The role of preconditioning for contractors

October 2, 2024

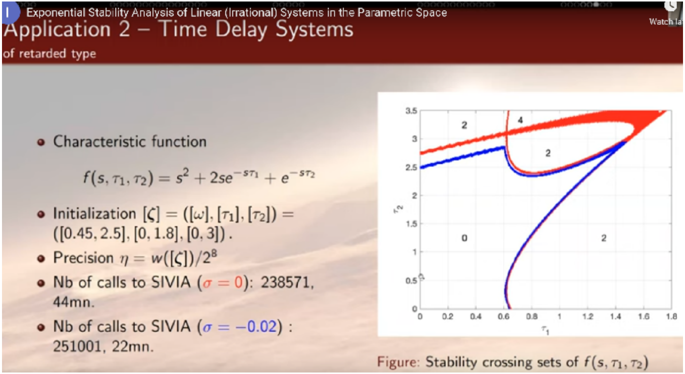
Abstract. Centered form is traditionally used to enclose the range of a function over narrow intervals. The quadratic approximation property guarantees an asymptotically small overestimation for sufficiently narrow boxes. In this presentation, I will show how the centered form can be included efficiently inside a propagation process to get more precise contractions. I will also show the fundamental role of the preconditioning.

1. Introduction

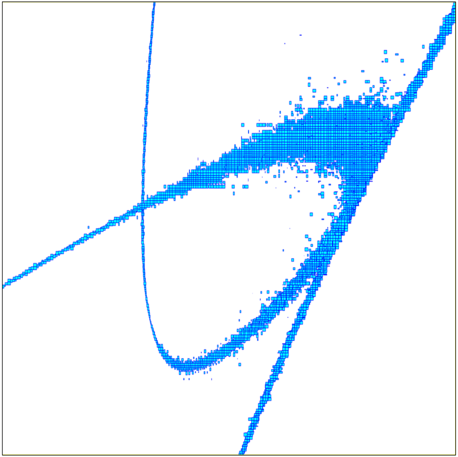
Take

$$\Leftrightarrow \quad \underbrace{\begin{pmatrix} -x_3^2 + 2x_3 \sin(x_3 x_1) + \cos(x_3 x_2) \\ 2x_3 \cos(x_3 x_1) - \sin(x_3 x_2) \end{pmatrix}}_{\mathbf{f}(x_1,x_2,x_3)} = \mathbf{0}$$

With $[x_1] = [0, 2.5]$, $[x_2] = [1, 4], [x_3] = [0, 10]$, with a Matlab implementation, with a forward-backward contractor, and $\varepsilon = 2^{-8}$, [?] got:



https://youtu.be/DaR2NZZIV10?t=2453



$\varepsilon = 2^{-8}$, Codac [4] generated 43173 boxes.

We still have a *Clustering effect*

Centered constraint

$$\left\{\begin{array}{c}\mathbf{f}(\mathbf{x})=\mathbf{0}\\ \mathbf{x}\in[\mathbf{x}]\\ \mathbf{m}=\mathsf{center}([\mathbf{x}])\end{array}\right. \Rightarrow \left\{\begin{array}{c}\mathbf{f}(\mathbf{m})+\mathbf{A}\cdot(\mathbf{x}-\mathbf{m})=\mathbf{0}\\ \mathbf{A}\in[\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}])\\ \mathbf{x}\in[\mathbf{x}]\end{array}\right.$$

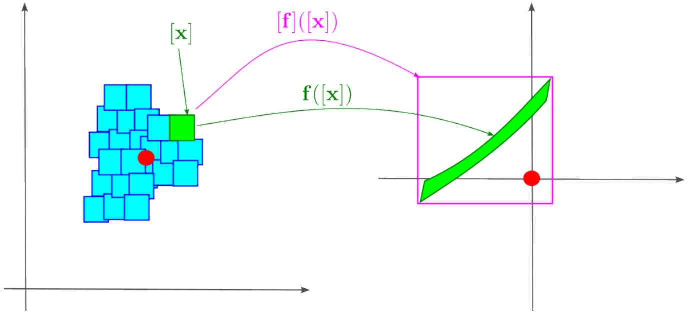
Centered constraint with preconditioning

$$\left\{\begin{array}{c} \mathbf{f}(\mathbf{x}) = \mathbf{0} \\ \mathbf{x} \in [\mathbf{x}] \\ \mathbf{m} = \mathsf{center}([\mathbf{x}]) \end{array}\right. \Rightarrow \left\{\begin{array}{c} \mathbf{Q} \cdot \mathbf{f}(\mathbf{m}) + \mathbf{Q} \cdot \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) = \mathbf{0} \\ \mathbf{A} \in [\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}]) \\ \mathbf{x} \in [\mathbf{x}] \end{array}\right.$$

