Optimal separator for an hyperbola
Application to localization

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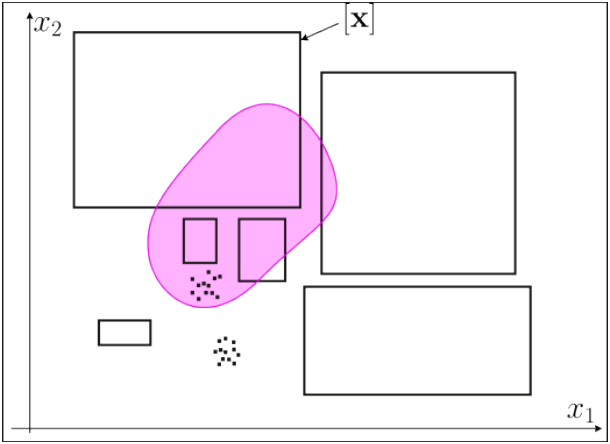
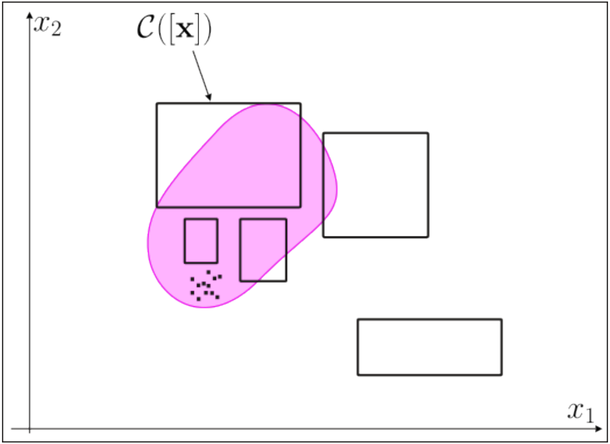
Motivation


x_2
[x]
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

We want a minimal contractor for the *hyperbola* constraint

$$f(\mathbf{q},\mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 = 0$$

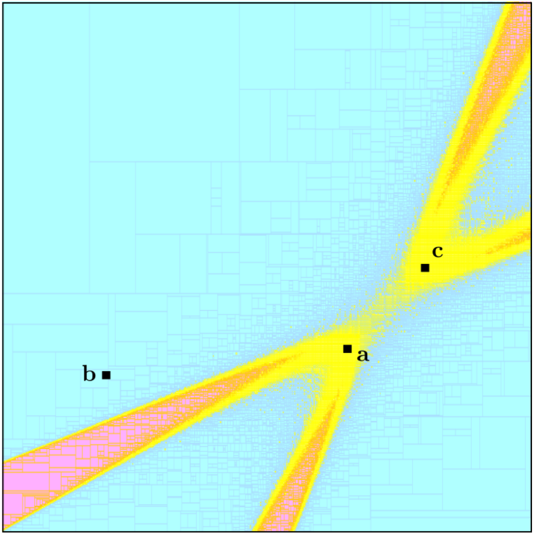
where

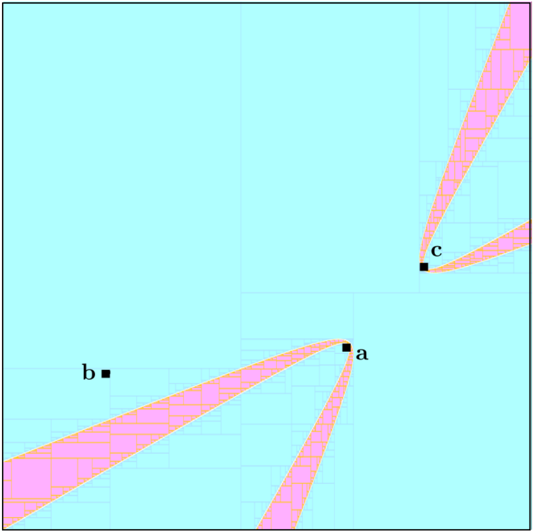
$$4 q_3 q_5 - q_4^2 < 0$$

- $\mathbf{q} = (q_0, \ldots, q_5)$ is the known parameter vector
- $\mathbf{x} = (x_1, x_2)$ is the vector of variables.

