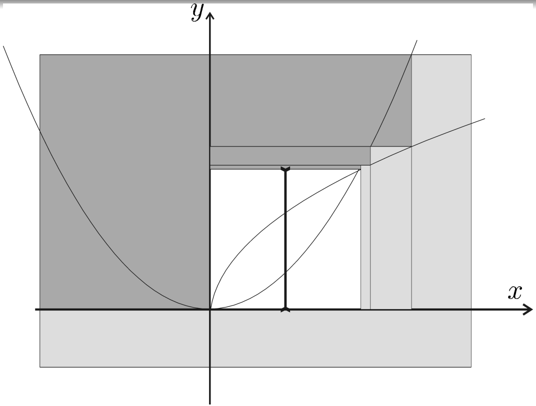

y
x


y
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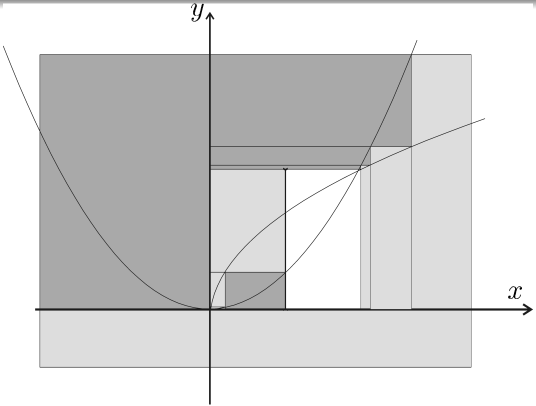
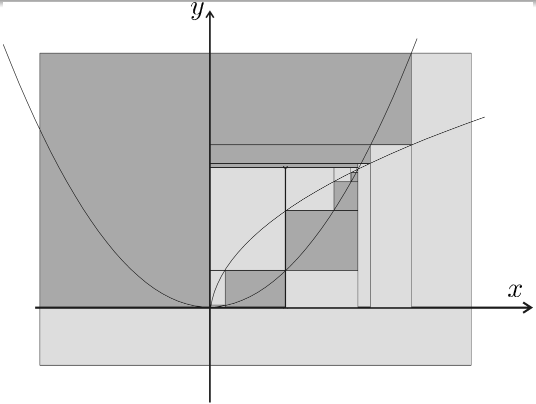

y
x


y
x

Constraint Network

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1) a set of variables $\mathscr{V} = \{x_1 \in \mathbb{X}_1, \dots, x_n \in \mathbb{X}_n\}$,

2) a set of constraints $\mathscr{C} = \{c_1, \dots, c_m\}$ and

3) a set of domains $\{[x_1], \dots, [x_n]\}$.

Classically, the $\mathbb{X}_i$ are lattices, but it is not necessary.

The domains $[x_i]$ should be representable in the machine.
The domains should be a Moore family.

An example with angles

The set of angles $\mathbb{A}$ is not a lattice. Thus, we cannot define intervals of angles.

The family $\mathbb{IA}$ is a Moore family (containing $\mathbb{A}$) if

$$\forall i, [a](i) \in \mathbb{IA} \;\; \Rightarrow \bigcap_i [a](i) \in \mathbb{IA}$$

E. H. Moore

Eliakim Hastings Moore

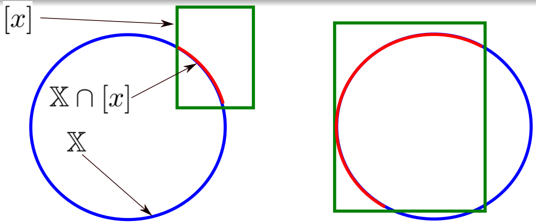
Born	January 26, 1862 Marietta, Ohio, U.S.
Died	December 30, 1932 (aged 70) Chicago, Illinois, U.S.
Nationality	American
Fields	Mathematics
Institutions	University of Chicago 1892–31 Yale University 1887–89

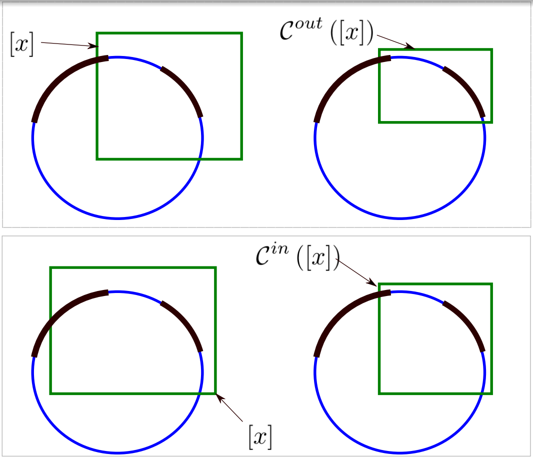
Embedding

Embedding. To have a Moore family, we perform an embedding:

$$\alpha \mapsto \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \in \mathbb{R}^2$$

Now, we introduce a pessimism (*Embedding effect*).