2. Minimal contractors

Given a function $\mathbf{f}: \mathbb{R}^n \mapsto \mathbb{R}^p$. An inclusion function for $\mathbf{f}$ is minimal if

$$[\mathbf{f}]([\mathbf{x}]) = [\![\{\mathbf{y} = \mathbf{f}(\mathbf{x}) \,|\, \mathbf{x} \in [\mathbf{x}]\}]\!].$$

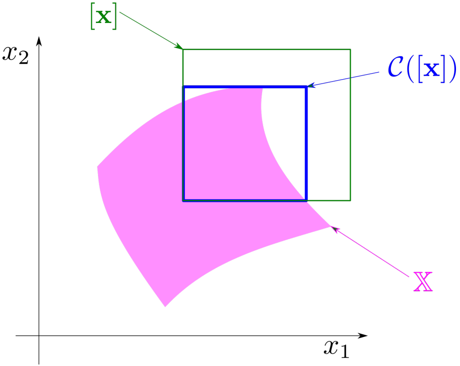


With a minimal inclusion, the clustering effect may exist, when solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$

A *contractor* associated to the set $\mathbb{X} \subset \mathbb{R}^n$ is a function $\mathscr{C} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$\begin{array}{cl} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contraction)} \\ [\mathbf{x}] \cap \mathbb{X} \subset \mathscr{C}([\mathbf{x}]) & \text{(consistency)} \end{array}$$

It is *minimal* if $\mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$.


$[\mathbf{x}]$
x_2
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$
x_1

Tree matrices

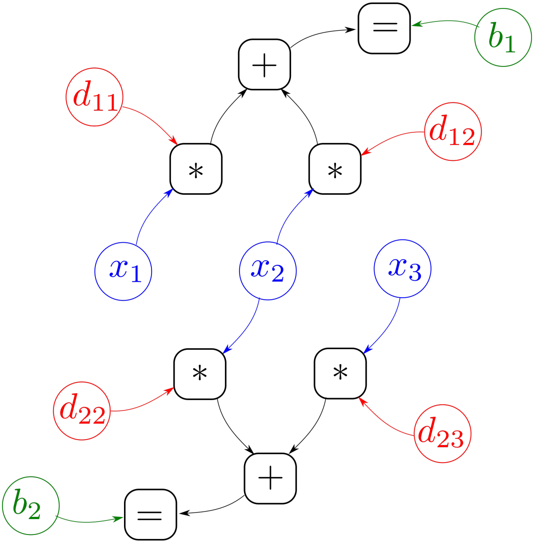
Consider the interval linear system:

$$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & d_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where

$$d_{ij} \in [d_{ij}], x_j \in [x_j], b_i \in [b_i]$$

The optimal contraction can be obtained by a simple interval propagation [1].


=
b_1
+
d_{11}
d_{12}
*
*
x_1
x_2
x_3
*
*
d_{22}
d_{23}
+
b_2
=

No cycle for:

$$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & 0 & d_{33} & d_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

A matrix $\mathbf{D}$ such that $\mathbf{D} \cdot \mathbf{x} = \mathbf{b}$ has no cycle is a *tree matrix*.

With a Gauss Jordan transformation:

$$\mathbf{A}\mathbf{x} = \mathbf{c} \Leftrightarrow \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{Q} \cdot \mathbf{c}$$

we may get a tree matrix: $\mathbf{D} = \mathbf{Q} \cdot \mathbf{A}$.

Simplex contractor

For the linear system

$$\mathbf{Ax} = \mathbf{c}, \mathbf{x} \in [\mathbf{x}], \mathbf{c} \in [\mathbf{c}]$$

we can use the simplex algorithm to build the minimal contractor. Guarantee can be obtained with an inflation [3]

