Underwater navigation with stable cycles

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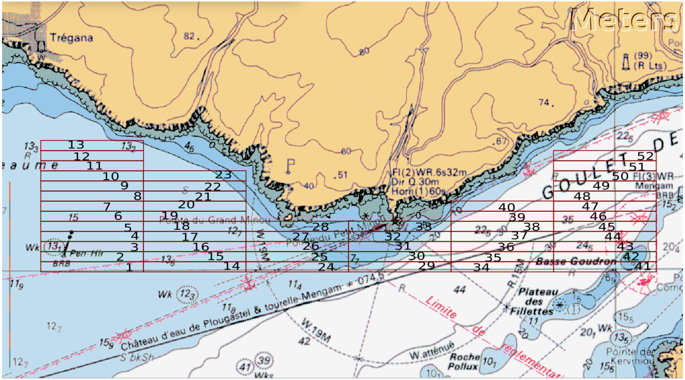
Présentation à Palaiseau, MRIS
Vendredi 13 mars 2020

Underwater robots to build maps


Meters
Trégana
82
80
74
67
60
51
40
Fl(2)WR.6s32m
Dir Q.30m
Horn(1)60s
GOULET DE
Fl(3)WR
Mengam
BRB
Pointe du Grand Minou
Pen-Hir
Basse Goudron
Plateau
des
Fillettes
Château d'eau de Plougastel & tourelle-Mengam
074,5
W.19M
Limite de réglementation
W.atténué
Roche
Pollux
Pointe de
Kerviniou
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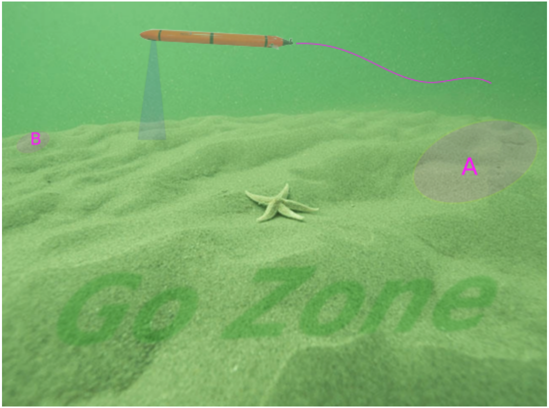
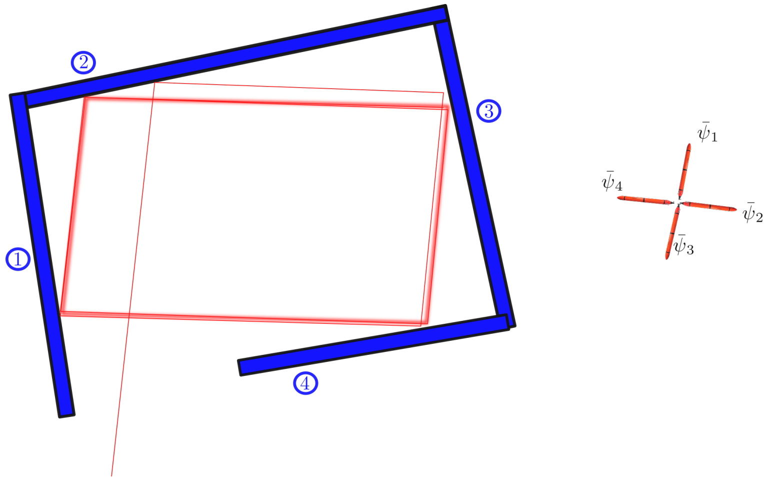
Stable cycles