More than this, we want a minimal separator for

$$\mathbb{X} = \{(x_1, x_2) | f(\mathbf{q}, \mathbf{x}) \in [y]\}.$$


c
a
b


c
a
b

Basic idea

We consider the product

$$x_3 = x_1 \cdot x_2$$

Equivalently

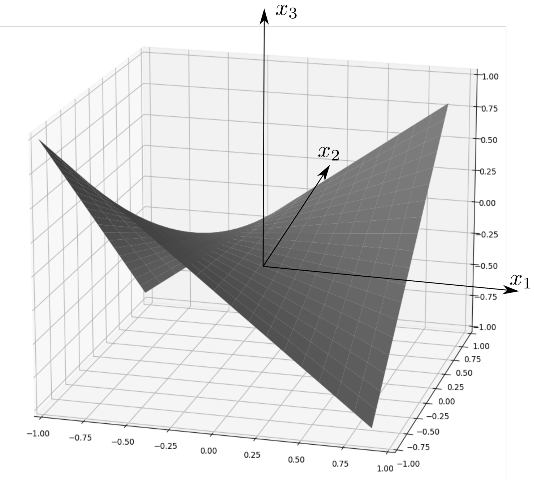
$$\mathbb{X} = \{(x_1, x_2, x_3) \,|\, x_1 \cdot x_2 = x_3\}$$

A (classical) contractor associated with $z = x \cdot y$ is:

Algorithm C^{mult} (inout: $[x_1],[x_2],[x_3]$)	
$[x_3] := [x_3] \cap ([x_1] \cdot [x_2])$	// $x_3 = x_1 \cdot x_2$
$[x_1] := [x_1] \cap \left([x_3] \cdot \frac{1}{[x_2]}\right)$	// $x_1 = x_3 / x_2$
$[x_2] := [x_2] \cap \left([x_3] \cdot \frac{1}{[x_1]}\right)$	// $x_2 = x_3 / x_1$

C^{mult} is not minimal.
Indeed, if $[x_1] = [-1,3], [x_2] = [-2,3], [x_3] = [9,10]$

$$
\begin{array}{rclcrcr}
[x_3] & := & [x_3] \cap ([x_1] \cdot [x_2]) & = & [9,10] \cap ([-1,3] \cdot [-2,3]) & = & [9,9] \\
[x_1] & := & [x_1] \cap \left([x_3] \cdot \frac{1}{[x_2]}\right) & = & [-1,3] \cap \left([9,9] \cdot \frac{1}{[-2,3]}\right) & = & [-1,3] \\
[x_2] & := & [x_2] \cap \left([x_3] \cdot \frac{1}{[x_1]}\right) & = & [-2,3] \cap \left([9,9] \cdot \frac{1}{[-1,3]}\right) & = & [-2,3]
\end{array}
$$


x_3
x_2
x_1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

We have

$$x_1 \cdot x_2 = x_3 \Leftrightarrow (-x_1) \cdot x_2 = -x_3$$

We say that $x_1 \cdot x_2 = x_3$ is invariant by the symmetry

$$\sigma_1 : \begin{cases} x_1 & \mapsto & -x_1 \\ x_2 & \mapsto & x_2 \\ x_3 & \mapsto & -x_3 \end{cases}$$

Equivalently, $x_1 \cdot x_2 = x_3$ is said to be invariant by

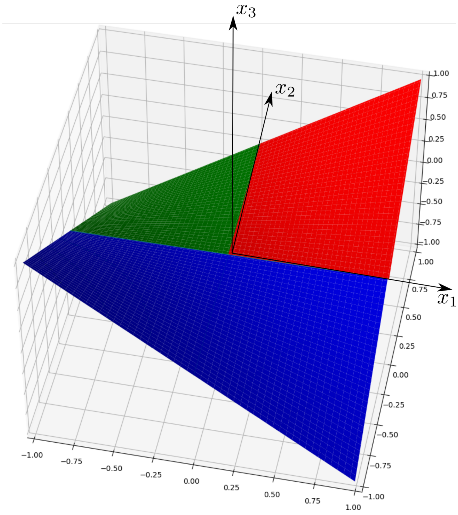
$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We also have

$$x_1 \cdot x_2 = x_3 \Leftrightarrow x_1 \cdot (-x_2) = -x_3$$

invariant with respect to

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$


x_3
x_2
x_1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

Due to the monotonicity, the minimal contractor (the seed) for the box

$$[\mathbf{x}] \subset \ [\mathbf{a}] \ = \ \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \ .$$

associated to $x_1 \cdot x_2 = x_3$ is

$$\mathscr{C}_0 \left(\begin{array}{c} [x_1] \\ [x_2] \\ [x_3] \end{array} \right) = \left(\begin{array}{c} [x_1] \cap [\frac{x_3^-}{x_2^+}, \frac{x_3^+}{x_2^-}] \\ [x_2] \cap [\frac{x_3^-}{x_1^+}, \frac{x_3^+}{x_1^-}] \\ [x_3] \cap [x_1^- \cdot x_2^-, x_1^+ \cdot x_2^+] \end{array} \right).$$