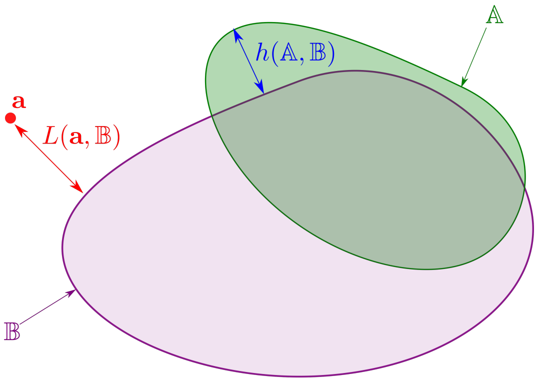
3. Asymptotic minimality

Proximity. Denote by $L(\mathbf{a},\mathbf{b})$ a distance between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{R}^n$ induced by the L-norm (L_∞ or L_2).
The *proximity* of $\mathbb{A}$ to $\mathbb{B}$ is

$$h(\mathbb{A},\mathbb{B}) = \sup_{\mathbf{a}\in\mathbb{A}} L(\mathbf{a},\mathbb{B})$$

where

$$L(\mathbf{a},\mathbb{B}) = \inf_{\mathbf{b}\in\mathbb{B}} L(\mathbf{a},\mathbf{b}).$$



Proximity of $\mathbb{A}$ to $\mathbb{B}$

Definition. The pessimism of an inclusion function $[\mathbf{f}]$ is

$$\eta([\mathbf{x}]) = h([\mathbf{f}]([\mathbf{x}]), [\![\mathbf{f}([\mathbf{x}])]\!])$$

Definition [2]. An inclusion function $[\mathbf{f}]$ is of order j if

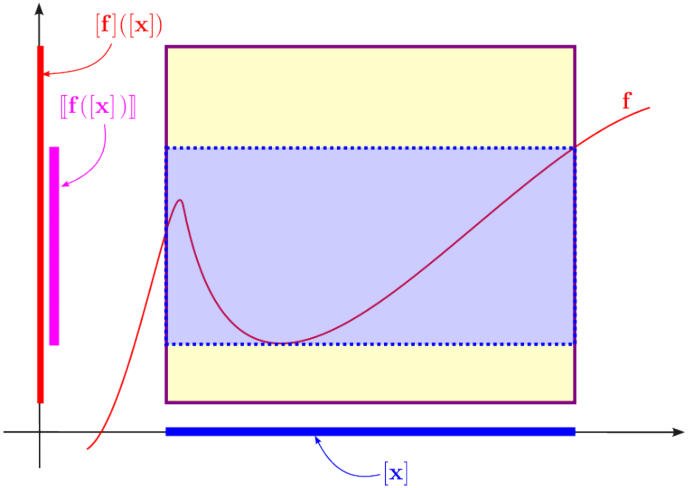
$$\eta([\mathbf{x}]) = o(w^j([\mathbf{x}]))$$

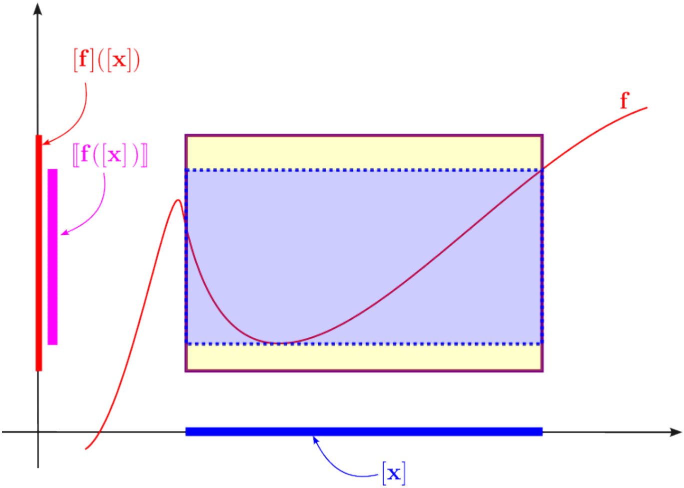
Definition. $[\mathbf{f}]$ is convergent if it is of order $j = 0$:

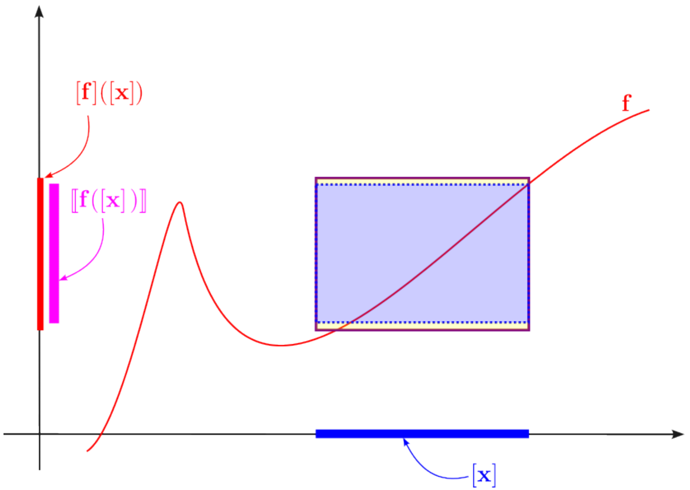
$$\eta([\mathbf{x}]) = o(w^0([\mathbf{x}])) = O(w([\mathbf{x}]))$$