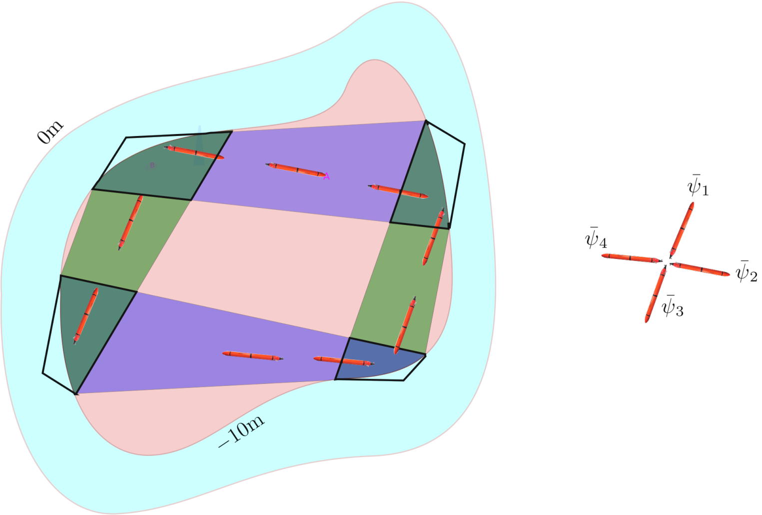
With Julien Damers, Simon Rohou, etc

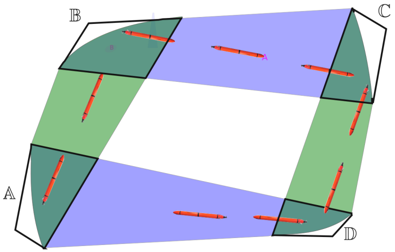
[Submeeting 2018]

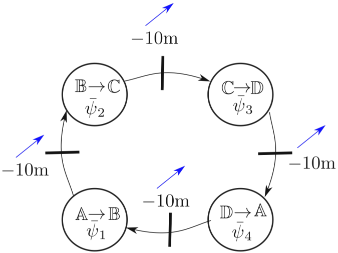
Cycles

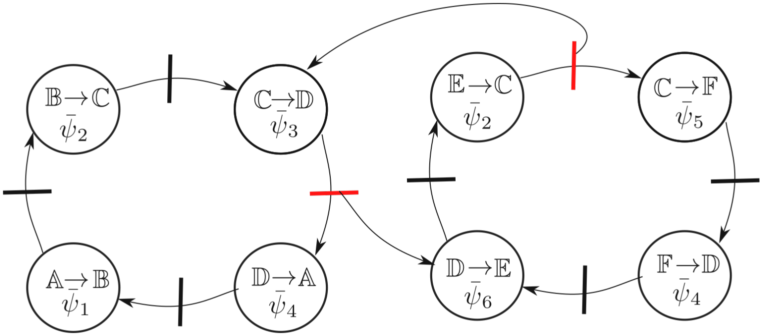

B
A
Go Zone


①
②
③
④
$\bar{\psi}_1$
$\bar{\psi}_2$
$\bar{\psi}_3$
$\bar{\psi}_4$

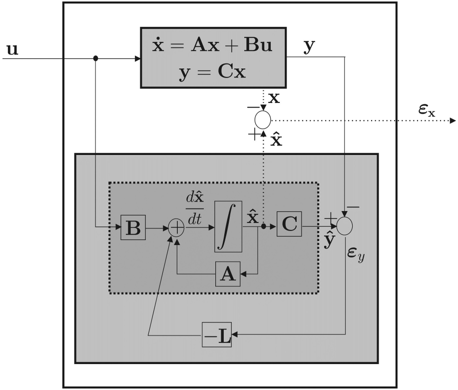

0m
−10m
$\bar{\psi}_1$
$\bar{\psi}_2$
$\bar{\psi}_3$
$\bar{\psi}_4$


𝔹
ℂ
𝔸
𝔻


−10m
𝔹→ℂ $\bar{\psi}_2$
ℂ→𝔻 $\bar{\psi}_3$
−10m
−10m
−10m
𝔸→𝔹 $\bar{\psi}_1$
𝔻→𝔸 $\bar{\psi}_4$


$\mathbb{B}\to\mathbb{C}$ $\bar{\psi}_2$
$\mathbb{C}\to\mathbb{D}$ $\bar{\psi}_3$
$\mathbb{A}\to\mathbb{B}$ $\bar{\psi}_1$
$\mathbb{D}\to\mathbb{A}$ $\bar{\psi}_4$
$\mathbb{E}\to\mathbb{C}$ $\bar{\psi}_2$
$\mathbb{C}\to\mathbb{F}$ $\bar{\psi}_5$
$\mathbb{D}\to\mathbb{E}$ $\bar{\psi}_6$
$\mathbb{F}\to\mathbb{D}$ $\bar{\psi}_4$

Stability is an old story



Luenberger observer : $\dot{\varepsilon}_x = (\mathbf{A} - \mathbf{LC})\,\varepsilon_x$

Poincaré map

System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

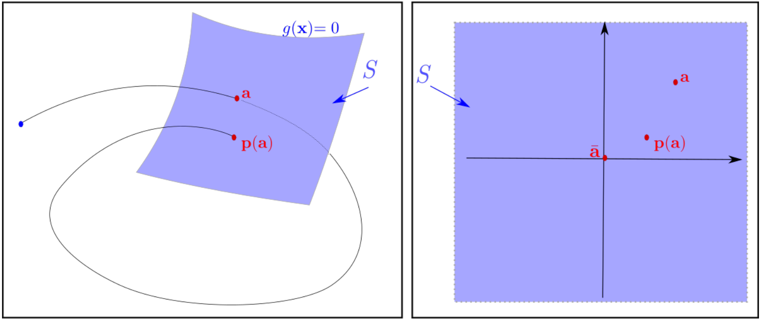
Poincaré section: $g(\mathbf{x}) = 0$

Transversality: $g(\mathbf{x}) = 0 \Rightarrow \left(\frac{\partial g}{\partial \mathbf{x}} \cdot \mathbf{f}\right)(\mathbf{x}) \neq 0$

