Underwater exploration by an autonomous robot with the method of stable cycles

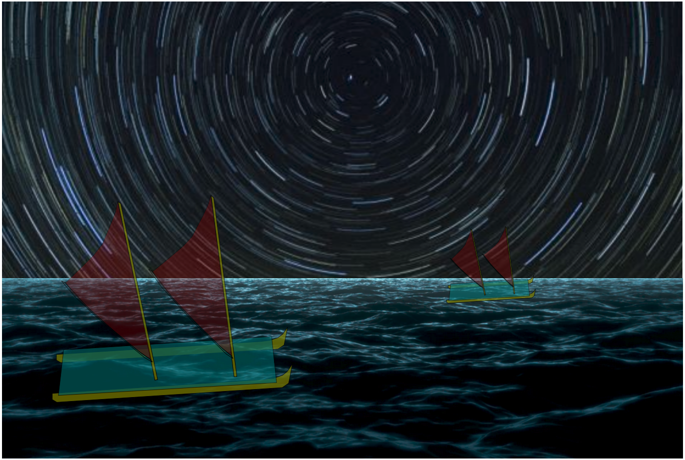
L. Jaulin

Hannover, October 14, 2021

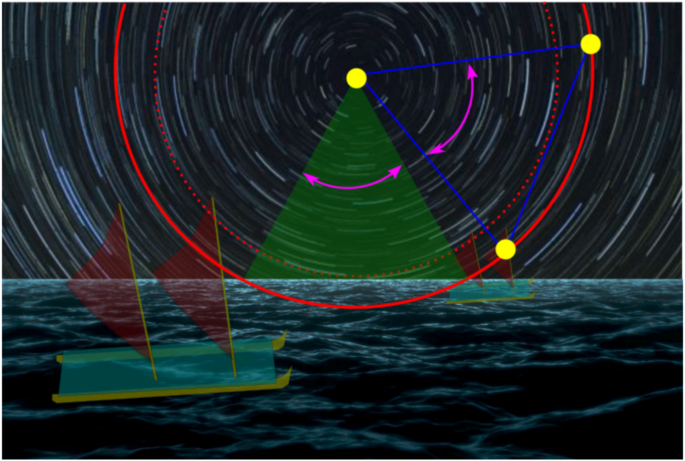
Ancestral method of navigation

Submeeting 2018

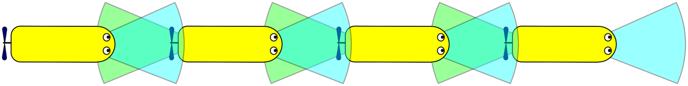
Polynesian navigation



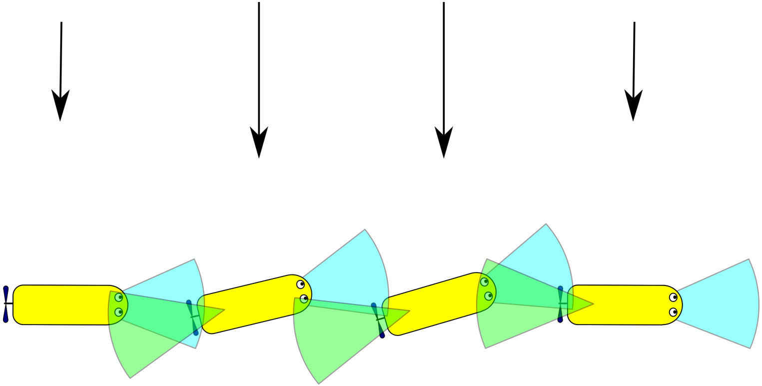
Find the route without GPS, compass and clocks with *wa'a kaulua*[4]



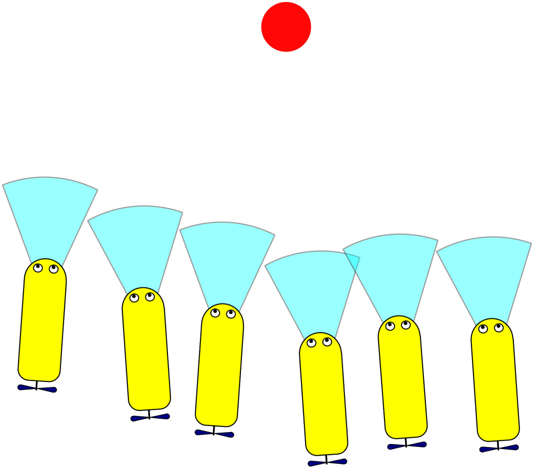
Alignment to keep the heading in case of clouds