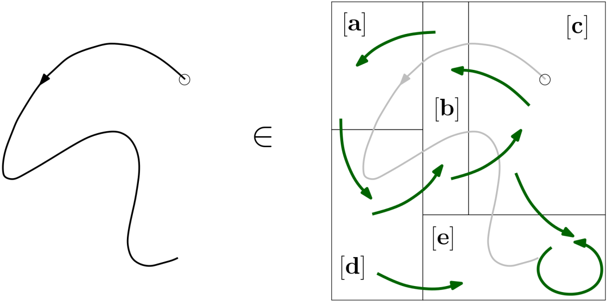

$[x]$
$\mathbb{X} \cap [x]$
$\mathbb{X}$



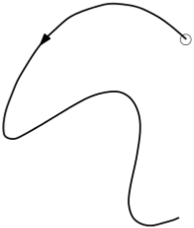
Inner and outer contractions

Mazes

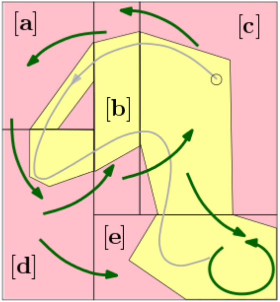
A maze is a set of trajectories.



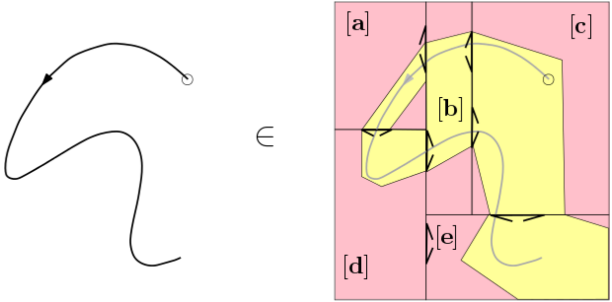
Mazes can be made more accurate by adding polygones.



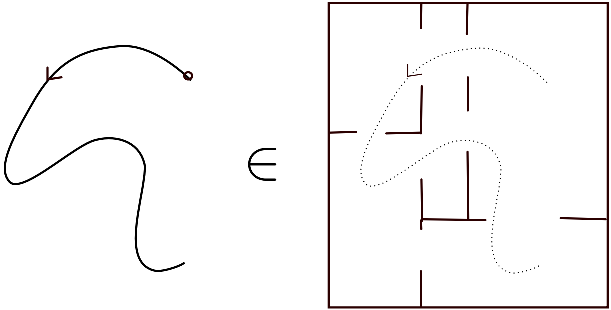
$\in$



Or using doors instead of a graph



Here, we use bi-directional doors



The trajectory $\mathbf{x}(\cdot)$ belongs to the maze $[\mathbf{x}](\cdot)$

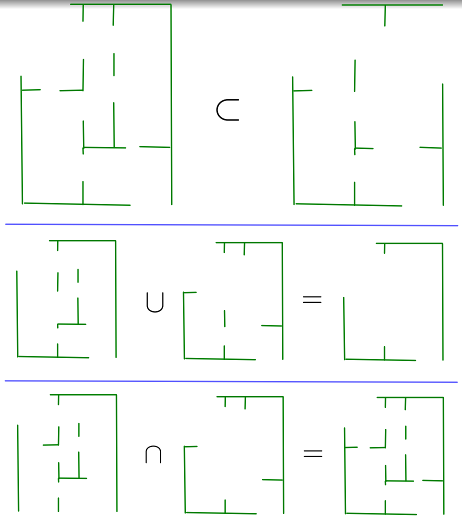
Here, a **maze** $\mathscr{L}$ is composed of [2][1]

- A paving $\mathscr{P}$
- Doors between adjacent boxes

The set of mazes forms a lattice with respect to $\subset$.