Define $\mathscr{G} = g^{-1}(0)$.

$$\mathbf{p}: \begin{array}{ccc} \mathscr{G} & \to & \mathscr{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where $\mathbf{p}(\mathbf{a})$ is the point of $\mathscr{G}$ such that the trajectory initialized at $\mathbf{a}$ intersects $\mathscr{G}$ for the first time.

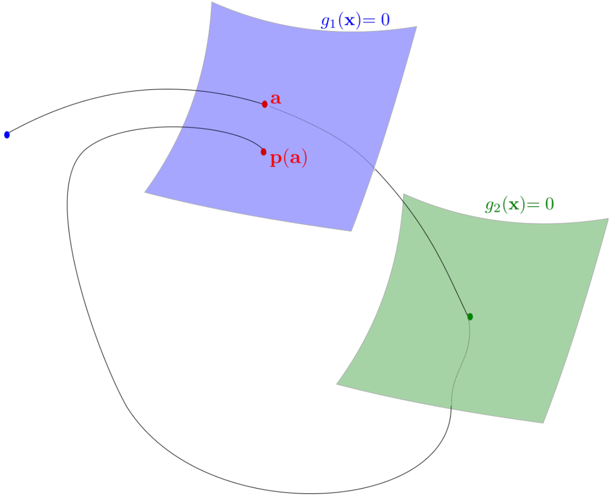

$g(\mathbf{x})=0$
S
$\mathbf{a}$
$\mathbf{p(a)}$
S
$\mathbf{a}$
$\mathbf{p(a)}$
$\bar{\mathbf{a}}$

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$

With hybrid systems

Systems: $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}), i \in \{1, \dots, m\}$
Section i: $g_i(\mathbf{x}) = 0$
Transversality: $g(\mathbf{x}) = 0 \Rightarrow \left(\frac{\partial g_i}{\partial \mathbf{x}} \cdot \mathbf{f}_i\right)(\mathbf{x}) \neq 0$
Automaton: $g_i(\mathbf{x}) = 0 \Rightarrow i := \mathrm{mod}(i+1, m)$


$g_1(\mathbf{x})= 0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$g_2(\mathbf{x})= 0$

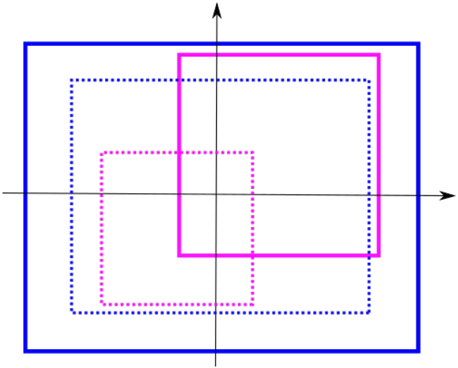
Stability contractor

With Auguste Bourgois

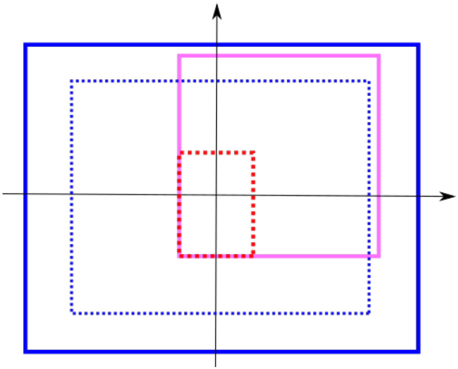
A *stability contractor* $\mathbf{\Psi} : \mathbb{IR}^n \to \mathbb{IR}^n$ of rate $\alpha < 1$ satisfies

$$
\begin{array}{c}
[\mathbf{a}] \subset [\mathbf{b}] \Rightarrow \mathbf{\Psi}([\mathbf{a}]) \subset \mathbf{\Psi}([\mathbf{b}]) \\
\mathbf{\Psi}([\mathbf{a}]) \subset [\mathbf{a}] \\
\mathbf{\Psi}(\mathbf{0}) = \mathbf{0} \\
\mathbf{\Psi}([\mathbf{a}]) \subset \alpha \cdot [\mathbf{a}] \Rightarrow \mathbf{\Psi}^2([\mathbf{a}]) \subset \alpha^2 \cdot [\mathbf{a}]
\end{array}
$$

Question. Is $\Psi : \mathbb{IR}^n \to \mathbb{IR}^n$, $[\mathbf{x}] \mapsto -0.9 \cdot [\mathbf{x}]$ a stability contractor of rate 0.9 ?



