Computing positive invariant tubes

Workshop Contredo, Nice, 3 mars 2017

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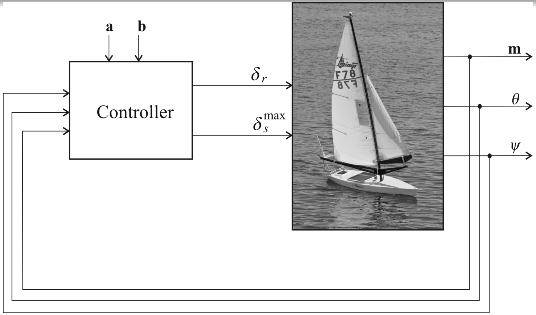
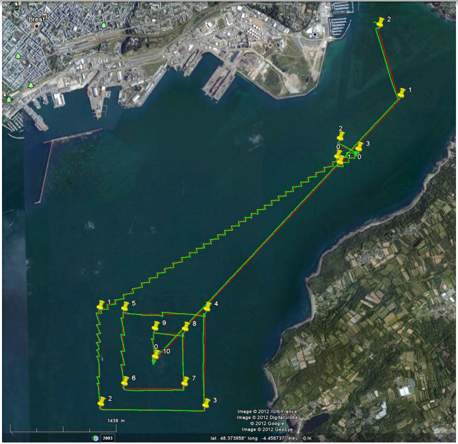
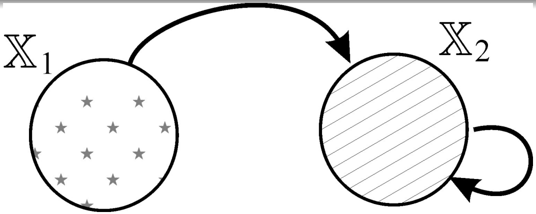
V-stability


F78


a
b
Controller
δ_r
$\delta_s^{\max}$
m
θ
ψ


Brest
2
1
2
0
3
1
0
1
5
4
9
8
0
10
6
7
2
3
1438 m
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
2003
lat 48.373958° long -4.458737° élév 0 m



$\mathbb{X}_1$: outside the corridor, $\mathbb{X}_2$: inside the corridor.

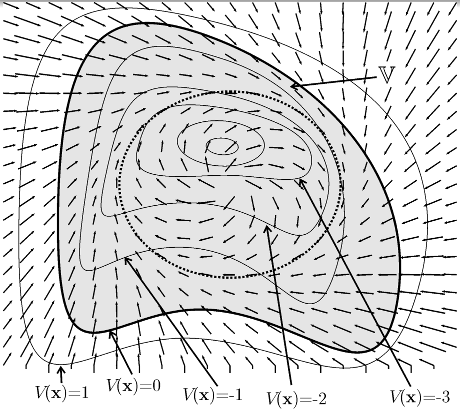
Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

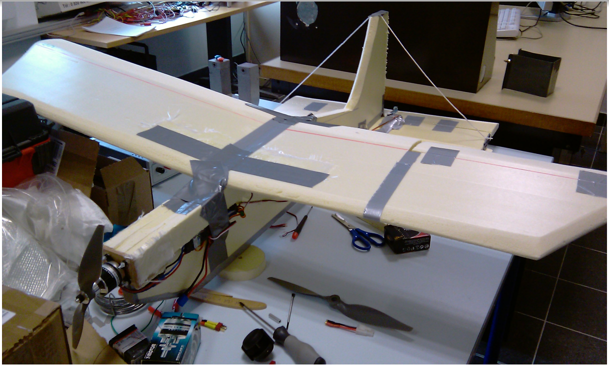
$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0 \right).$$

Since

$$\dot{V}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

Checking the V-stability can be done using using interval analysis.