$\mathscr{L}_a \subset \mathscr{L}_b$ means :

- the boxes of $\mathscr{L}_a$ are subboxes of the boxes of $\mathscr{L}_b$.
- The doors of $\mathscr{L}_a$ are thinner than those of $\mathscr{L}_b$.



We consider a constraint network composed of

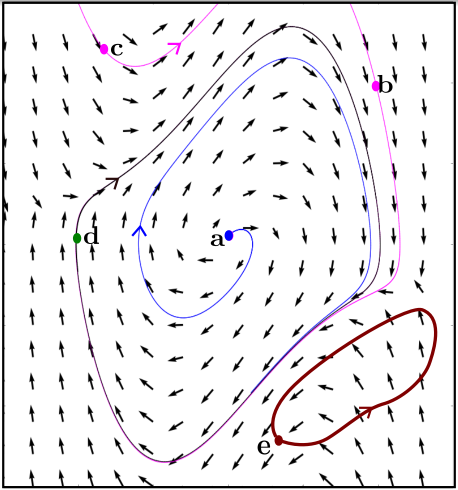
1) One trajectory $\mathscr{V} = \{\mathbf{x}(\cdot) \in \mathbb{X} = \{\mathbf{x}(\cdot) \,|\, \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\}\}$,

2) One constraint $\mathscr{C} = \{x([0,\infty]) \subset \mathbb{A}\}$

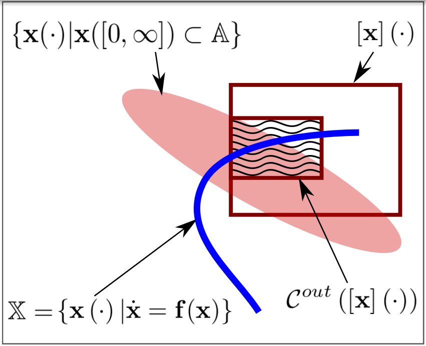
3) One maze $\{[\mathbf{x}]\}$.

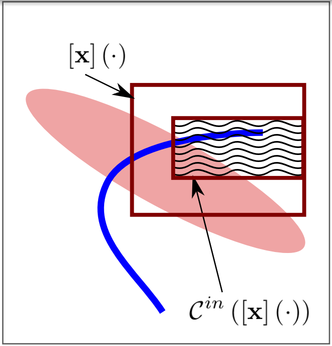
Example: The Van der Pol system

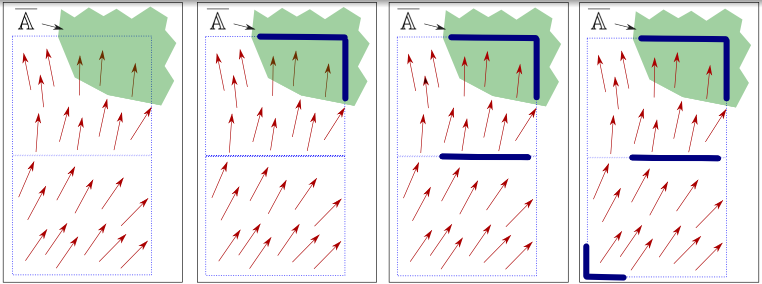
$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$



Trajectories (a),(b),(d) are variables that are solution. (c) is a variable which is not a solution.
The trajectory (e) satisfies the constraint, but is not a variable.


$\{\mathbf{x}(\cdot)|\mathbf{x}([0,\infty]) \subset \mathbb{A}\}$
$[\mathbf{x}](\cdot)$
$\mathbb{X} = \{\mathbf{x}(\cdot)\,|\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\}$
$\mathcal{C}^{out}([\mathbf{x}](\cdot))$


$[\mathbf{x}](\cdot)$
$\mathcal{C}^{in}([\mathbf{x}](\cdot))$



We search for a trajectory which never reach $\mathbb{A}$.

Abstract interpretation

Abstract interpretation

If $\mathbf{x}_{k+1} = \mathbf{h}(\mathbf{x}_k)$, $\mathbf{x}_0 \in \mathbb{X}_0 \subset \mathbb{A}$.
Show that $\mathbf{x}_k$ will never leave $\mathbb{A}$.

Forward method (inflations)

1. Repeat $\mathbb{X}_{k+1} = \mathbf{h}(\mathbb{X}_k) \cup \mathbb{X}_k$, until $\mathbb{X}_{k+1} = \mathbb{X}_k$.
2. Check that $\mathbb{X}_k \subset \mathbb{A}$.