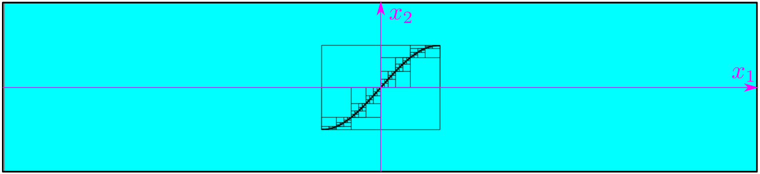
We will see later that

$$\mathscr{C} = \sigma_2 \bullet \sigma_1 \bullet \mathscr{C}_0$$

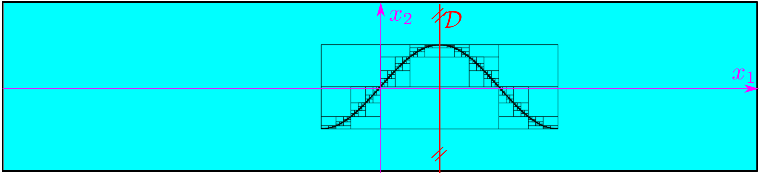
is an optimal contractor for $\mathbb{X}$ as soon as $\mathscr{C}_0$ is a minimal contractor for $\mathbb{X}_0$.

Sine

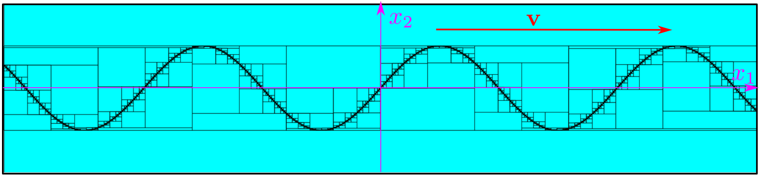
$$x_2 = \sin x_1$$



$\mathscr{C}_0$



$$\sigma_D \bullet \mathscr{C}_0$$



$$\mathscr{C}_{\sin} = \boldsymbol{\sigma}_{\mathbf{v}} \bullet \boldsymbol{\sigma}_D \bullet \mathscr{C}_0$$

Hyperoctahedral symmetries

The hyperoctahedral group B_n is the group of symmetries of the unit hypercube of $\mathbb{R}^n$.
It contains $2^n \cdot n!$ elements.

For $n = 2$, we have $2^2 \cdot 2! = 8$ elements:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\sigma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_7 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

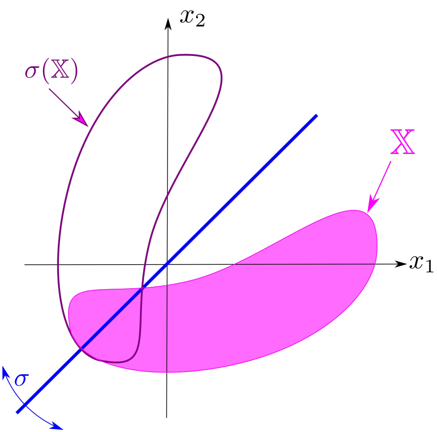
We will write equivalently

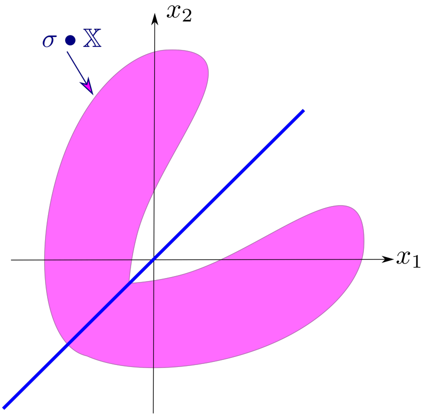
$$\sigma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \sigma : \begin{cases} \mathbb{R}^2 & \mapsto & \mathbb{R}^2 \\ (x_1, x_2) & \mapsto & (x_2, -x_1) \end{cases}$$