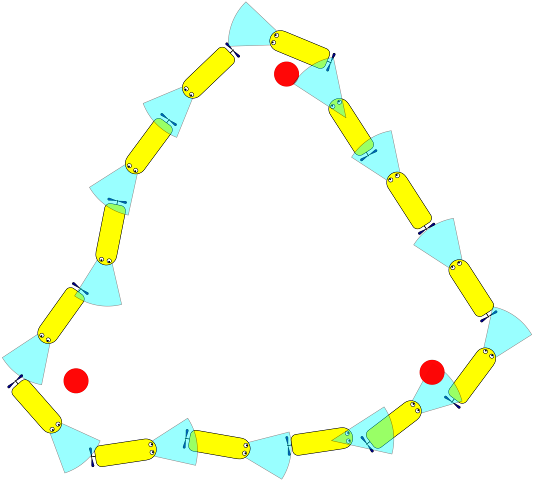
More inertia, more predictable



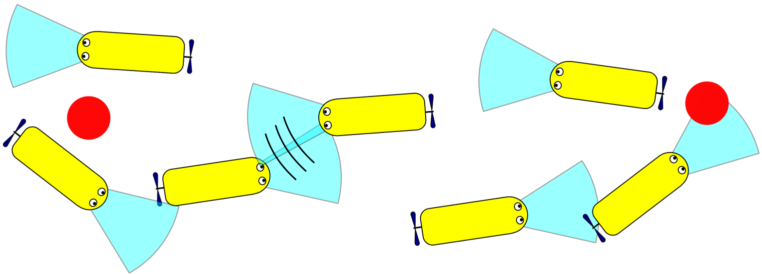
Internal deformations provide information



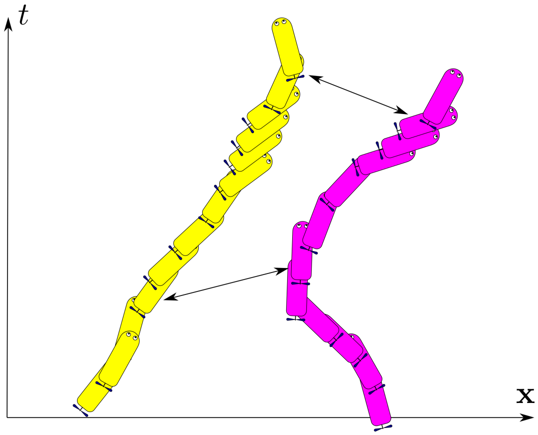
Explore further



Virtual chain: localization ↔ proprioception

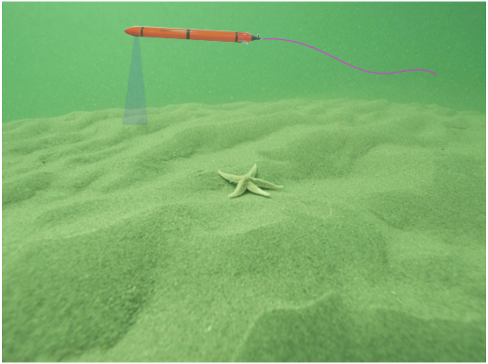


With communication we can do more

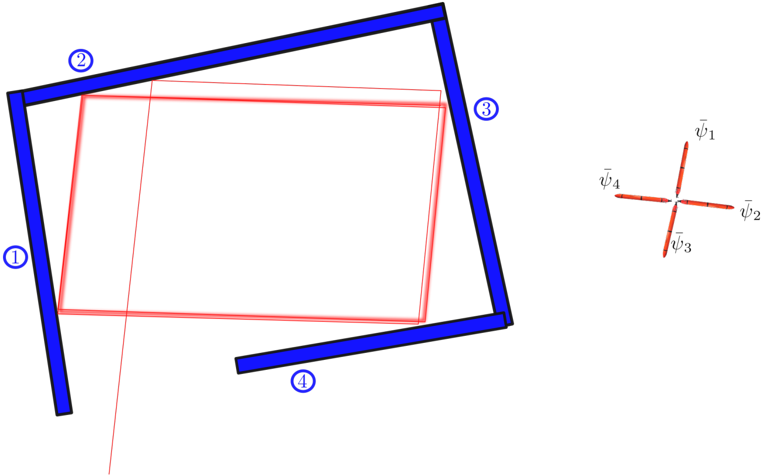


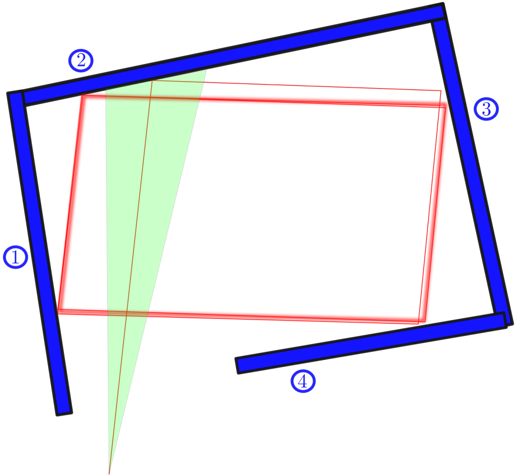
Perception of others rigidifies the evolution of the group