Question. Is $\Psi : \mathbb{IR}^n \rightarrow \mathbb{IR}^n, [\mathbf{x}] \mapsto [\mathbf{x}] \cap -0.9 \cdot [\mathbf{x}]$ a stability contractor of rate 0.9 ?



If $\mathbf{\Psi}$ is a stability contractor of rate $\alpha < 1$ then we have

$$\mathbf{\Psi}([\mathbf{x}])) \subset \alpha \cdot [\mathbf{x}] \Rightarrow \lim_{k \to \infty} \mathbf{\Psi}^k([\mathbf{x}]) \to \mathbf{0}.$$

Centered form

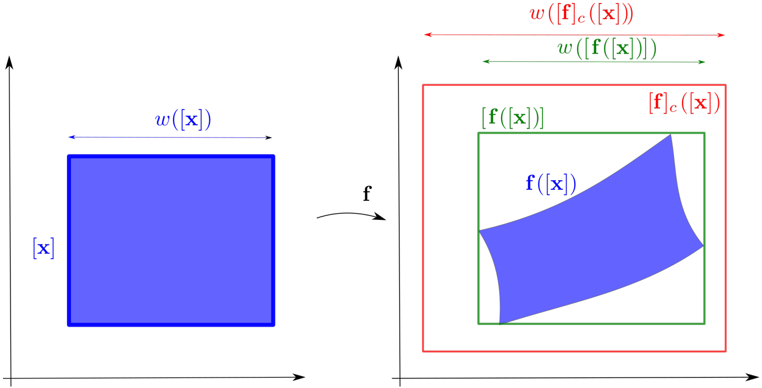
Consider a function $\mathbf{f}:\mathbb{R}^n \rightarrow \mathbb{R}^n$, with $\mathbf{J}(\mathbf{x}) = \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x})$, a box $[\mathbf{x}]$ and one point $\bar{\mathbf{x}}$ in $[\mathbf{x}] \ni \mathbf{0}$. We assume that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

$$[\mathbf{f}_c]([\mathbf{x}]) = ([\mathbf{J}]([\mathbf{x}])) \cdot [\mathbf{x}]$$

We have

$$(i) \qquad \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}_c]([\mathbf{x}])$$

$$(ii) \qquad \lim_{w([\mathbf{x}])\to 0} \frac{w([\mathbf{f}_c]([\mathbf{x}])) - w([\mathbf{f}([\mathbf{x}])])}{w([\mathbf{x}])} = 0$$


$w([\mathbf{x}])$
$[\mathbf{x}]$
$\mathbf{f}$
$w([\mathbf{f}]_c([\mathbf{x}]))$
$w([\mathbf{f}([\mathbf{x}])])$
$[\mathbf{f}]_c([\mathbf{x}])$
$[\mathbf{f}([\mathbf{x}])]$
$\mathbf{f}([\mathbf{x}])$

Proving stability

With Auguste Bourgois

Consider the discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Theorem. If $\exists[\mathbf{x}_0] \ni \mathbf{0}$, with $[\mathbf{f}_c]([\mathbf{x}_0]) \subset \alpha \cdot [\mathbf{x}_0]$, $\alpha < 1$ then $[\mathbf{f}_c]([\mathbf{x}])$ is a stability contractor inside $[\mathbf{x}_0]$.

Theorem. Consider a function $\mathbf{f}$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{J}(\mathbf{x}) = \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x})$. The centered form $[\mathbf{f}_c^k]$ associated to $\mathbf{f}^k = \mathbf{f} \circ \mathbf{f} \circ \cdots \circ \mathbf{f}$ is:

$$\begin{array}{rcl}
[\mathbf{z}](0) & = & [\mathbf{x}] \\
[\mathbf{A}](0) & = & \mathrm{Id} \\
[\mathbf{z}](k) & = & [\mathbf{f}]\,([\mathbf{z}](k-1)) \\
[\mathbf{A}](k) & = & [\mathbf{J}]([\mathbf{z}](k-1)) \cdot [\mathbf{A}](k-1) \\
[\mathbf{f}_c^k]([\mathbf{x}]) & = & [\mathbf{A}](k) \cdot [\mathbf{x}]
\end{array}$$

