Characterizing discontinuities of a dynamical system

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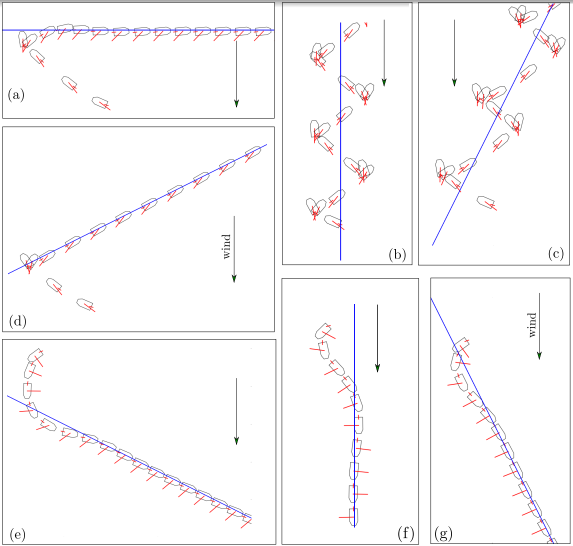
Montpellier, LIRMM, Journée Contredo, 23 janvier 2019

Brave

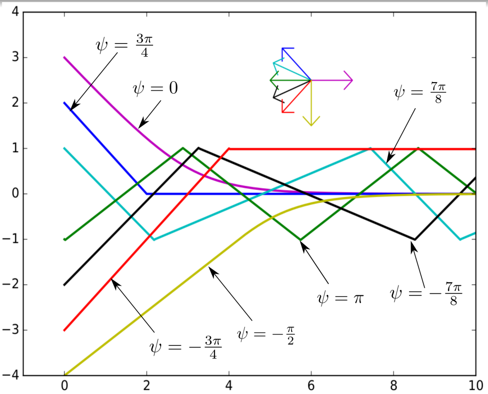

(a)
(b)
(c)
(d)
wind
(e)
(f)
(g)
wind

Easy-boat

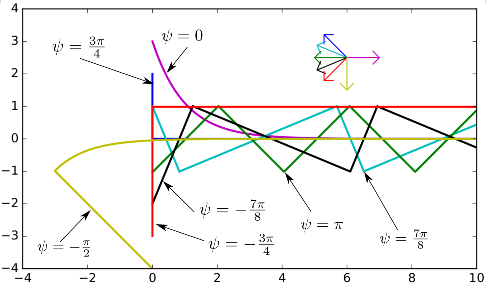
$$\dot{d} = \sin u$$
$$\cos(\psi - u) + \cos\frac{\pi}{5} > 0$$

Controller in: (d, ψ, q); out: u
if $d^2 - 1 > 0$ then $q :=$ sign(d) if $\cos(\psi + \text{atan}(d)) + \cos\frac{\pi}{4} \leq 0 \vee \left(d^2 \leq 1 \wedge \cos\psi + \cos\frac{\pi}{4} \leq 0\right)$ then $u := \pi + \psi - q\frac{\pi}{4}$. else $u := -\text{atan}(d)$.

Simulations



Simulation in the (t, d)-space



Simulation in the $(\int^t \cos u, d)$-space

Formalism

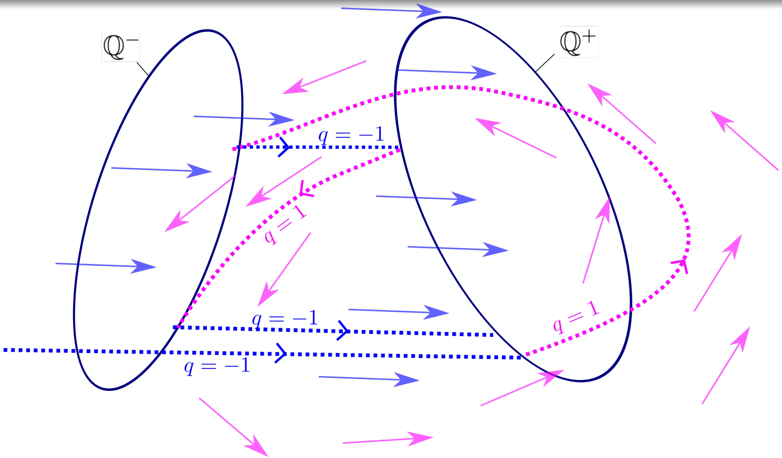
Given $\mathbb{Q}^-$, $\mathbb{Q}^+$ two disjoint closed subsets of $\mathbb{R}^n$, two smooth functions $\mathbf{f}_a, \mathbf{f}_b : \mathbb{R}^n \to \mathbb{R}^n$.
We define [3]