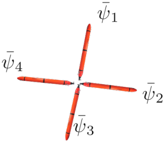
Stable cycles

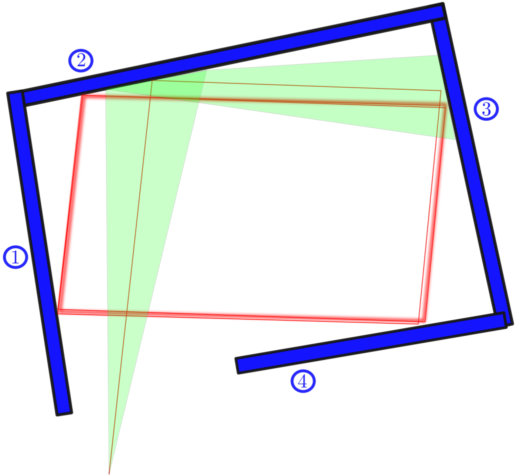


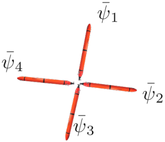
No route exist

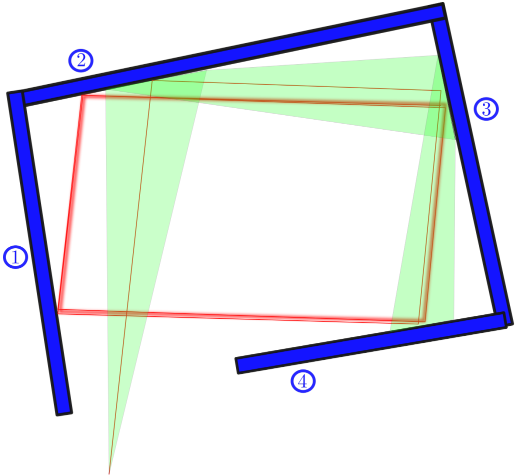

②
③
①
④
$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


2
3
1
4


$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


②
③
①
④


$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


②
③
①
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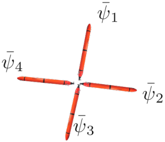
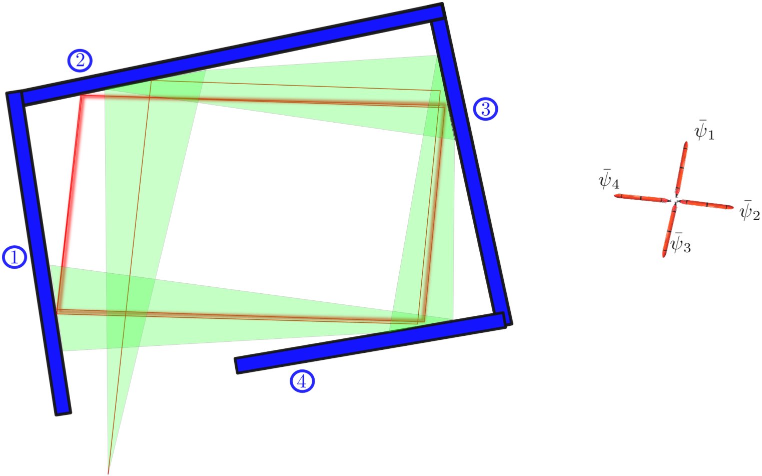
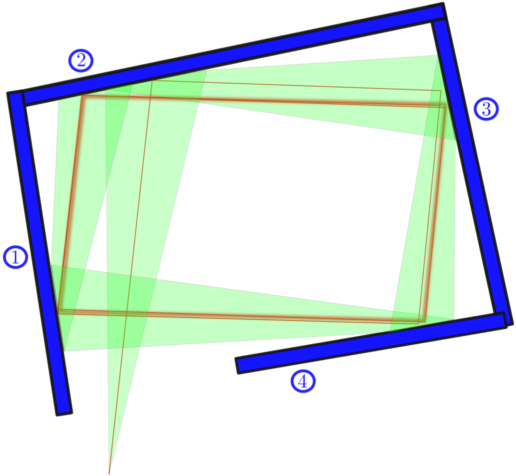
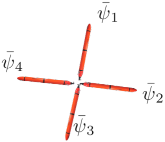
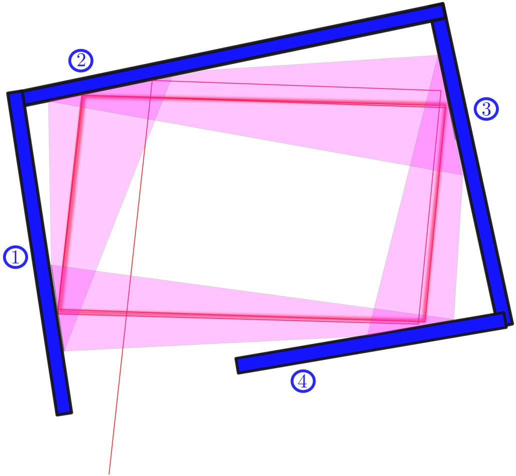