Acts

For $\sigma \in B_n$, we define the *act* operator:

$$\sigma \bullet \mathbb{X} = \mathbb{X} \cup \sigma(\mathbb{X}).$$

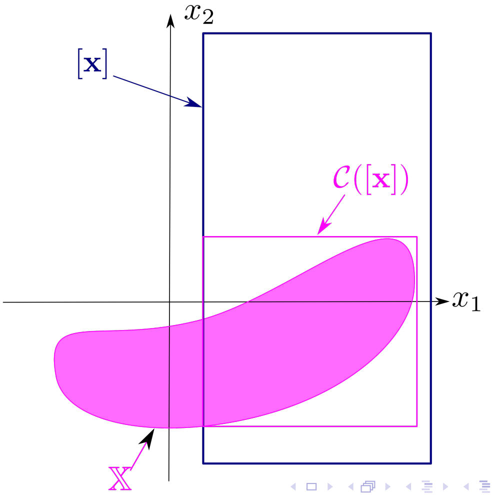

x2
σ(𝕏)
𝕏
x1
σ

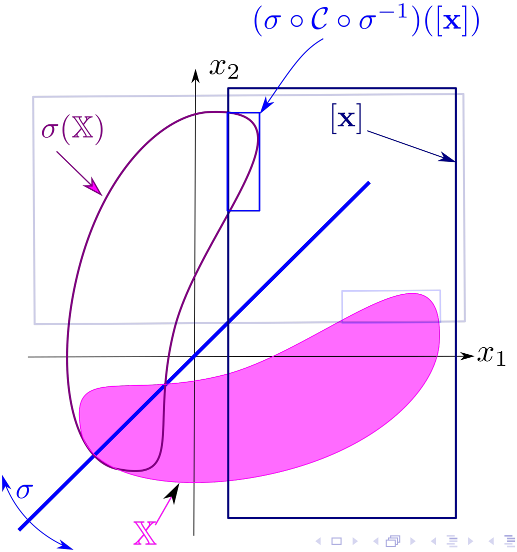

x_2
$\sigma \bullet \mathbb{X}$
x_1

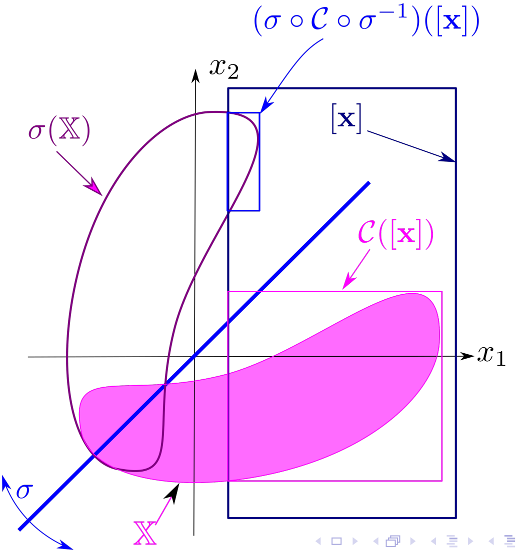
Contractor Acts

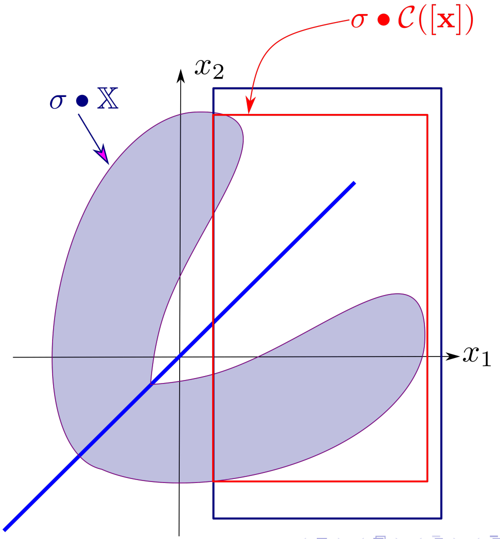
If $\mathscr{C}$ is a contractor in $\mathbb{R}^n$, and $\sigma \in B_n$, we define the *contractor act* of σ on $\mathscr{C}$ as

$$\sigma \bullet \mathscr{C} = \mathscr{C} \sqcup \sigma \circ \mathscr{C} \circ \sigma^{-1}.$$


x_2
$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
x_1
$\mathbb{X}$


$(\sigma \circ \mathcal{C} \circ \sigma^{-1})([\mathbf{x}])$
x_2
$\sigma(\mathbb{X})$
$[\mathbf{x}]$
x_1
σ
$\mathbb{X}$


$(\sigma \circ \mathcal{C} \circ \sigma^{-1})([\mathbf{x}])$
x_2
$\sigma(\mathbb{X})$
$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
x_1
σ
$\mathbb{X}$