The principle is to add the $\mathbf{x}$ that can be reached from $\mathbb{X}_k$ until no more can be added.

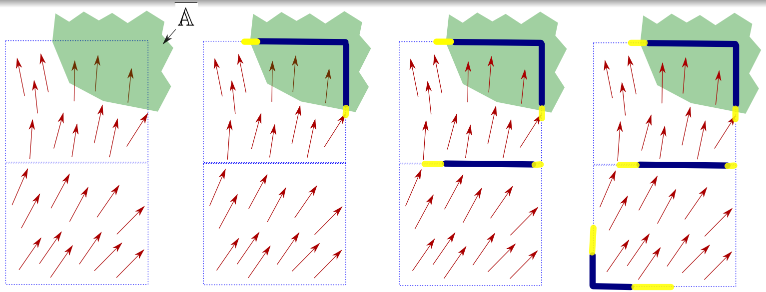
Backward method (contractions)

1. Set $\mathbb{X}_0' = \mathbb{A}$.
2. Repeat $\mathbb{X}_{k+1}' = \mathbf{h}^{-1}(\mathbb{X}_k') \cap \mathbb{X}_k'$, until $\mathbb{X}_{k+1}' = \mathbb{X}_k'$.
3. Check that $\mathbb{X}_0 \subset \mathbb{X}_k'$.

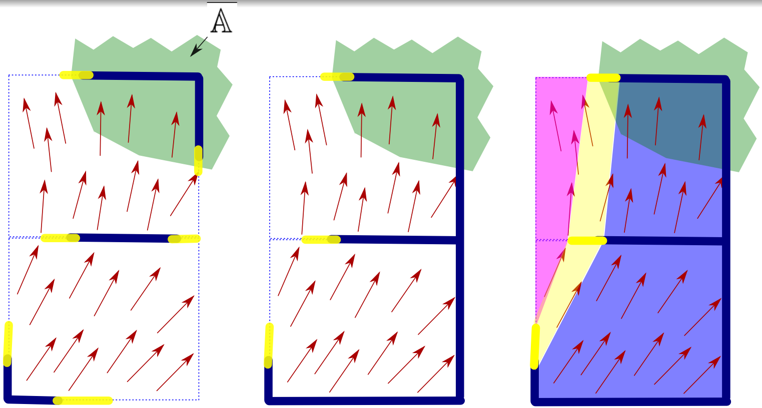
The principle is to remove the $\mathbf{x}$ that leave $\mathbb{X}_k'$ until no more can be removed.

Backward method for mazes

Take a maze $[\mathbf{x}](\cdot)$ and close door in $\overline{\mathbb{A}}$.
Remove from $[\mathbf{x}](\cdot)$ paths that may leave $[\mathbf{x}](\cdot)$ until no more can be removed.