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


②
③
①
④
$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


②
③
①
④


$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


①
②
③
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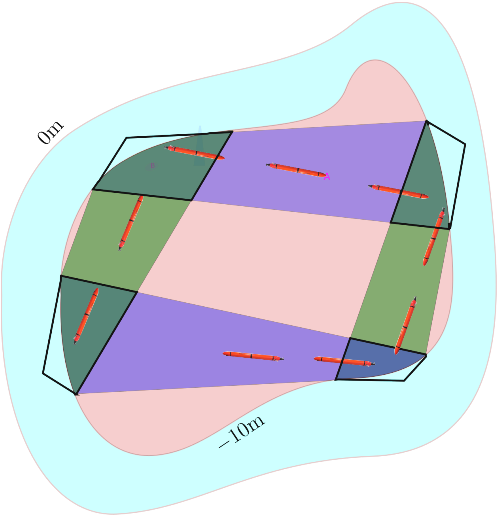
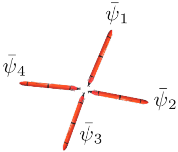
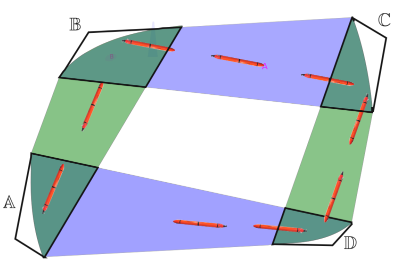
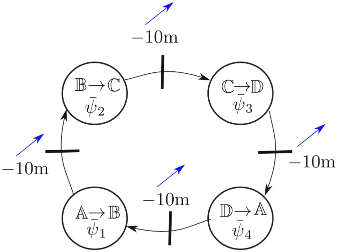
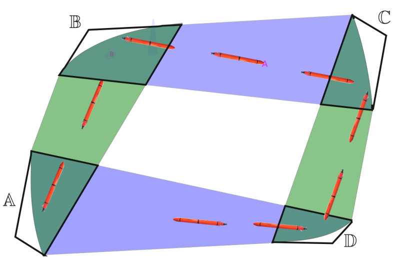
https://youtu.be/TsvEUGa-XAs?t=73

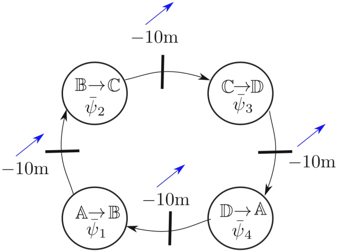

0m
−10m
A
B

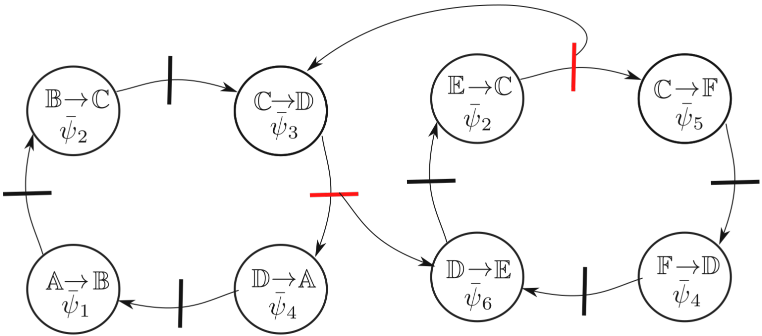

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


$\mathbb{B}$
$\mathbb{C}$
$\mathbb{A}$
$\mathbb{D}$

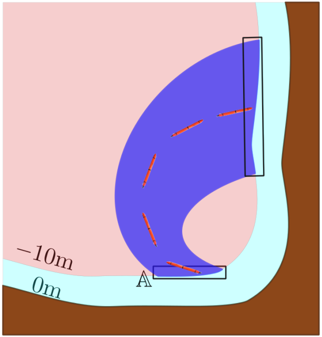

-10m
$\mathbb{B}\to\mathbb{C}$ $\bar{\psi}_2$
$\mathbb{C}\to\mathbb{D}$ $\bar{\psi}_3$
-10m
-10m
-10m
$\mathbb{A}\to\mathbb{B}$ $\bar{\psi}_1$
$\mathbb{D}\to\mathbb{A}$ $\bar{\psi}_4$

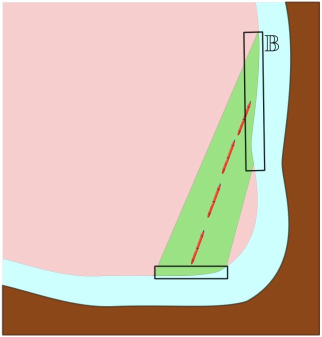

𝔹
ℂ
𝔸
𝔻

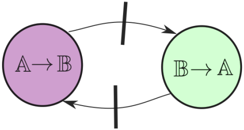

−10m
𝔹→ℂ
$\bar{\psi}_2$
ℂ→𝔻
$\bar{\psi}_3$
−10m
−10m
−10m
𝔸→𝔹
$\bar{\psi}_1$
𝔻→𝔸
$\bar{\psi}_4$


$\mathbb{B} \to \mathbb{C}$ $\bar{\psi}_2$
$\mathbb{C} \to \mathbb{D}$ $\bar{\psi}_3$
$\mathbb{E} \to \mathbb{C}$ $\bar{\psi}_2$
$\mathbb{C} \to \mathbb{F}$ $\bar{\psi}_5$
$\mathbb{A} \to \mathbb{B}$ $\bar{\psi}_1$
$\mathbb{D} \to \mathbb{A}$ $\bar{\psi}_4$
$\mathbb{D} \to \mathbb{E}$ $\bar{\psi}_6$
$\mathbb{F} \to \mathbb{D}$ $\bar{\psi}_4$