$\sigma \bullet \mathcal{C}([\mathbf{x}])$
x_2
$\sigma \bullet \mathbb{X}$
x_1

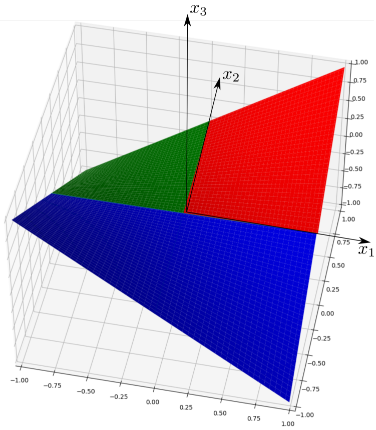
Product constraint

We consider the constraint

$$\mathbb{X}: \; x_1 x_2 = x_3.$$

i.e.

$$\mathbb{X} = \{\mathbf{x} = (x_1, x_2, x_3) | x_1 x_2 = x_3\}$$



Product constraint: $x_1 x_2 = x_3$

A minimal contractor over

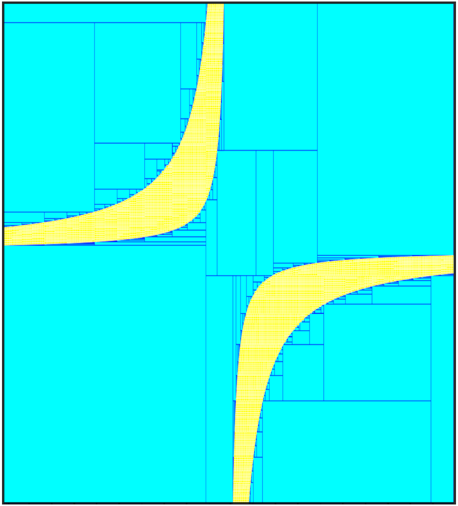
$$[\mathbf{a}] \quad = \quad \mathbb{R}^{+} \times \mathbb{R}^{+} \times \mathbb{R}^{+}.$$

is:

$$\mathscr{C}_0 \left(\begin{array}{c} [x_1] \\ [x_2] \\ [x_3] \end{array} \right) = \left(\begin{array}{c} [x_1] \cap [\frac{x_3^-}{x_2^+}, \frac{x_3^+}{x_2^-}] \\ [x_2] \cap [\frac{x_3^-}{x_1^+}, \frac{x_3^+}{x_1^-}] \\ [x_3] \cap [x_1^- \cdot x_2^-, x_1^+ \cdot x_2^+] \end{array} \right).$$

Generators are the stabilizers

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$



Conjugation

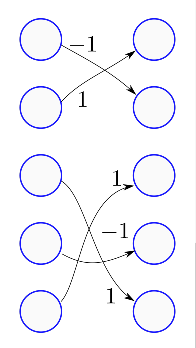
Consider an equation of the form

$$f(\mathbf{q},\mathbf{x}) = 0.$$

The pair of transformations (σ,γ) is *conjugate* with respect to f if

$$f(\gamma(\mathbf{q}),\sigma(\mathbf{x})) = 0 \Leftrightarrow f(\mathbf{q},\mathbf{x}) = 0.$$

Transformations that will be consider are limited to the *hyperoctahedral group* B_n.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 5 & -4 & 3 \end{pmatrix}$$

A symmetry of B_2 in a matrix form, satisfies

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

with $\sigma_{ij}^2 \in \{0,1\}, \sigma_{i1}^2 + \sigma_{i2}^2 = 1, \sigma_{1j}^2 + \sigma_{1j}^2 = 1.$

The Cauchy form is obtained from the matrix form by left multiplying by the line vector $(1,2)$:

$$\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = (\sigma_{11} + 2\sigma_{21}, \sigma_{12} + 2\sigma_{22}).$$

Hyperbolic symmetries

Proposition. Take a point $\mathbf{x} = (x_1, x_2)$ such

$$f(\mathbf{q},\mathbf{x}) = q_0 + q_1x_1 + q_2x_2 + q_3x_1^2 + q_4x_1x_2 + q_5x_2^2 = 0$$

and a symmetry

$$\sigma = (\sigma_{11} + 2\sigma_{21}, \sigma_{12} + 2\sigma_{22}) \in B_2.$$

If

$$\begin{aligned}\gamma \ = \ & (q_0, \sigma_{11}q_1 + \sigma_{21}q_2, \sigma_{12}q_1 + \sigma_{22}q_2, \sigma_{11}^2 q_3 + \sigma_{21}^2 q_5 \\ & , (\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21})q_4, \sigma_{12}^2 q_3 + \sigma_{22}^2 q_5)\end{aligned}$$

the pair (σ^{-1}, γ) is conjugate with respect to $f(\mathbf{q},\mathbf{x})$.