For any given k, we have

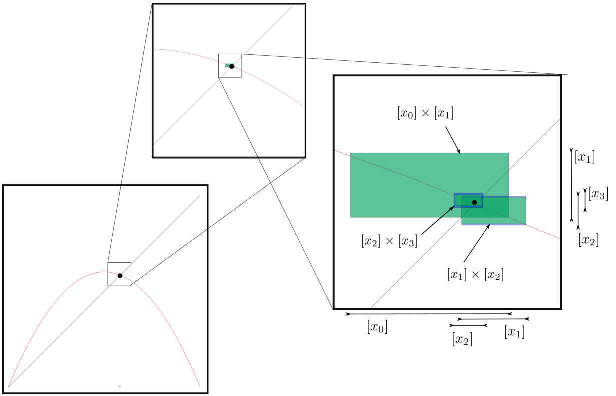
$$\lim_{w([\mathbf{x}])\to 0} \frac{w([\mathbf{f}_c^k]([\mathbf{x}])) - w([\mathbf{f}^k([\mathbf{x}])])}{w([\mathbf{x}])} = 0$$

Theorem. Define $\mathscr{B}_\eta$, the set of all cubes $[\mathbf{x}]$ centered at $\mathbf{0}$, $w([\mathbf{x}]) \leq \eta$. If the system is exponentially stable around $\mathbf{0}$, then

$$\exists \eta > 0, \forall [\mathbf{x}] \in \mathscr{B}_\eta, \exists k > 0, \exists \alpha < 1, [\mathbf{f}_c^k]([\mathbf{x}]) \subset \alpha \cdot [\mathbf{x}]$$

Example 1: logistic map

$$x_{k+1} = \rho \cdot x_k \cdot (1 - x_k), \quad \rho = 2.4$$

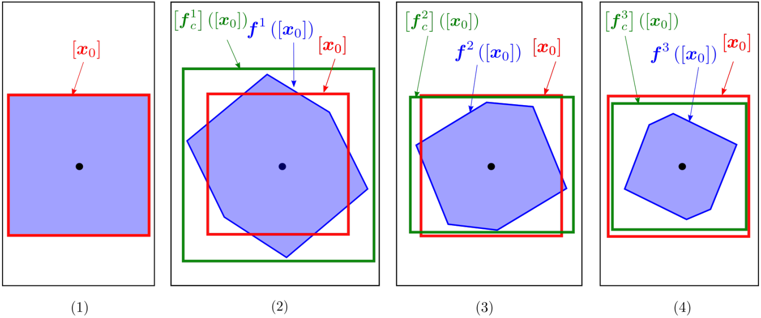

$[x_0] \times [x_1]$
$[x_2] \times [x_3]$
$[x_1] \times [x_2]$
$[x_1]$
$[x_3]$
$[x_2]$
$[x_0]$
$[x_2]$
$[x_1]$

Example 2. Three dimensional map

$$\begin{array}{rcl}\mathbf{x}_{k+1} & = & 0.8\cdot\mathbf{R}\left(\frac{\pi}{6}+x_1,\,\frac{\pi}{4}+x_2,\,\frac{\pi}{3}+x_3\right)\cdot\mathbf{x}_k\\ & = & \mathbf{f}\left(\mathbf{x}_k\right)\end{array}$$

where

$$\mathrm{R}\left(\varphi,\theta,\psi\right)=\left(\begin{array}{ccc}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{array}\right)\cdot\left(\begin{array}{ccc}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{array}\right)\cdot\left(\begin{array}{ccc}1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{array}\right)$$


$[\boldsymbol{x}_0]$
$[\boldsymbol{f}_c^1]([\boldsymbol{x}_0])$
$\boldsymbol{f}^1([\boldsymbol{x}_0])$
$[\boldsymbol{x}_0]$
$[\boldsymbol{f}_c^2]([\boldsymbol{x}_0])$
$\boldsymbol{f}^2([\boldsymbol{x}_0])$
$[\boldsymbol{x}_0]$
$[\boldsymbol{f}_c^3]([\boldsymbol{x}_0])$
$\boldsymbol{f}^3([\boldsymbol{x}_0])$
$[\boldsymbol{x}_0]$
(1)
(2)
(3)
(4)

Example.