Definition. $[\mathbf{f}]$ is asymptotically minimal if it is of order $j = 1$:

$$\eta([\mathbf{x}]) = o(w([\mathbf{x}]))$$


[f]([x])
[[f([x])]]
f
[x]


$[f]([x])$
$[\![f([x])]\!]$
f
$[x]$


[f]([x])
[f([x])]
f
[x]

Proposition [2]. The centered form

$$[\mathbf{f}]([\mathbf{x}]) = \mathbf{f}(\mathbf{m}) + [\mathbf{f}']([\mathbf{x}]) \cdot ([\mathbf{x}] - \mathbf{m})$$

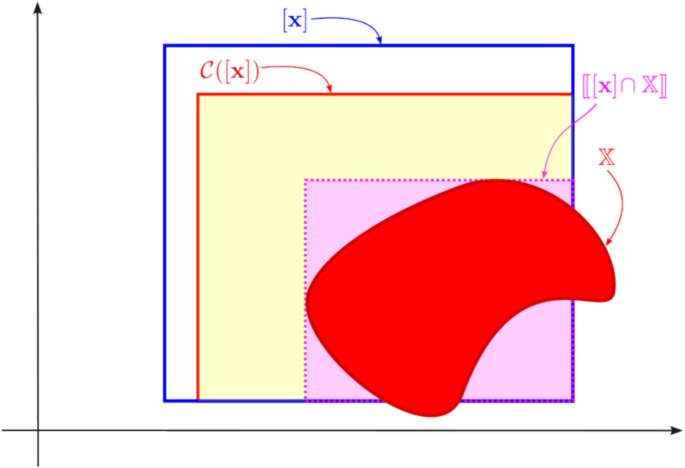
where $\mathbf{m} = \text{center}([\mathbf{x}])$ is asymptotically minimal.

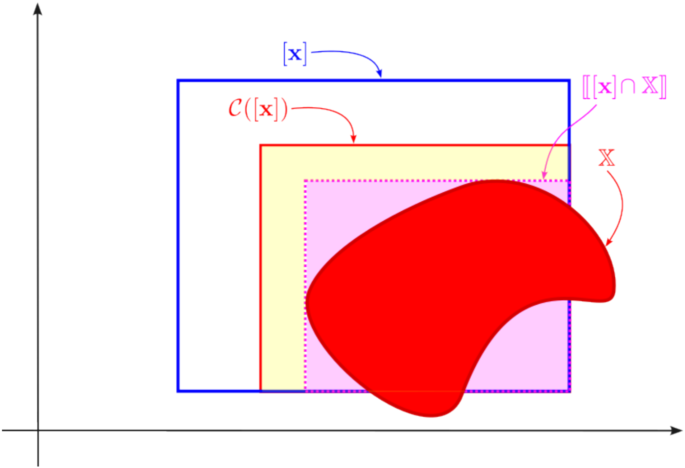
Definition. The *pessimism* of a contractor $\mathscr{C}$ for $\mathbb{X}$ at $[\mathbf{x}]$ is

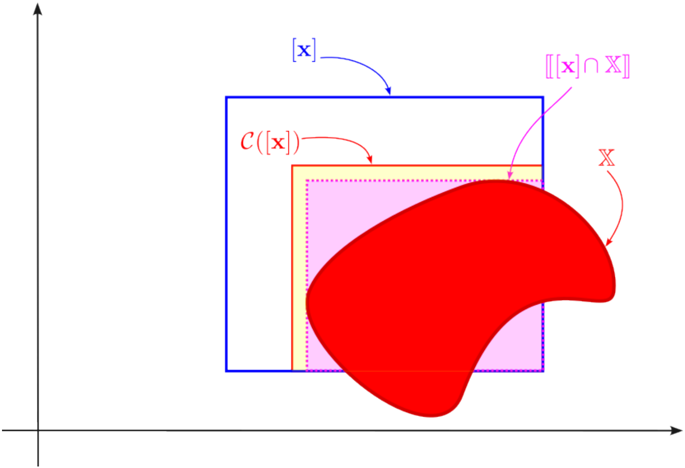
$$\eta([\mathbf{x}]) = h(\mathscr{C}([\mathbf{x}]), [\![[\mathbf{x}] \cap \mathbb{X}]\!])$$