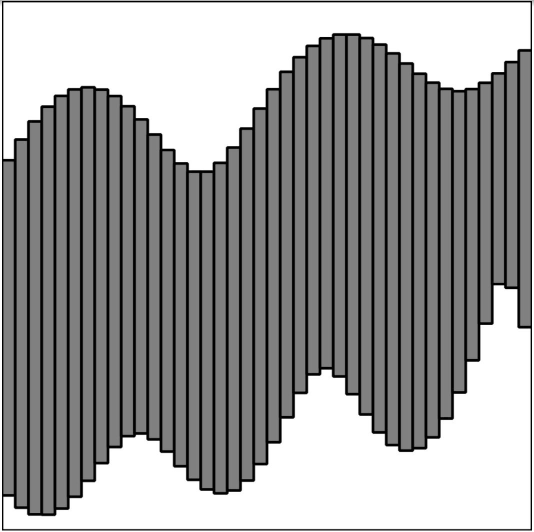

$\mathbb{V}$
$V(\mathbf{x})$=1
$V(\mathbf{x})$=0
$V(\mathbf{x})$=-1
$V(\mathbf{x})$=-2
$V(\mathbf{x})$=-3



Non-holonomic system

Tubes

A tube is a function which associates to any $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$.



Example of tubes

$$
\begin{aligned}
[f](t) &= [1,2]\cdot t + \sin([1,3]\cdot t) \\
[g](t) &= [a_0] + [a_1]\,t + [a_2]\,t^2 + [a_3]\,t^3 \\
\int_0^t [g](\tau)\,d\tau &= [a_0]\,t + [a_1]\,\frac{t^2}{2} + [a_2]\,\frac{t^3}{3} + [a_3]\,\frac{t^4}{4}.
\end{aligned}
$$

Positive invariant tubes

Consider the time dependant system

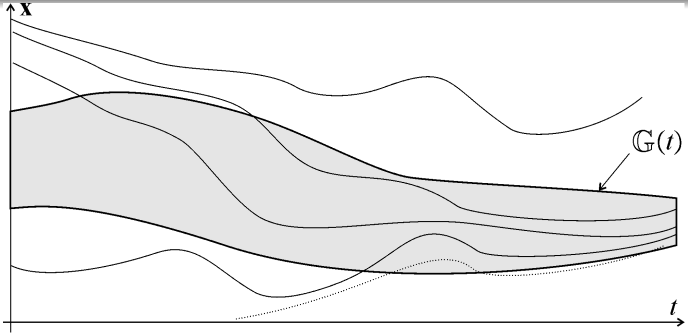
$$\mathscr{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

and a *tube*

$$\mathbb{G}(t) \subset \mathbb{R}^n, t \in \mathbb{R}.$$

The tube $\mathbb{G}(t)$ is said to be a *positive invariant* if

$$\mathsf{x}(t) \in \mathbb{G}(t), \tau > 0 \Rightarrow \mathsf{x}(t+\tau) \in \mathbb{G}(t+\tau).$$


x
$\mathbb{G}(t)$
t

Theorem. Consider the tube

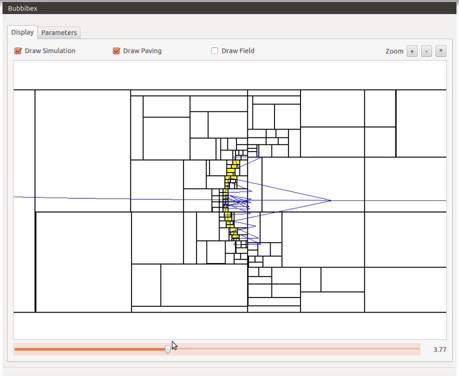
$$\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$$

where $\mathbf{g} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$. If the *cross out* condition

$$\begin{cases} \underbrace{\frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t).\mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t)}_{\dot{g}_i(\mathbf{x}, t)} \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases}$$

is inconsistent for all $(\mathbf{x}, t, i)$, then $\mathbb{G}(t)$ is a capture tube for $\mathscr{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

A software Bubbibex (using Ibex) made by students from ENSTA Bretagne for MBDA uses interval analysis to prove the inconsistency.