$$\mathscr{S}(\mathbb{A}) : \begin{cases} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x}, q) & = & \begin{cases} \mathbf{f}_a(\mathbf{x}, q) & \text{if } \mathbf{x} \in \mathbb{A} \\ \mathbf{f}_b(\mathbf{x}, q) & \text{if } \mathbf{x} \in \mathbb{B} = \overline{\mathbb{A}} \end{cases} \\ q & = & -1 & & \text{as soon as } \mathbf{x} \in \mathbb{Q}^- \\ & = & +1 & & \text{as soon as } \mathbf{x} \in \mathbb{Q}^+ \end{cases}$$

The pair $(\mathbf{x}, q)$ always satisfies the constraint

$$\begin{array}{ccc} \mathbf{x} \in \mathbb{Q}^{+} & \Rightarrow & q = 1 \\ \mathbf{x} \in \mathbb{Q}^{-} & \Rightarrow & q = -1 \end{array}$$

or equivalently, $\mathbf{x} \in \overline{\mathbb{Q}^{-q}}$.

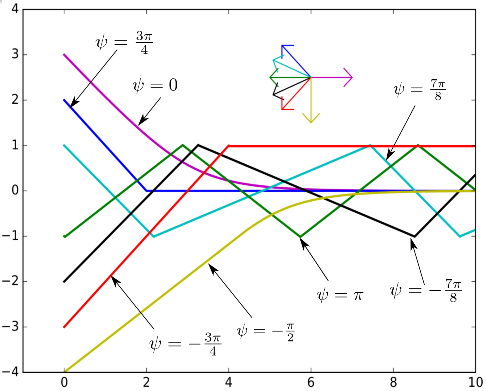

$\mathbb{Q}^-$
$\mathbb{Q}^+$
$q=-1$
$q=1$
$q=-1$
$q=-1$
$q=1$

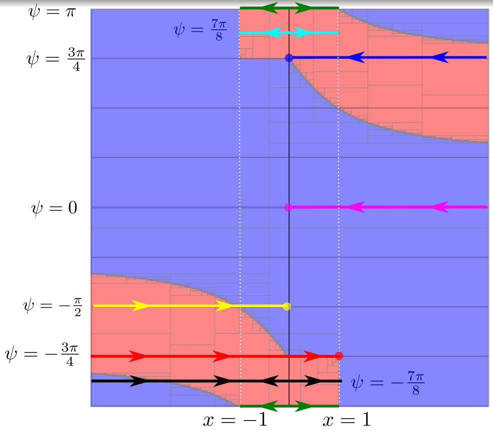
With easy-boat

We take $\mathbf{x} = (d, \psi)$,

Function $\mathbf{f}(\mathbf{x}, q)$
if $\cos(x_2 + \text{atan}\, x_1) + \cos\frac{\pi}{4} \leq 0 \vee \left(x_1^2 - 1 \leq 0 \wedge \cos x_2 + \cos\frac{\pi}{4} \leq 0\right)$
then $u := \pi + x_2 - q\frac{\pi}{4}$
else $u := -\text{atan}\, x_1$.
Return $\sin u$

$$\mathbf{x} = (d, \psi)$$
$$\mathbf{f}_a(\mathbf{x}, q) = \begin{pmatrix} \sin\left(\pi + x_2 - q\frac{\pi}{4}\right) \\ 0 \end{pmatrix}$$
$$\mathbf{f}_b(\mathbf{x}) = \begin{pmatrix} \sin(-\operatorname{atan} x_1) \\ 0 \end{pmatrix}$$
$$\mathbb{A}_1 = \left\{\mathbf{x} \,|\, \cos(x_2 + \operatorname{atan} x_1) + \cos\frac{\pi}{4} \leq 0\right\}$$
$$\mathbb{A}_2 = \left\{\mathbf{x} \,|\, x_1^2 - 1 \leq 0\right\}$$
$$\mathbb{A}_3 = \left\{\mathbf{x} \,|\, \cos x_2 + \cos\frac{\pi}{4} \leq 0\right\}$$
$$\mathbb{A} = \mathbb{A}_1 \cup (\mathbb{A}_2 \cap \mathbb{A}_3)$$
$$\mathbb{Q}^- = \{\mathbf{x} \,|\, x_1 + 1 \leq 0\}$$
$$\mathbb{Q}^+ = \{\mathbf{x} \,|\, 1 - x_1 \leq 0\}$$