Border: $x_2 = h(x_1) = 20\,(1 - \exp\,(-0.25 x_1))$

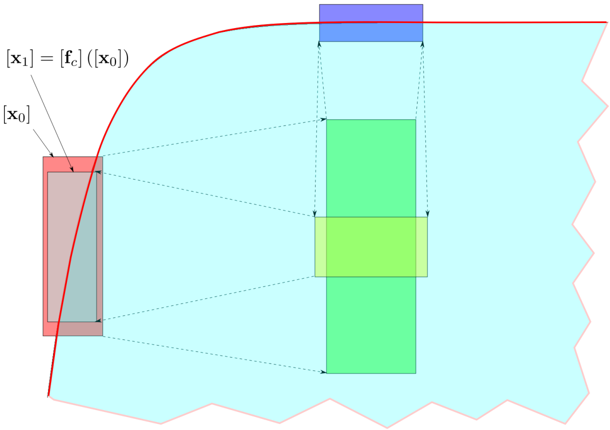
Transition functions :

$$\mathbf{f}_1(\mathbf{x}) = \begin{pmatrix} x_1 + 25 \\ x_2 \end{pmatrix} \qquad \mathbf{f}_2(\mathbf{x}) = \begin{pmatrix} x_1 \\ h(x_1) \end{pmatrix}$$

$$\mathbf{f}_3(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 - 7.5 \end{pmatrix} \qquad \mathbf{f}_4(\mathbf{x}) = \begin{pmatrix} h^{-1}(x_2) \\ x_2 \end{pmatrix}$$

Poincaré recurrence:

$$\mathbf{p}(\mathbf{x}) = \mathbf{f}_4 \circ \mathbf{f}_3 \circ \mathbf{f}_2 \circ \mathbf{f}_1(\mathbf{x})$$

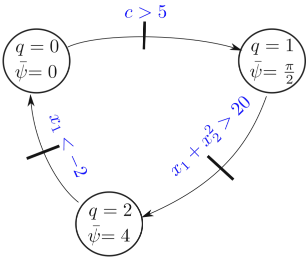

$[\mathbf{x}_1] = [\mathbf{f}_c]([\mathbf{x}_0])$
$[\mathbf{x}_0]$

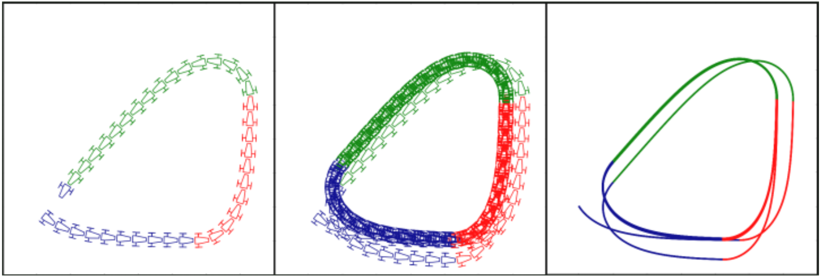
Poincaré scan

Consider the robot

$$\begin{cases} \dot{x}_1 = \cos x_3 \\ \dot{x}_2 = \sin x_3 \\ \dot{x}_3 = u \end{cases}$$

with the heading control $u = \sin(\bar{\psi} - x_3)$.





Consider the section

$$\mathscr{S} : x_1 + 2 = 0$$

Since

$$\left(\frac{dg}{d\mathbf{x}}(\mathbf{x})\right) \cdot (\mathbf{f}(\mathbf{x})) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ \sin(\bar{\psi} - x_3) \end{pmatrix} = \cos x_3$$

the field is transverse except if $\cos x_3 = 0$.

After 10 laps, we get approximately $\bar{\mathbf{a}} = (-2, 2.3, -2.29)$.

To conclude about the stability, we compute the Jacobian matrix

$$\mathbf{J}(\bar{\mathbf{a}}) = \frac{d\mathbf{p}}{d\mathbf{a}}(\bar{\mathbf{a}}) =$$

$$\begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_3} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \frac{\partial p_2}{\partial x_3} \\ \frac{\partial p_3}{\partial x_1} & \frac{\partial p_3}{\partial x_2} & \frac{\partial p_3}{\partial x_3} \end{pmatrix}$$

We get

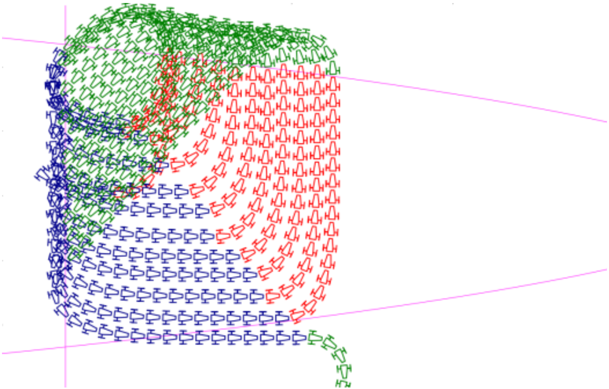
$$\begin{pmatrix} \frac{\partial p_2}{\partial x_2}(\bar{\mathbf{a}}) & \frac{\partial p_2}{\partial x_3}(\bar{\mathbf{a}}) \\ \frac{\partial p_3}{\partial x_2}(\bar{\mathbf{a}}) & \frac{\partial p_3}{\partial x_3}(\bar{\mathbf{a}}) \end{pmatrix} \simeq \begin{pmatrix} 0.045 & 2.66 \\ 0.0 & 0.02 \end{pmatrix}.$$