Bubbibex
Display
Parameters
Draw Simulation
Draw Paving
Draw Field
Zoom
3.77

Lattice and capture tubes

Consider $\mathscr{S}: \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.
If $\mathbb{T}$ is the set of tubes and $\mathbb{T}_c$ is the set of all capture tubes of $\mathscr{S}$ then $(\mathbb{T}_c, \subset)$ is a sublattice of $(\mathbb{T}, \subset)$.

We have indeed

$$\left\{\begin{array}{lcl}\mathbb{G}_1(t) & \in & \mathbb{T}_c \\ \mathbb{G}_2(t) & \in & \mathbb{T}_c\end{array}\right. \Rightarrow \left\{\begin{array}{lcl}\mathbb{G}_1(t)\cap\mathbb{G}_2(t) & \in & \mathbb{T}_c \\ \mathbb{G}_1(t)\cup\mathbb{G}_2(t) & \in & \mathbb{T}_c\end{array}\right.$$

Computing Capture Tube

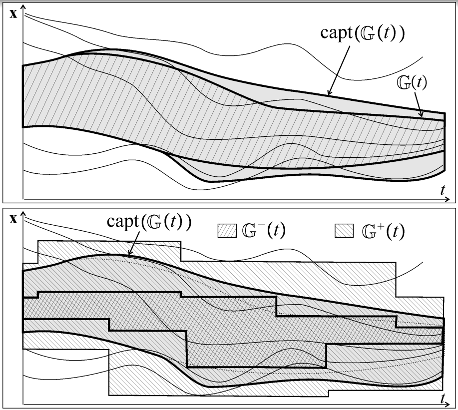
If $\mathbb{G}(t) \in \mathbb{T}$, define

$$\text{capt}\left(\mathbb{G}(t)\right) = \bigcap \left\{ \overline{\mathbb{G}}(t) \in \mathbb{T}_c \mid \mathbb{G}(t) \subset \overline{\mathbb{G}}(t) \right\}.$$

This set is the smallest capture tube enclosing $\mathbb{G}(t)$.

Problem. Given $\mathbb{G}(t) \in \mathbb{T}$, compute an interval $[\mathbb{G}^-(t), \mathbb{G}^+(t)] \in \mathbb{IT}$ such that

$$\text{capt}\,(\mathbb{G}(t)) \in \left[\mathbb{G}^-(t), \mathbb{G}^+(t)\right].$$


x
capt(𝔾(t))
𝔾(t)
t
x
capt(𝔾(t))
$\mathbb{G}^-(t)$
$\mathbb{G}^+(t)$
t

Flow. The flow associated with $\mathscr{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is a function $\phi_{t_0,t_1} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \Rightarrow \phi_{t_0,t_1}(\mathbf{x}(t_0)) = \mathbf{x}(t_1).$$

Proposition. For the system $\mathscr{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ and the tube $\mathbb{G}(t)$, we have

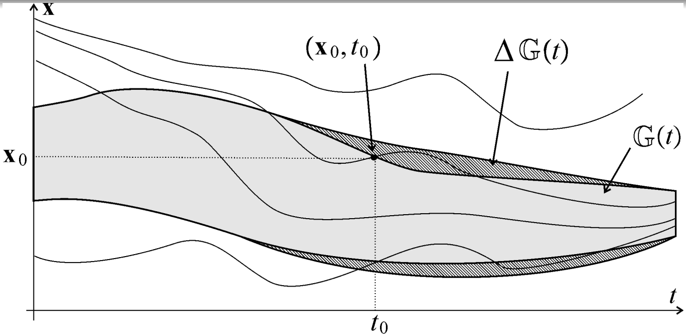
$$\text{capt}(\mathbb{G}(t)) = \mathbb{G}(t) \cup \Delta\mathbb{G}(t),$$

with

$$\Delta\mathbb{G}(t) = \left\{ (\mathbf{x}, t) \;\middle|\; \begin{array}{l} \exists (\mathbf{x}_0, t_0) \text{ satisfying the cross out condition} \\ t \geq t_0,\ \phi_{t_0,t}(\mathbf{x}_0) \notin \mathbb{G}(t) \end{array} \right\}.$$