Example. If

$$\sigma = (2,1)$$

we get

$$\gamma \;=\; (q_0, q_2, q_1, q_5, q_4, q_3)$$

This means that

$$q_0 + q_1x_1 + q_2x_2 + q_3x_1^2 + q_4x_1x_2 + q_5x_2^2 = 0$$

is equivalent to

$$q_0 + q_2x_2 + q_1x_1 + q_5x_2^2 + q_4x_2x_1 + q_3x_1^2 = 0$$

Given a symmetry σ, the choice function $\psi_\sigma(\mathbf{q})$ returns the symmetry γ such that (σ, γ) is a conjugate pair.

Cardinal functions

Define

$$\varphi_1(\mathbf{q}, x_2) = \max\{x_1 \,|f(\mathbf{q}, \mathbf{x}) = 0)\}$$

If $q_3 > 0$, we have

$$\varphi_1(\mathbf{q}, x_2) = \frac{-(q_1 + q_4 x_2) + \sqrt{(q_1 + q_4 x_2)^2 - 4q_1(q_0 + q_2 x_2 + q_5 x_2^2)}}{2q_3}$$

The minimal interval extension function of $\varphi_1(\mathbf{q}, x_2)$ is

$$[\varphi_1](\mathbf{q}, [x_2]) = [\{x_1 | \exists x_2 \in [x_2], x_1 = \varphi_1(\mathbf{q}, x_2)\}]$$

Seed contractor

We can build the contractor

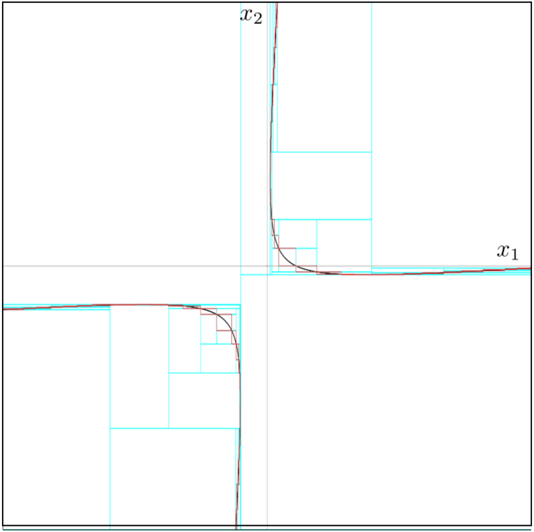
$$C_0^{\mathbf{q}} : [\mathbf{x}] \to [x_1] \times [\varphi_1](\mathbf{q}, [x_2]).$$

$C_0^{\mathbf{q}}([\mathbf{x}])$ contracts the box $[\mathbf{x}]$ with respect to a small portion of the hyperbola.

Proposition. A minimal contractor associated to the hyperbola set $\mathbb{X}$ is

$$\bigcup_{\sigma\in\{(1,2),(1,-2),(-1,2),(-1,-2)\}} \sigma \bullet \left((2,1) \bullet C_0^{\psi_{(2,1)}\cdot\psi_\sigma(\mathbf{q})} \cap C_0^{\psi_\sigma(\mathbf{q})} \right).$$

where $\psi_\sigma(\mathbf{q})$ is the choice function.

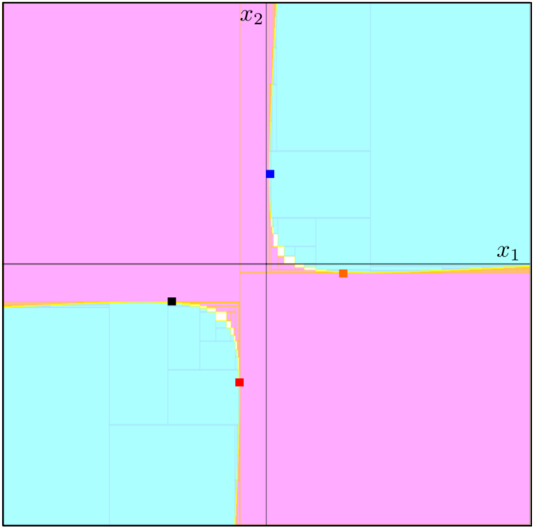

x_2
x_1

Hyperbola separator

For an hyperbola area defined by

$$\mathbb{X} = \{\mathbf{x} | q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \leq 0\}$$

a minimal separator can be derived.


x_2
x_1

Application to localization

Consider two points $\mathbf{a}, \mathbf{b}$ of the plane. The set $\mathbb{X}$ of all points such that

$$\|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| \leq \ell$$

is an hyperbola area with foci $\mathbf{a}, \mathbf{b}$. We have

$$\mathbb{X} = \{\mathbf{x} | \mathbf{f}_{\mathbf{a},\mathbf{b},\ell}(\mathbf{x}) \leq 0\}$$