A simple cycle


−10m
0m
A


B

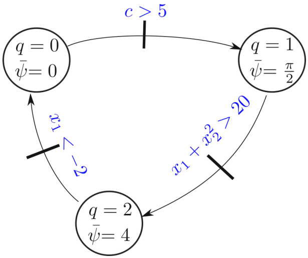

A → B
B → A

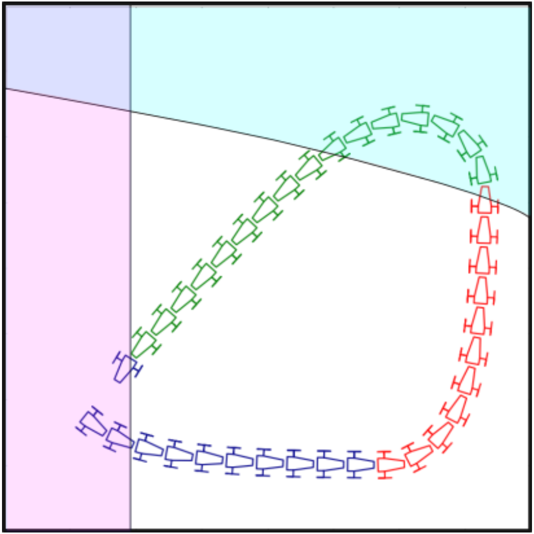
Test-case

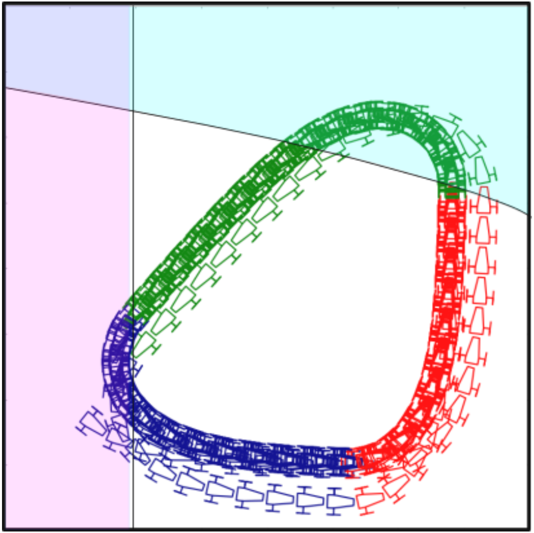
Consider the robot [3]

$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

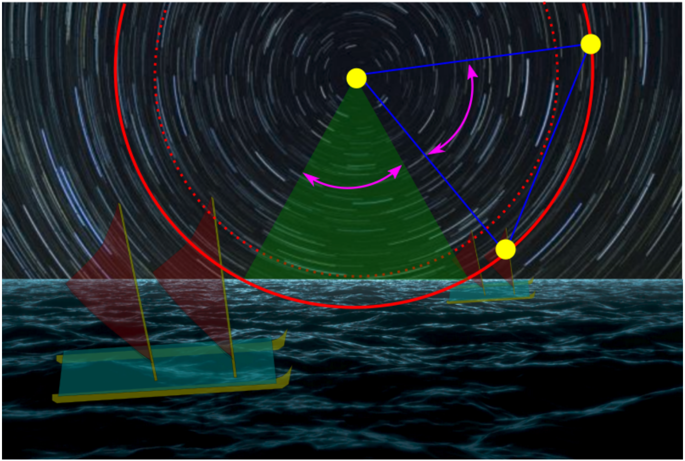
with the heading control $u = \sin(\bar{\psi} - x_3)$.

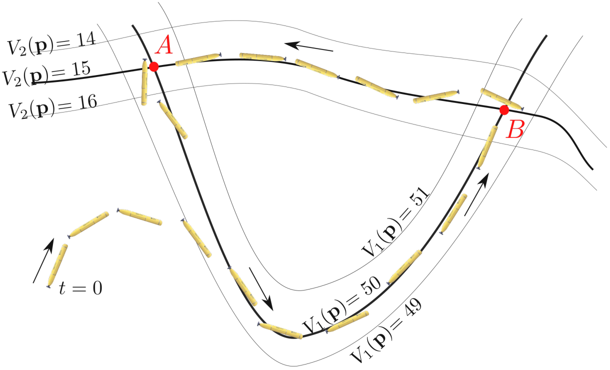


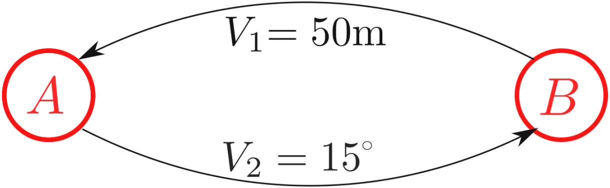




Metric maps ? Topological maps ? Other ?