Definition. A contractor $\mathscr{C}$ for $\mathbb{X}$ is of order j if

$$\eta([\mathbf{x}]) = o(w^j([\mathbf{x}]))$$


$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
$[[\mathbf{x}] \cap \mathbb{X}]$
$\mathbb{X}$


$[\mathbf{x}]$
$[[\mathbf{x}] \cap \mathbb{X}]$
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$

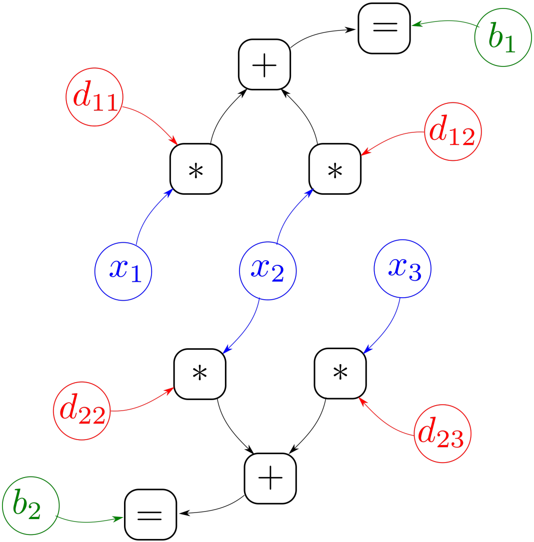

$[\mathbf{x}]$
$[[\mathbf{x}] \cap \mathbb{X}]$
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$

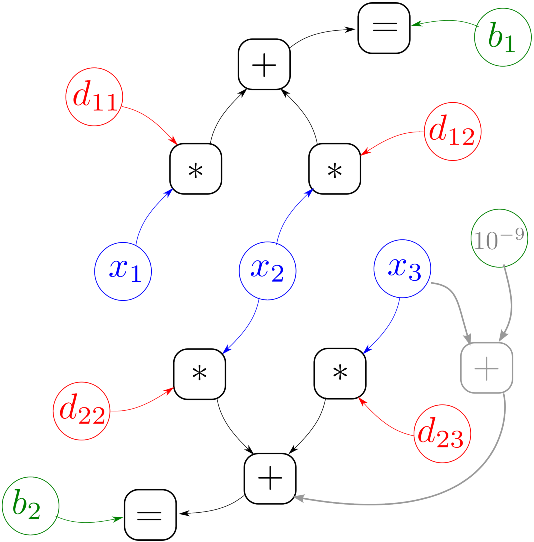
Proposition. Consider a set $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$. Take $[\mathbf{x}]$ with center $\mathbf{m}$. Define $\mathbf{Q}$ s.t. $\mathbf{Q} \cdot \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$ is a tree matrix.
An interval propagation on;

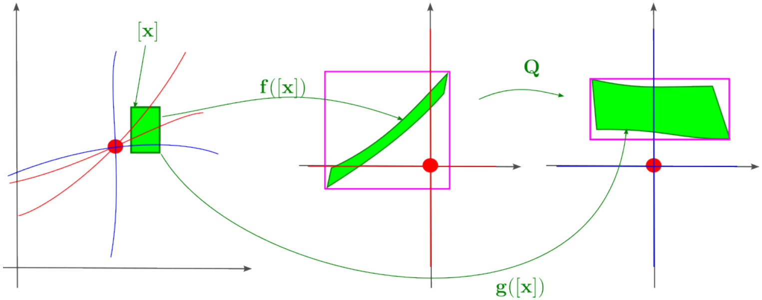
$$\begin{gathered} \mathbf{Q} \cdot \mathbf{f}(\mathbf{m}) + \mathbf{Q} \cdot \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) = \mathbf{0} \\ \mathbf{A} \in [\tfrac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}]) \\ \mathbf{x} \in [\mathbf{x}] \end{gathered}$$

yields an asymptotically minimal contractor for $\mathbb{X}$.

Proof. ...