$\psi = \frac{3\pi}{4}$
$\psi = 0$
$\psi = \frac{7\pi}{8}$
$\psi = \pi$
$\psi = -\frac{7\pi}{8}$
$\psi = -\frac{\pi}{2}$
$\psi = -\frac{3\pi}{4}$
4
3
2
1
0
−1
−2
−3
−4
0
2
4
6
8
10


$\psi=\pi$
$\psi=\frac{7\pi}{8}$
$\psi=\frac{3\pi}{4}$
$\psi=0$
$\psi=-\frac{\pi}{2}$
$\psi=-\frac{3\pi}{4}$
$\psi=-\frac{7\pi}{8}$
$x=-1$
$x=1$

Three problems

Three problems:

- Safety : the sailboat never goes upwind. [1]
- Capture : The boat will be captured by its corridor [2]
- Characterize of the sliding surface.

Safety

Show that we never have

$$h(\mathbf{x},q) \leq 0.$$

Since $\mathbf{f}_a$ and $\mathbf{f}_b$ are continuous, we have

$$\begin{cases} h(\mathbf{x},q) = h_a(\mathbf{x},q) & \text{if } \mathbf{x} \in \mathbb{A} \\ \quad\quad\quad\;\; = h_b(\mathbf{x},q) & \text{if } \mathbf{x} \in \mathbb{B} \end{cases}$$

where h_a, h_b are continuous.

Proposition 1. If

$$\begin{array}{rcl}\mathbb{H} & = & \cup_{q\in\{-1,1\}}\left(\{\mathbf{x}|h_a(\mathbf{x},q)\leq 0)\}\cap\mathbb{A}\cap\overline{\mathbb{Q}^{-q}}\right) \\ & & \cup\left(\{\mathbf{x}|h_b(\mathbf{x},q)\leq 0)\}\cap\mathbb{B}\cap\overline{\mathbb{Q}^{-q}}\right)\end{array}$$

is empty then we cannot have $h(\mathbf{x},q)\leq 0$.

We want to prove that we never have

$$h(\mathbf{x},q)=\cos(x_2-u)+\cos\frac{\pi}{5}\leq 0.$$

with

$$\begin{cases} u = \pi + x_2 - q\frac{\pi}{4} & \text{if } \mathbf{x}\in\mathbb{A} \\ = -\operatorname{atan} x_1 & \text{otherwise} \end{cases}$$

The set $\mathbb{H}$ has no solution. We conclude that the forbidden constraint $\cos(x_2-u)+\cos\frac{\pi}{5}\leq 0$ is never reached.

```
c1=Function("x1","x2","cos(atan(x1)+x2)+sqrt(2)/2")
c2=Function("x1","x2","x1^2-1")
c3=Function("x1","x2","cos(x2)+sqrt(2)/2")
A1=SepFwdBwd(c1,[-oo,0])
A2=SepFwdBwd(c2,[-oo,0])
A3=SepFwdBwd(c3,[-oo,0])
A=A1|A2&A3
B=~A
Hb=SepFwdBwd(Function("x1","x2",
"cos(x2+atan(x1))+cos(2*asin(1)/5)"),[-oo,0])
H=Hb&B
pySIVIA(H)
```

Capture

The set $\mathbb{C} = \{\mathbf{x} | V(\mathbf{x}) \leq 0\}$, with $V : \mathbb{R}^n \to \mathbb{R}$ is a *capture set* if all trajectories that enter inside $\mathbb{C}$ stays inside forever.