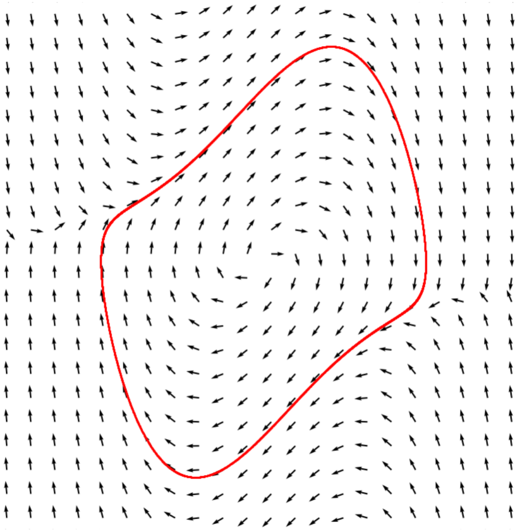



$V_2(\mathbf{p}) = 14$
$V_2(\mathbf{p}) = 15$
$V_2(\mathbf{p}) = 16$
A
B
$V_1(\mathbf{p}) = 51$
$V_1(\mathbf{p}) = 50$
$V_1(\mathbf{p}) = 49$
$t = 0$


$V_1 = 50\text{m}$
A
B
$V_2 = 15^\circ$

Stability with Poincaré map

System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
How to prove that the system has a cycle ?
How to prove that the system is stable ? [2][6]

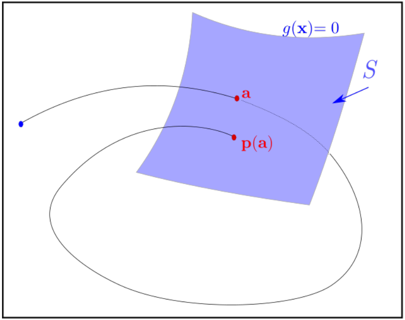


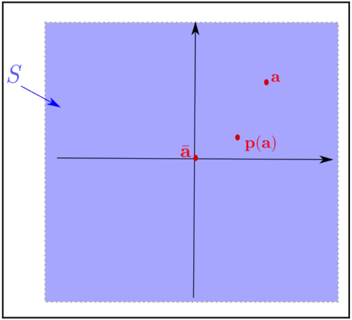
System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
Poincaré section $\mathscr{G}$: $g(\mathbf{x}) = 0$

We define

$$\mathbf{p} : \begin{array}{ccc} \mathscr{G} & \to & \mathscr{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where $\mathbf{p}(\mathbf{a})$ is the point of $\mathscr{G}$ such that the trajectory initialized at $\mathbf{a}$ intersects $\mathscr{G}$ for the first time.


g(x)= 0
S
a
p(a)

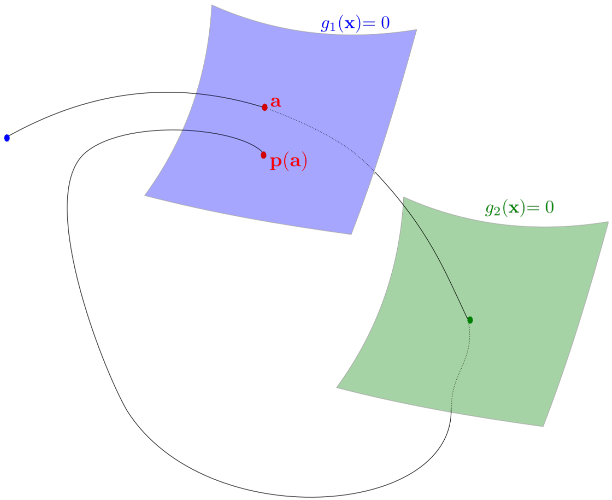

S
a
p(a)
ā

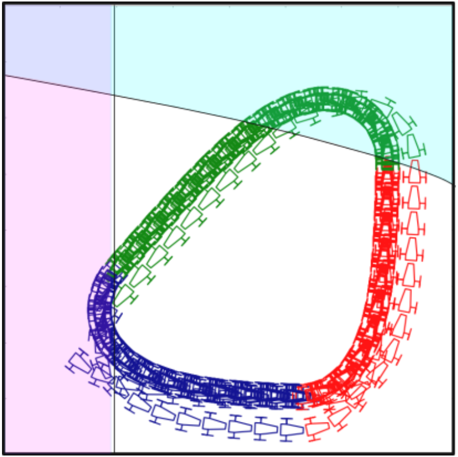
The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$

With hybrid systems

Systems: $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}), i \in \{1, \ldots, m\}$
Section i: $g_i(\mathbf{x}) = 0$


$g_1(\mathbf{x})=0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$g_2(\mathbf{x})=0$