Recall the cross out condition:

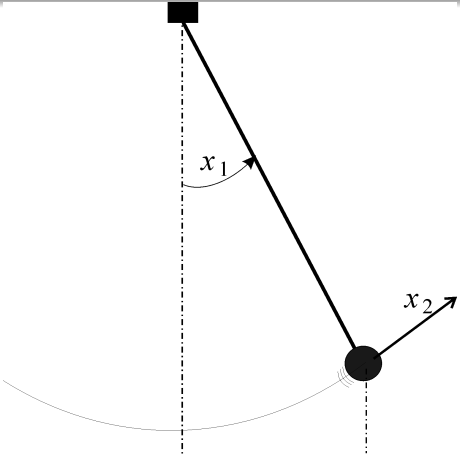
$$\left\{ \begin{array}{c} \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) . \mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{array} \right.$$


x
$(\mathbf{x}_0, t_0)$
$\Delta\,\mathbb{G}(t)$
$\mathbb{G}(t)$
$\mathbf{x}_0$
t
t_0

Pendulum

Pendulum:

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\sin x_1 - 0.15 \cdot x_2 \end{cases}$$


x_1
x_2

The energy

$$E(\mathbf{x}) = \frac{1}{2}\dot{x}_1^2 - \cos x_1 + 1 = \frac{1}{2}x_2^2 - \cos x_1 + 1$$

allows us to find candidate for the positive invariant tube:

$$g(\mathbf{x}, t) = E(\mathbf{x}) - 1 = \frac{1}{2}x_2^2 - \cos x_1.$$

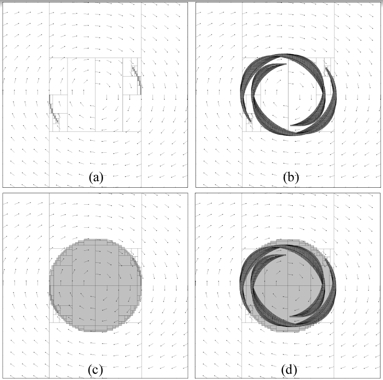
The cross-out conditions

$$\left\{ \begin{array}{ll} \text{(i)} & \begin{pmatrix} \sin x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -\sin x_1 - 0.15 \cdot x_2 \end{pmatrix} = -0.15 \cdot x_2^2 \geq 0, \\ \text{(ii)} & \frac{1}{2} x_2^2 - \cos x_1 = 0. \end{array} \right.$$

has two solutions: $\mathbf{x} = \left(\pm\frac{\pi}{2}, 0\right)$.

Without considering the energy, we consider, as an candidate tube:

$$g(\mathbf{x}, t) = x_1^2 + x_2^2 - 1.$$



(a) encloses the cross-out points; (b) integration $\Delta\mathbb{G}$ this set; (c) $\mathbb{C}^{-} \subset \text{Capt}(\mathbb{G})$; (d) $\mathbb{C}^{+} \supset \text{Capt}(\mathbb{G})$

Dubins car

Consider the Dubin's car

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = u \end{cases}$$

where $u \in [-2, 2]$.