$=$
b_1
$+$
d_{11}
d_{12}
$*$
$*$
x_1
x_2
x_3
$*$
$*$
d_{22}
d_{23}
$+$
b_2
$=$


=
b_1
+
d_{11}
d_{12}
*
*
10^{-9}
x_1
x_2
x_3
*
*
+
d_{22}
d_{23}
+
b_2
=

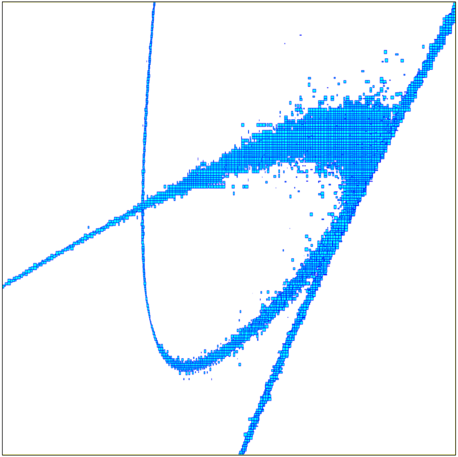

[x]
f([x])
Q
g([x])

Centered contractor

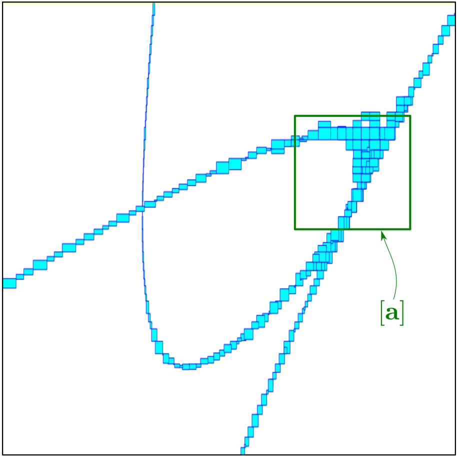
Input: $\mathsf{f}, [\mathbf{x}]$

$\mathbf{m} = \mathrm{center}([\mathbf{x}])$
Compute the Gauss-Jordan matrix $\mathbf{Q}$ for $\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$
Define $\mathbf{g}(\mathbf{x}) = \mathbf{Q} \cdot \mathbf{f}(\mathbf{x})$
For $i \in \{1, \ldots, p\}$
  For $j \in \{1, \ldots, n\}$
    $[\mathbf{a}] = [\frac{\partial g_i}{\partial \mathbf{x}}]([\mathbf{x}])$
    $[s] = \sum_{k \neq j} [a_k] \cdot ([x_k] - m_k)$
    $[x_j] = [x_j] \cap (-g_i(\mathbf{m}) - [s])$
Return $[\mathbf{x}]$

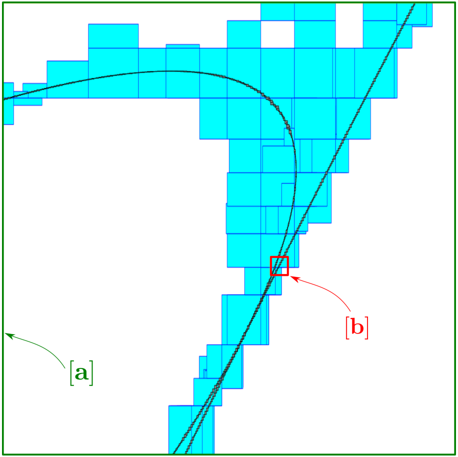
4. Results



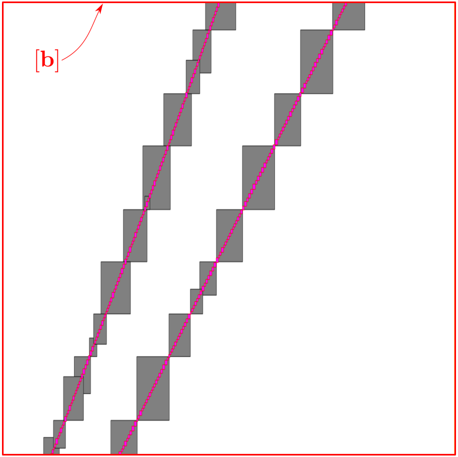
With a forward-backward contractor and $\varepsilon = 2^{-8}$



With the centered contractor $\varepsilon = 2^{-4}$



Blue: $\varepsilon = 2^{-4}$; Thin: $\varepsilon = 2^{-8}$



Gray: $\varepsilon = 2^{-8}$; Magenta: $\varepsilon = 2^{-12}$

Contributions

Notion of asymptotic minimal contractor
Link between the preconditioning and acyclic constraint networks
Better results than the basic affine arithmetic
No use of guaranteed linear programming

Perspectives

Compare with modern affine arithmetic
Improve the tree preconditioning
Use linear programming with an order 1 inflation
Sequential centered contractor (for state estimation)

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