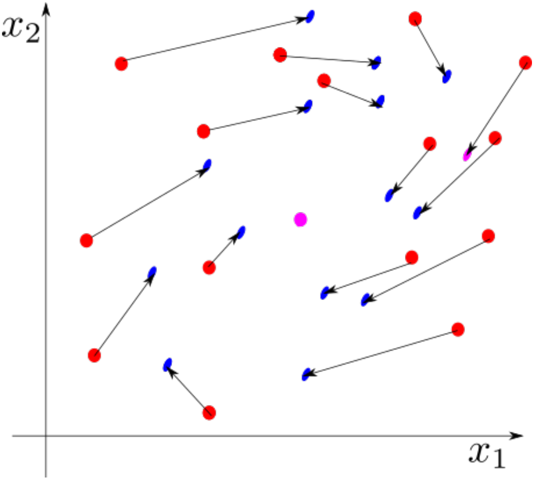


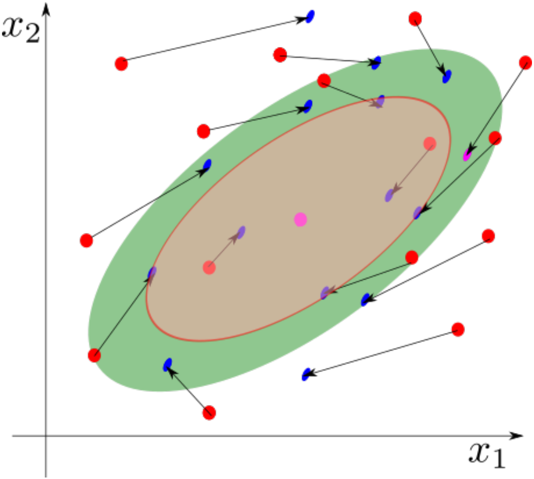
Proving the stability

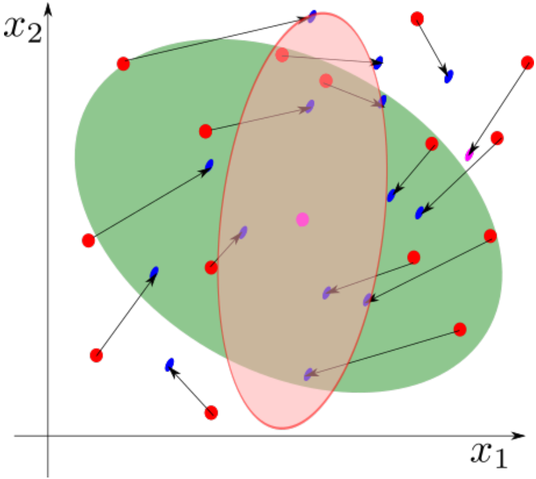
Consider the discrete time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.


x_2
x_1


x_2
x_1


x_2
x_1

We have to find

$$\mathscr{E}_{\mathbf{x}} : \mathbf{x}^{\mathsf{T}} \cdot \mathbf{P} \cdot \mathbf{x} \leq \varepsilon$$

Such that

$$\mathbf{f}(\mathscr{E}_{\mathbf{x}}) \subset \mathscr{E}_{\mathbf{x}}$$

If the system is stable and linear

$$\mathbf{x}_{k+1} = \mathbf{A} \cdot \mathbf{x}_k$$

we can find $\mathbf{P} \succ 0$ such that $V(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}$ is a Lyapunov function

$$\begin{aligned} & V(\mathbf{x}_{k+1}) = V(\mathbf{x}_k) - \mathbf{x}_k^\mathsf{T}\mathbf{x}_k \\ \Leftrightarrow \quad & \mathbf{x}_{k+1}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}_{k+1} = \mathbf{x}_k^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}_k - \mathbf{x}_k^\mathsf{T}\mathbf{x}_k \\ \Leftrightarrow \quad & \mathbf{x}_k^\mathsf{T} \cdot \mathbf{A}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{x}_k - \mathbf{x}_k^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}_k = -\mathbf{x}_k^\mathsf{T}\mathbf{x}_k \end{aligned}$$

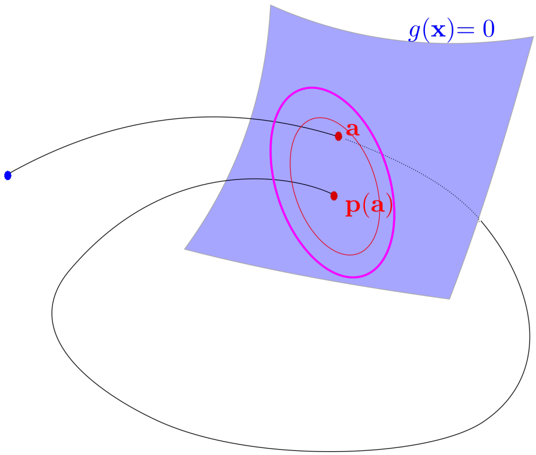
We have to solve the Lyapunov equation

$$\mathbf{A}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{A} - \mathbf{P} = -\mathbf{I}$$

Stability of cycles

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$



See [5]