$\mathbb{A}$

Positive invariant set

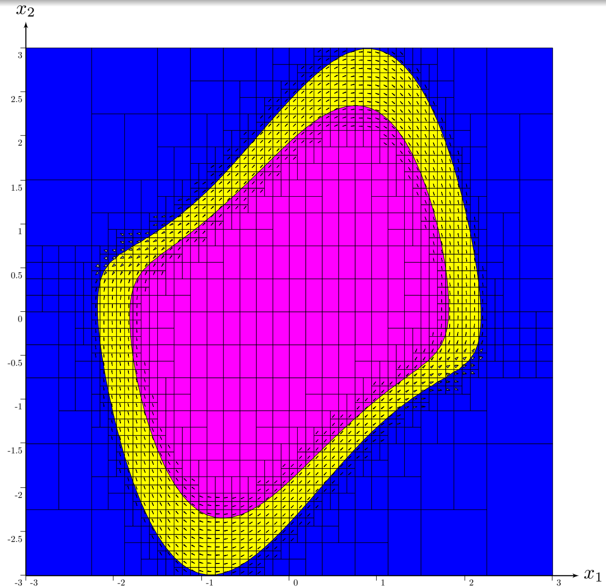

$\mathbb{A}$

Van der Pol system

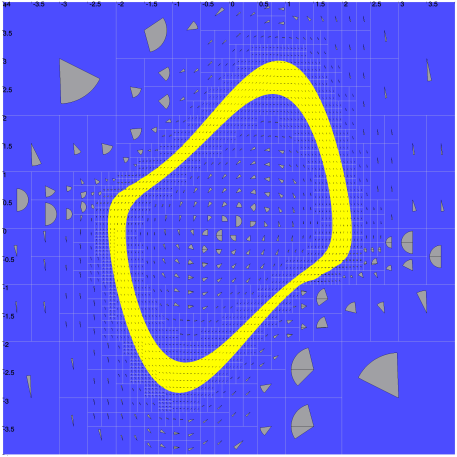
Consider the system

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$

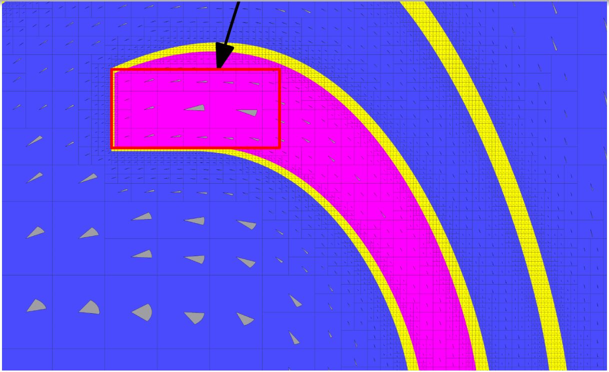
and the box $\mathbb{X}_0 = [-4, 4] \times [-4, 4]$.