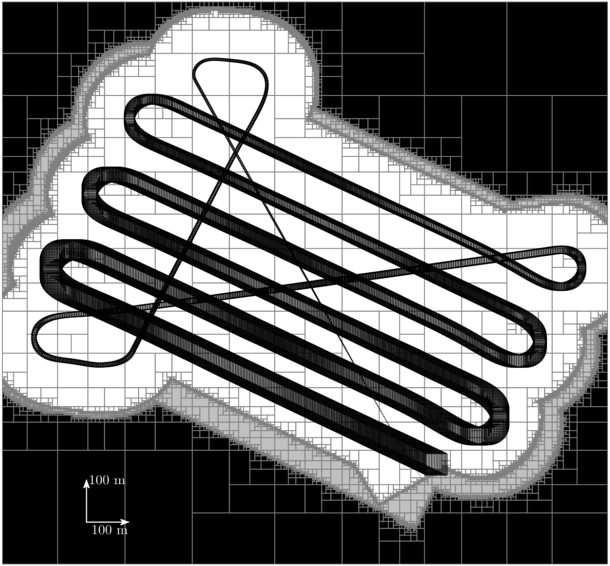
The eigen values are approximately

$$\{0.06, 0.009\}$$

Both are in the unit circle. We conclude that the limit cycle is stable (with no guarantee here).

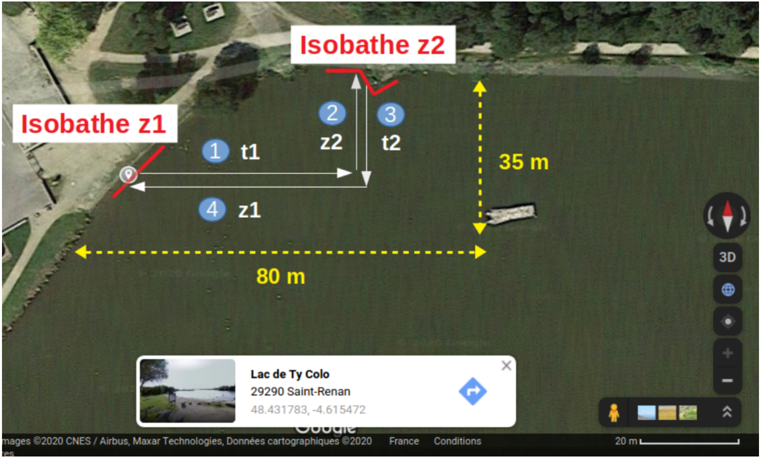
For scanning, at each lap, we replace the transition $c > c_{max}$ by $c > c_{max} + 1$.