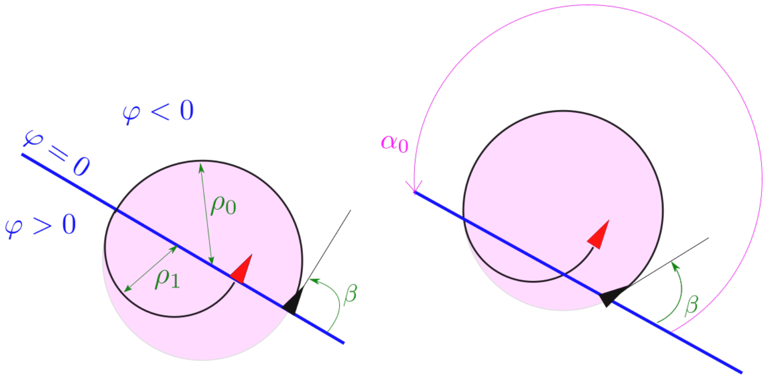
Rolling

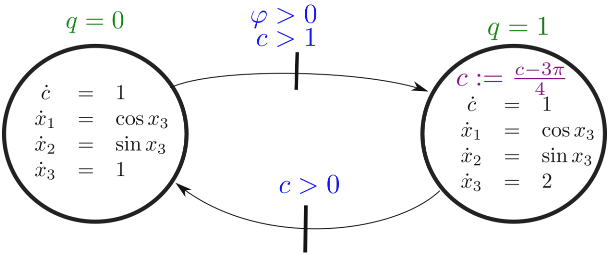
Rolling stability problem

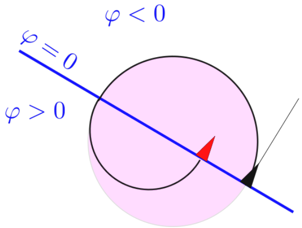
Robot moving on a plane described by

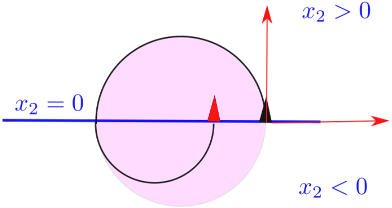
$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

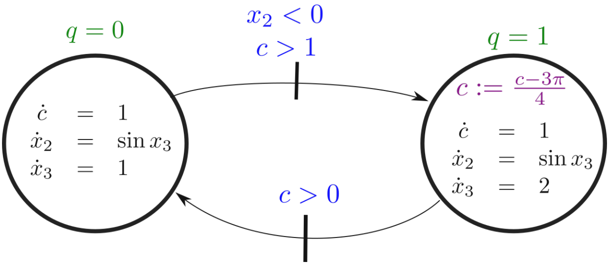
The robot is able to measure a function $\varphi(x_1, x_2)$ has to moves along $\varphi(x_1, x_2) = 0$. [1]


$\varphi < 0$
$\varphi = 0$
$\varphi > 0$
ρ_0
ρ_1
β
α_0
β


$q = 0$
$\dot{c} = 1$
$\dot{x}_1 = \cos x_3$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 1$
$\varphi > 0$
$c > 1$
$q = 1$
$c := \frac{c-3\pi}{4}$
$\dot{c} = 1$
$\dot{x}_1 = \cos x_3$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 2$
$c > 0$

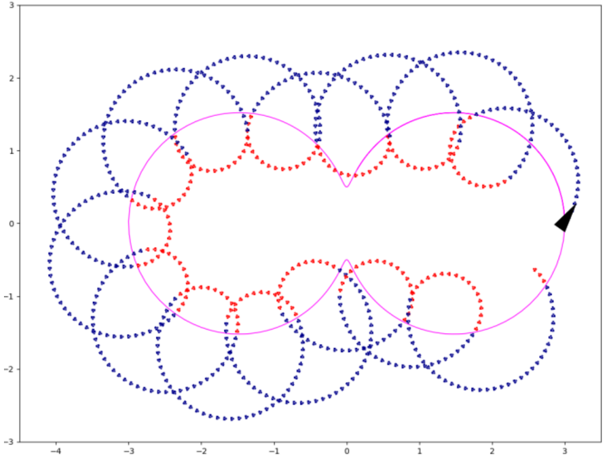

$\varphi < 0$
$\varphi = 0$
$\varphi > 0$


$x_2 > 0$
$x_2 = 0$
$x_2 < 0$


$q = 0$
$x_2 < 0$
$c > 1$
$q = 1$
$\dot{c} = 1$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 1$
$c := \frac{c-3\pi}{4}$
$\dot{c} = 1$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 2$
$c > 0$

We consider the Hippopede of Proclus given by $\varphi(x_1,x_2)=0$ where

$$\varphi(x_1,x_2)=9x_1^2+x_2^2-\left(x_1^2+y_2^2\right)^2.$$



The online Python program can be found here:
https://replit.com/@aulin/rolling

A. Bourgois, A. Chaabouni, A. Rauh, and L. Jaulin.
Proving the stability of navigation cycles.
In *SCAN*, 2021.

A. Bourgois and L. Jaulin.
Interval centred form for proving stability of non-linear discrete-time system.
In *SNR, Vienna*, 2020.

L. Jaulin.
Mobile Robotics.
ISTE editions, 2015.

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