The *Lie derivative* of V with respect to $\mathbf{f}$ is

$$\mathscr{L}_{\mathbf{f}}^{V}(\mathbf{x}) = \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

The Lie set as

$$\mathbb{L}_{\mathbf{f}}^{V} = \left\{ \mathbf{x} | \mathscr{L}_{\mathbf{f}}^{V}(\mathbf{x}) \leq 0 \right\}.$$

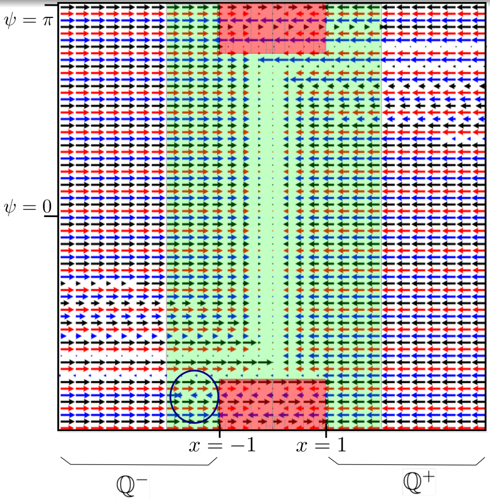
Proposition 2. Define the set

$$\mathbb{V}=\bigcup_{q\in\{-1,1\}}\left(\overline{\mathbb{L}_a^V(q)}\cap\mathbb{A}\cap\overline{\mathbb{Q}^{-q}}\right)\cup\left(\overline{\mathbb{L}_b^V(q)}\cap\mathbb{B}\cap\overline{\mathbb{Q}^{-q}}\right)$$

If $\mathbb{V}\cap\overline{\mathbb{C}}=\emptyset$ then $\mathbb{C}$ is a capture set.

Take $V(\mathbf{x}) = x_1^2 - 4$. We have

$$\begin{array}{rcccl} \mathscr{L}_a^V(\mathbf{x},q) & = & \frac{dV}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x},q) & = & 2x_1\cdot\sin(\frac{q\pi}{4}-x_2) \\ \mathscr{L}_b^V(\mathbf{x}) & = & \frac{dV}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & \frac{-2x_1^2}{\sqrt{x_1^2+1}} \end{array}$$



Fields $\mathbf{f}_a(\mathbf{x}, q)$, $\mathbf{f}_b(\mathbf{x})$, the sets $\mathbb{C}$ (green) and $\mathbb{V}$ (red)

Sliding surface

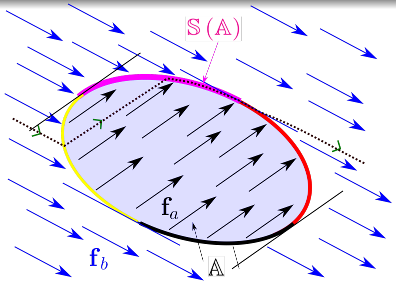
The *sliding surface* $\mathbb{S}(\mathbb{A})$ for $\mathscr{S}(\mathbb{A})$ is largest subset of $\partial\mathbb{A}$ such that the state can slide inside for a non degenerated interval of time.

If $\mathbb{A}$:$c(\mathbf{x}) \leq 0$, then

$$\begin{array}{rcl} \mathbb{S}(\mathbb{A}) & = & \partial\mathbb{A} \cap \left\{\mathbf{x} \mid \exists q, \mathbf{x} \in \overline{\mathbb{Q}^{-q}}, \mathscr{L}_a^c(\mathbf{x},q) \geq 0 \wedge \mathscr{L}_b^c(\mathbf{x},q) \leq 0\right\} \\ & = & \partial\mathbb{A} \cap \bigcup_{q\in\{-1,1\}} \overline{\mathbb{Q}^{-q}} \cap \overline{\mathbb{L}_a^V(q)} \cap \mathbb{L}_b^V(q) \end{array}$$

Without the discrete variable q.

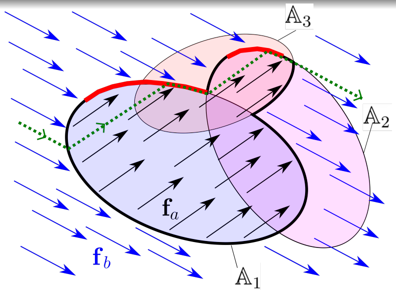
$$\mathbb{S}(\mathbb{A}) = \partial\mathbb{A} \cap \{\mathbf{x} \,|\, \mathscr{L}_a^c(\mathbf{x}) \geq 0 \wedge \mathscr{L}_b^c(\mathbf{x}) \leq 0\}.$$



Sliding set $\mathbb{S}(\mathbb{A})$ (magenta) for $\mathbb{A} = \{\mathbf{x} | c(\mathbf{x}) \leq 0\}$