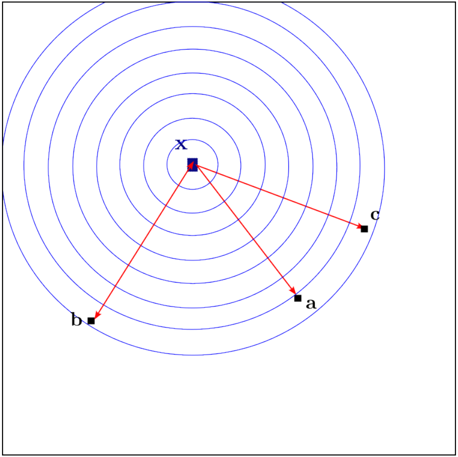
where

$$\mathbf{f}_{\mathbf{a},\mathbf{b},\ell}(\mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2$$

with ...

$$
\begin{aligned}
q_0 &= - \quad a_1^4 - 2a_1^2a_2^2 + 2a_1^2b_1^2 + 2a_1^2b_2^2 + 2a_1^2\ell^2 \\
&\quad -a_2^4 + 2a_2^2b_1^2 + 2a_2^2b_2^2 \\
&\quad +2a_2^2\ell^2 - b_1^4 - 2b_1^2b_2^2 + 2b_1^2\ell^2 - b_2^4 + 2b_2^2\ell^2 - \ell^4 \\
q_1 &= 4a_1^3 - 4a_1^2b_1 + 4a_1a_2^2 - 4a_1b_1^2 - 4a_1b_2^2 \\
&\quad -4a_1\ell^2 - 4a_2^2b_1 + 4b_1^3 + 4b_1b_2^2 - 4b_1\ell^2 \\
q_2 &= 4a_1^2a_2 - 4a_1^2b_2 + 4a_2^3 - 4a_2^2b_2 - 4a_2b_1^2 \\
&\quad -4a_2b_2^2 - 4a_2\ell^2 + 4b_1^2b_2 + 4b_2^3 - 4b_2\ell^2 \\
q_3 &= -4a_1^2 + 8a_1b_1 - 4b_1^2 + 4\ell^2 \\
q_4 &= -8a_1a_2 + 8a_1b_2 + 8a_2b_1 - 8b_1b_2 \\
q_5 &= -4a_2^2 + 8a_2b_2 - 4b_2^2 + 4\ell^2
\end{aligned}
$$

Localization



x emits a sound received later by three microphones **a**, **b** and **c**

We have

$$\begin{aligned} \|\mathbf{x}-\mathbf{a}\| &= c\cdot(t_a-t_0) \\ \|\mathbf{x}-\mathbf{b}\| &= c\cdot(t_b-t_0) \\ \|\mathbf{x}-\mathbf{c}\| &= c\cdot(t_c-t_0) \end{aligned}$$

where c is the sound speed and t_a, t_b, t_c is the detection time for microphones $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

We eliminate t_0 which is unknown to get

$$\begin{aligned}\|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| &= c\cdot(t_a - t_b) = \ell_{ab} \\ \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{c}\| &= c\cdot(t_a - t_c) = \ell_{ac}\end{aligned}$$

Measure the two pseudo distances: $\ell_{ab} = 8 \pm 0.1$ and $\ell_{ac} = 4 \pm 0.1$.

The set $\mathbb{X}$ of all feasible locations is defined by

$$\begin{array}{llcl} \text{(i)} & \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| & \in & [7.9, 8.1] \\ \text{(ii)} & \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{c}\| & \in & [3.9, 4.1] \end{array}$$

We get

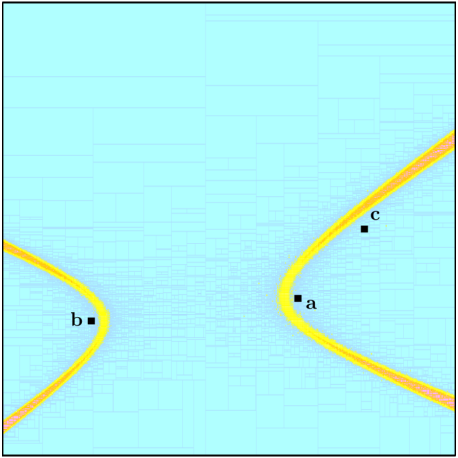
$$\mathbb{X} = \mathbb{X}_{ab} \cap \mathbb{X}_{ac}$$

where:

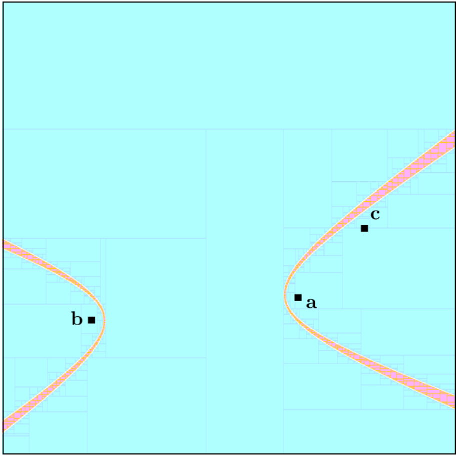
$$\mathbb{X}_{ab} : \begin{cases} \mathbf{f}_{\mathbf{a},\mathbf{b},8.1}(\mathbf{x}) \leq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{b},7.9}(\mathbf{x}) \geq 0 \end{cases}$$

and

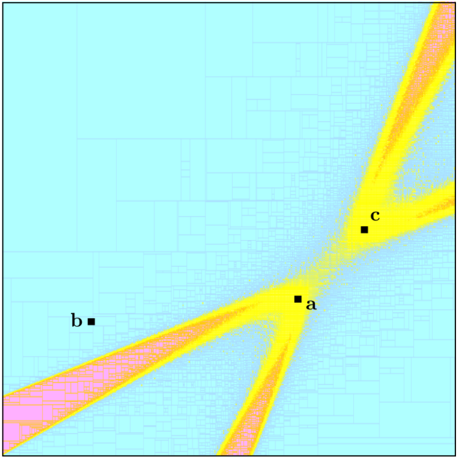
$$\mathbb{X}_{ac} : \begin{cases} \mathbf{f}_{\mathbf{a},\mathbf{c},4.1}(\mathbf{x}) \leq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{c},3.9}(\mathbf{x}) \geq 0 \end{cases}$$



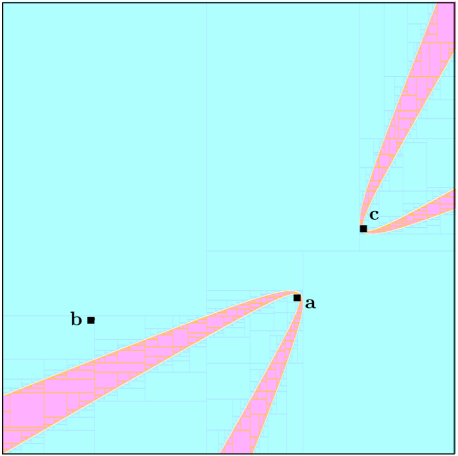
Set $\mathbb{X}_{ab}$ of positions consistent with microphones $\mathbf{a}, \mathbf{b}$ (classic)



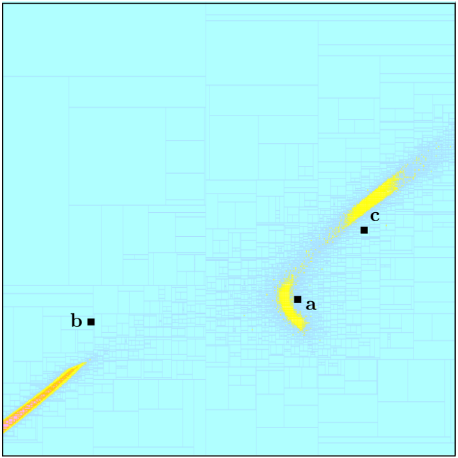
$\mathbb{X}_{ab}$ (with the hyperbola separator)



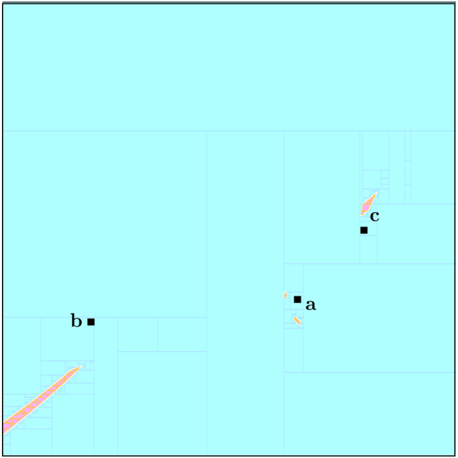
$\mathbb{X}_{ac}$ (classic)



$\mathbb{X}_{ac}$ (with the hyperbola separator)



Set $\mathbb{X} = \mathbb{X}_{ab} \cap \mathbb{X}_{ac}$ (classic)



Set $\mathbb{X} = \mathbb{X}_{ab} \cap \mathbb{X}_{ac}$ (with the hyperbola separator)

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