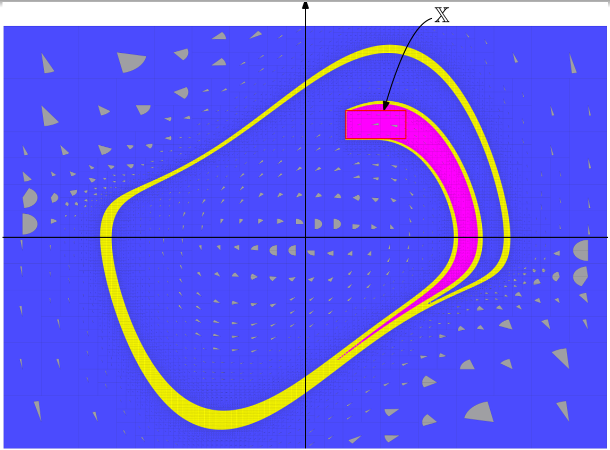

x_2
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2
-1
0
1
2
3
x_1

www.ensta-bretagne.fr/lemezo/pyinvariant/pyinvariant.html


-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5

Guaranteed integration

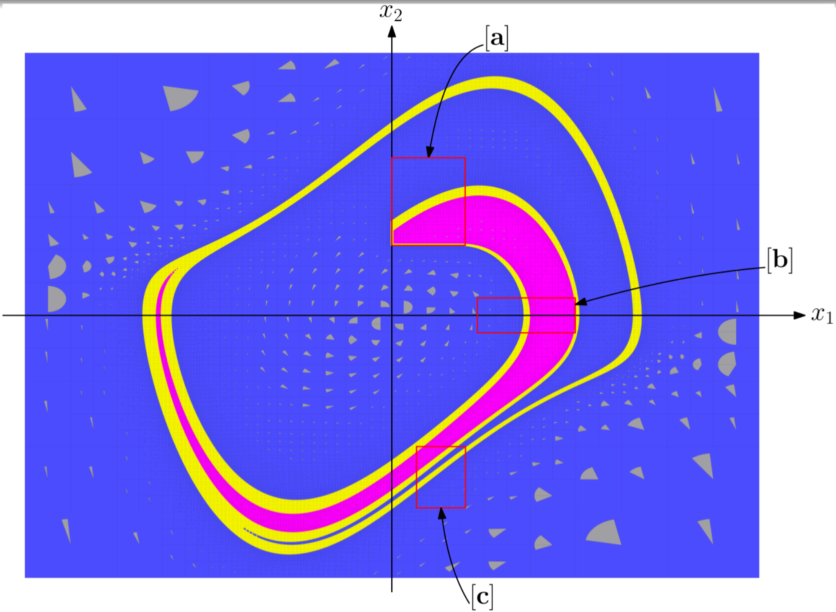


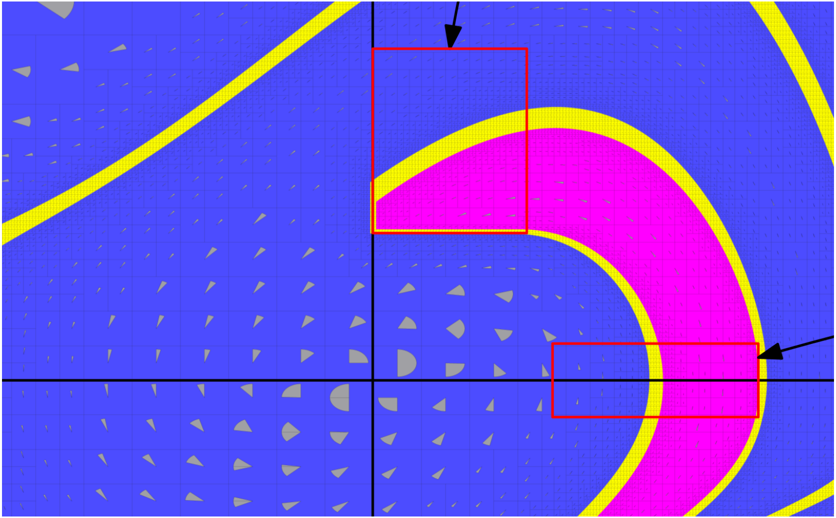

X

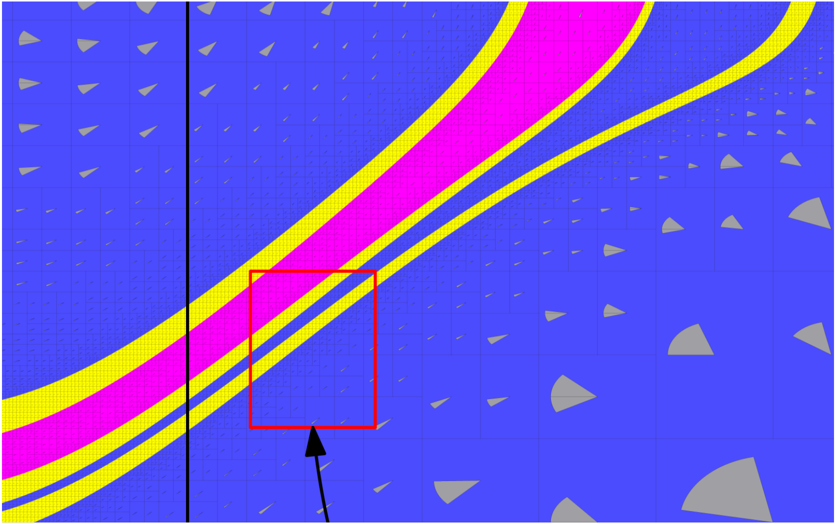
Eulerian smoother

Take the Van der Pol system with

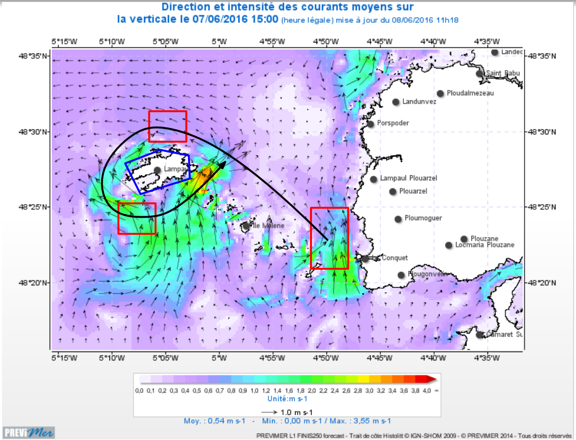
$$\begin{aligned}\mathbb{X}_0 &= [\mathbf{a}] = [0, 0.6] \times [0.8, 1.8] \\ \mathbb{X}_1 &= [\mathbf{b}] = [0.7, 1.5] \times [-0.2, 0.2] \\ \mathbb{X}_2 &= [\mathbf{c}] = [0.2, 0.6] \times [-2.2, -1.5]\end{aligned}$$


x_2
[a]
[b]
x_1
[c]





An application of Eulerian state estimation moving taking advantage of ocean currents.



Visiting the three red boxes using a buoy that follows the currents is an Eulerian state estimation problem

T. Le Mézo L. Jaulin and B. Zerr.
Bracketing the solutions of an ordinary differential equation with uncertain initial conditions.
Applied Mathematics and Computation, 2017.

T. Le Mézo, L. Jaulin, and B. Zerr.
An interval approach to compute invariant sets.
IEEE Transaction on Automatic Control, 2017.