Proposition 3. If we have two closed sets $\mathbb{A}_1$ and $\mathbb{A}_2$. We have

$$\begin{array}{llll} (i) & \mathbb{S}(\mathbb{A}_1 \cap \mathbb{A}_2) & = & (\mathbb{S}(\mathbb{A}_1) \cap \mathbb{A}_2) \cup (\mathbb{S}(\mathbb{A}_2) \cap \mathbb{A}_1) \\ (ii) & \mathbb{S}(\mathbb{A}_1 \cup \mathbb{A}_2) & = & \left(\mathbb{S}(\mathbb{A}_1) \cap \text{clo}\overline{\mathbb{A}_2}\right) \cup \left(\mathbb{S}(\mathbb{A}_2) \cap \text{clo}\overline{\mathbb{A}_1}\right) \end{array}$$



$$\mathbb{S}(\mathbb{A}_1 \cup (\mathbb{A}_2 \cap \mathbb{A}_3))$$

For our boat, the sliding surface for $\mathbb{A}_i : c_i(\mathbf{x}) \leq 0$ is

$$\begin{array}{rcl} \mathbb{S}(\mathbb{A}_i) & = & \partial\mathbb{A}_i \cap \bigcup_{q\in\{-1,1\}} \overline{\mathbb{Q}^{-q}} \cap \overline{\mathbb{L}_a^i(q)} \cap \mathbb{L}_b^i \\ & = & \partial\mathbb{A}_i \cap \mathbb{L}_b^i \cap \left(\overline{\mathbb{L}_a^i(1)} \cap \overline{\mathbb{Q}^-} \cup \overline{\mathbb{L}_a^i(-1)} \cap \overline{\mathbb{Q}^+} \right) \end{array}$$

where

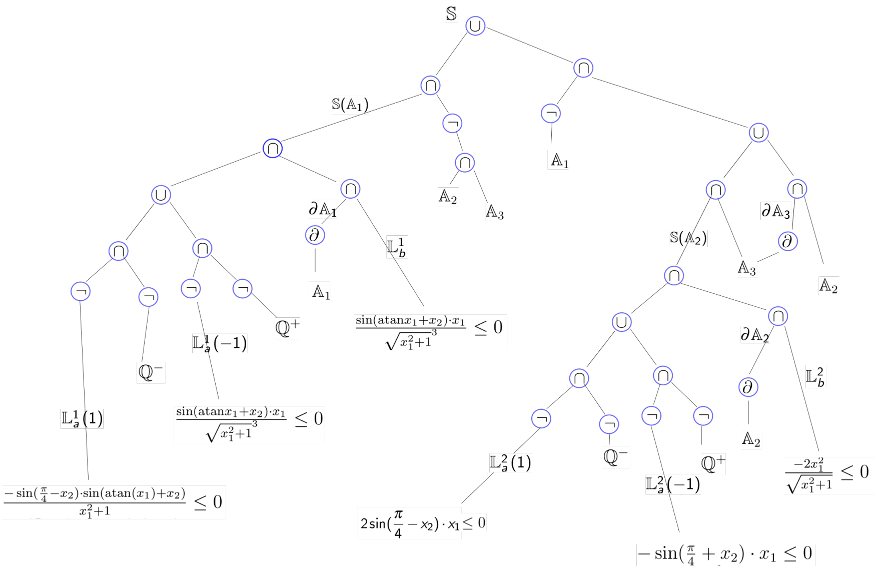
$$\begin{array}{rcl} \mathbb{L}_a^i(q) & = & \{\mathbf{x} | \mathscr{L}_a^{c_i}(\mathbf{x}, q) \leq 0\} \\ \mathbb{L}_b^i & = & \left\{\mathbf{x} | \mathscr{L}_b^{c_i}(\mathbf{x}) \leq 0\right\} \end{array}$$