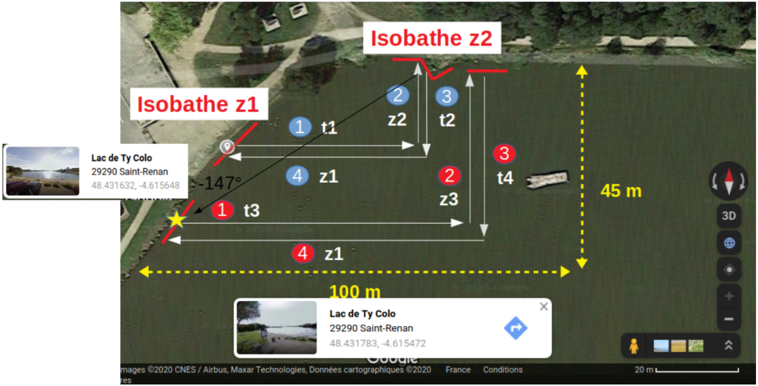


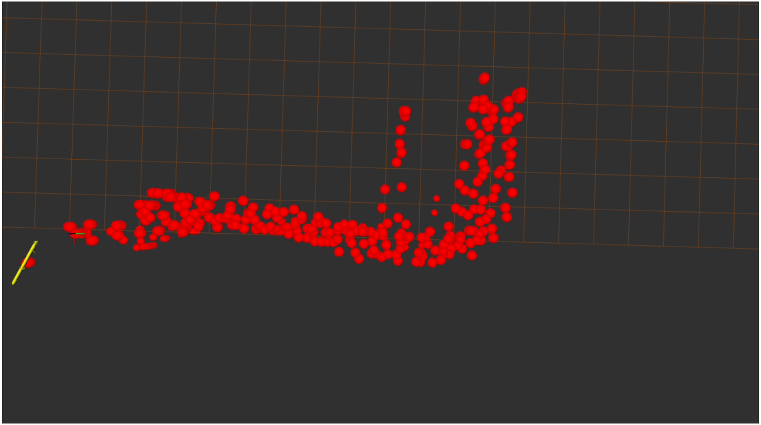

100 m
100 m

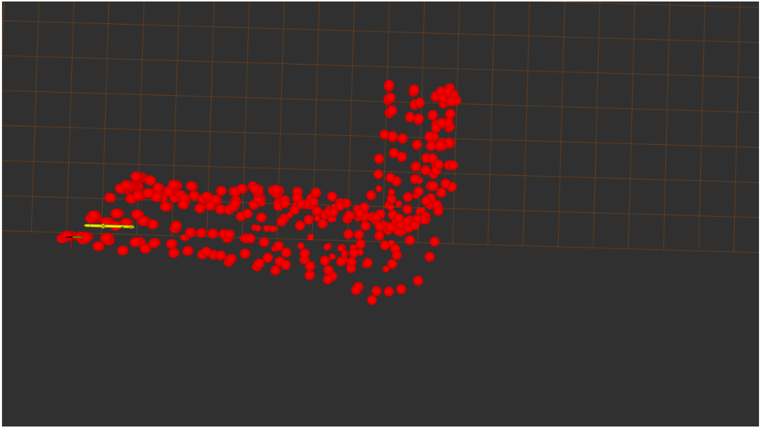
How to guarantee the robot will no get lost ?
How the robot can know it has been kidnapped ?
How to compute the prior set that will be explored ?
How to find the cycles ?

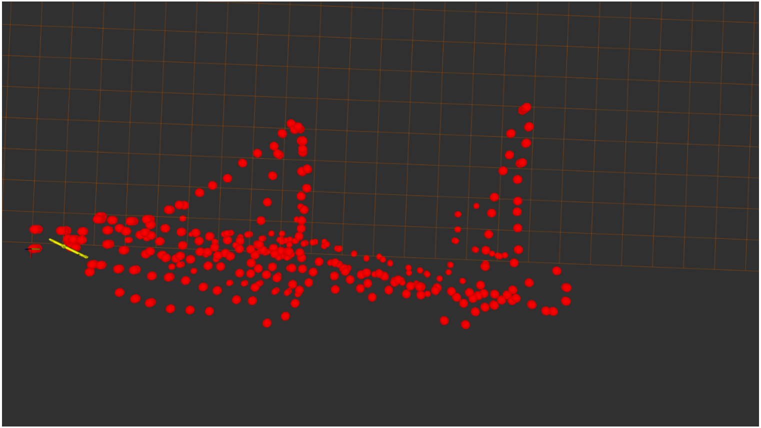
Experiment

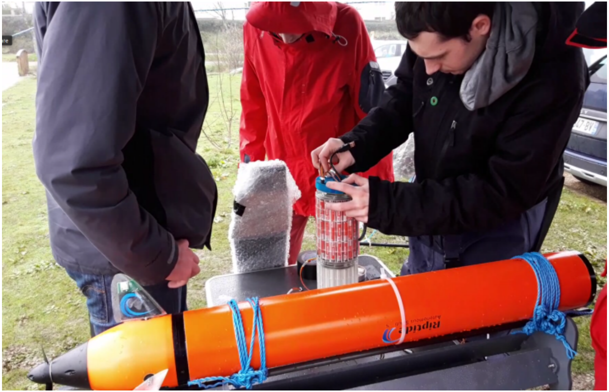

Isobathe z2
Isobathe z1
1 t1
2 z2
3 t2
4 z1
35 m
80 m
3D
Lac de Ty Colo
29290 Saint-Renan
48.431783, -4.615472
mages ©2020 CNES / Airbus, Maxar Technologies, Données cartographiques ©2020
France
Conditions
20 m


Isobathe z2
Isobathe z1
2
z2
3
t2
1
t1
4
z1
2
z3
3
t4
-147°
1
t3
4
z1
45 m
100 m
Lac de Ty Colo
29290 Saint-Renan
48.431632, -4.615648
Lac de Ty Colo
29290 Saint-Renan
48.431783, -4.615472
3D
mages ©2020 CNES / Airbus, Maxar Technologies, Données cartographiques ©2020
France
Conditions
20 m









Stable cycles

[Youtube]