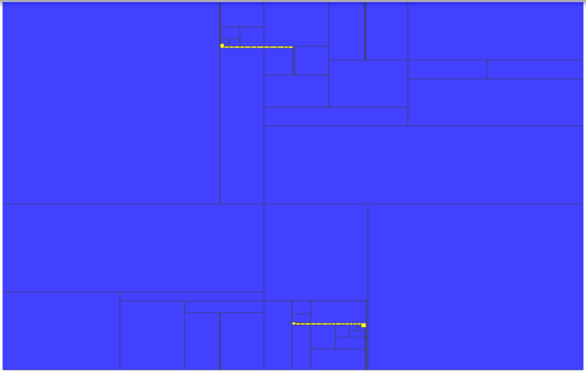
$$
\begin{array}{rclcl}
\mathcal{L}_a^{c_1}(\mathbf{x},q) & = & \frac{dc_1}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x},q) & = & \frac{-\sin(\frac{q\pi}{4}-x_2)\cdot\sin(\operatorname{atan}(x_1)+x_2)}{x_1^2+1} \\
\mathcal{L}_b^{c_1}(\mathbf{x}) & = & \frac{dc_1}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & \frac{\sin(\operatorname{atan}x_1+x_2)\cdot x_1}{\sqrt{x_1^2+1}^3} \\
\mathcal{L}_a^{c_2}(\mathbf{x},q) & = & \frac{dc_2}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x}) & = & 2\sin(\frac{q\pi}{4}-x_2)\cdot x_1 \\
\mathcal{L}_b^{c_2}(\mathbf{x}) & = & \frac{dc_2}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & \frac{-2x_1^2}{\sqrt{x_1^2+1}} \\
\mathcal{L}_a^{c_3}(\mathbf{x},q) & = & \frac{dc_3}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x},q) & = & 0 \\
\mathcal{L}_b^{c_3}(\mathbf{x}) & = & \frac{dc_3}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & 0
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{S}(\mathbb{A}_1) & = & \partial\mathbb{A}_1\cap\mathbb{L}_b^1\cap\left(\overline{\mathbb{L}_a^1(1)}\cap\overline{\mathbb{Q}^-}\cup\overline{\mathbb{L}_a^1(-1)}\cap\overline{\mathbb{Q}^+}\right) \\
\mathbb{S}(\mathbb{A}_2) & = & \partial\mathbb{A}_2\cap\mathbb{L}_b^2\cap\left(\overline{\mathbb{L}_a^2(1)}\cap\overline{\mathbb{Q}^-}\cup\overline{\mathbb{L}_a^2(-1)}\cap\overline{\mathbb{Q}^+}\right) \\
\mathbb{S}(\mathbb{A}_3) & = & \partial\mathbb{A}_3
\end{array}
$$

```
La1Q=Function("x1","x2","-sin(asin(1)/2-x2)*sin(atan(x1)
La1R=Function("x1","x2","sin(asin(1)/2+x2)*sin(atan(x1)+
Lb1=Function("x1","x2","sin(atan(x1)+x2)*x1/sqrt((x1^2+1
La2Q=Function("x1","x2","2*sin(asin(1)/2-x2)*x1")
La2R=Function("x1","x2","-2*sin(asin(1)/2-x2)*x1")
Lb2=Function("x1","x2","-2*x1^2/sqrt(x1^2+1)")
dA1=A1&~A1
SLa1Q=SepFwdBwd(La1Q,[-oo,0])
SLa1R=SepFwdBwd(La1R,[-oo,0])
SLb1=SepFwdBwd(Lb1,[-oo,0])
S1=dA1 & SLb1 & ((~SLa1Q)&~R | (~SLa1R)&~Q)
dA2=A2&~A2
SLa2Q=SepFwdBwd(La2Q,[-oo,0])
SLa2R=SepFwdBwd(La2R,[-oo,0])
SLb2=SepFwdBwd(Lb2,[-oo,0])
S2=dA2 & SLb2 & ((~SLa2Q)&~R | (~SLa2R)&~Q)
```

```
dA3=A3&~A3
S23=S2&A3|dA3&A2
S=S1&~(A2&A3) | S23&~A1
```


𝕊
𝕊(𝔸₁)
∂𝔸₁
𝔸₁
𝔸₂
𝔸₃
𝕃¹_b
𝕃¹_a(−1)
𝕃¹_a(1)
ℚ⁺
ℚ⁻
sin(atan x₁+x₂)·x₁ / √(x₁²+1)³ ≤ 0
− sin(π/4 − x₂)·sin(atan(x₁)+x₂) / (x₁²+1) ≤ 0
𝕊(𝔸₂)
∂𝔸₃
∂𝔸₂
𝕃²_b
𝕃²_a(1)
𝕃²_a(−1)
2 sin(π/4 − x₂)·x₁ ≤ 0
− sin(π/4 + x₂)·x₁ ≤ 0
−2x₁² / √(x₁²+1) ≤ 0



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