To move toward the target (x_d, y_d), we take the controller:

$$\left\{\begin{array}{lll}\mathbf{n} & = & \frac{1}{\sqrt{(x_d-x)^2+(y_d-y)^2}}\left(\begin{array}{c}x_d-x\\ y_d-y\end{array}\right)+\frac{2}{\sqrt{\dot{x}_d^2+\dot{y}_d^2}}\left(\begin{array}{c}\dot{x}_d\\ \dot{y}_d\end{array}\right)\\ \theta_d & = & \operatorname{atan2}(\mathbf{n})\\ u & = & -2\cdot\sin(\theta-\theta_d).\end{array}\right.$$

Target

$$\begin{cases} x_d(t) & = & \rho_x \cos t \\ y_d(t) & = & \rho_y \sin t. \end{cases}$$

For the derivative, we get

$$\begin{cases} \dot{x}_d(t) & = & -\rho_x \sin t \\ \dot{y}_d(t) & = & \rho_y \cos t. \end{cases}$$

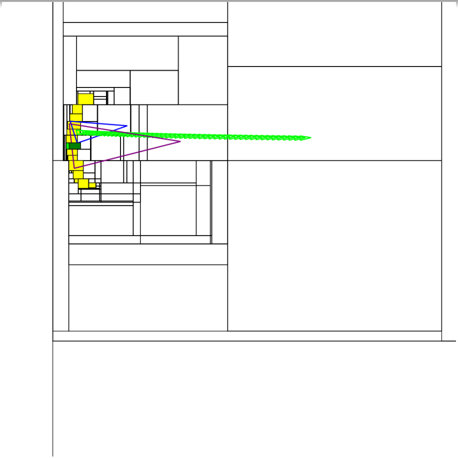
Target tube. We want the robot to stay inside the set

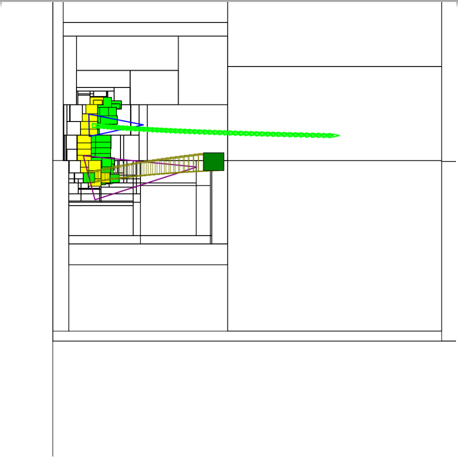
$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x},t) \leq \mathbf{0}\},$$

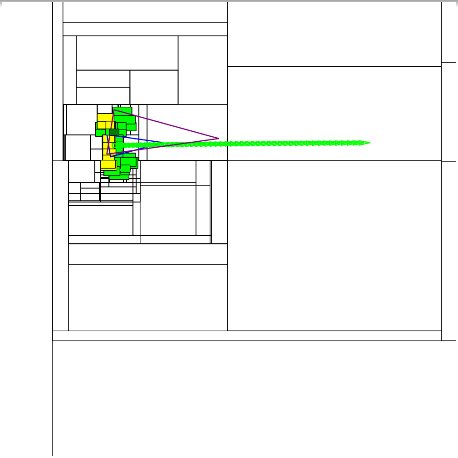
with

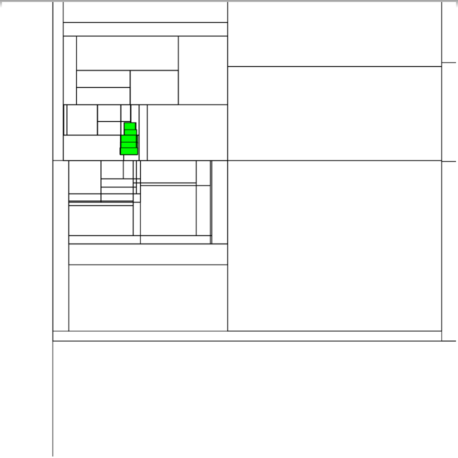
$$\begin{cases} g_1(\mathbf{x},t) & = & (x-x_d)^2 + (y-y_d)^2 - \rho^2 \\ g_2(\mathbf{x},t) & = & \left(\cos\theta - \frac{n_x}{\|\mathbf{n}\|}\right)^2 + \left(\sin\theta - \frac{n_y}{\|\mathbf{n}\|}\right)^2 - \alpha^2. \end{cases}$$

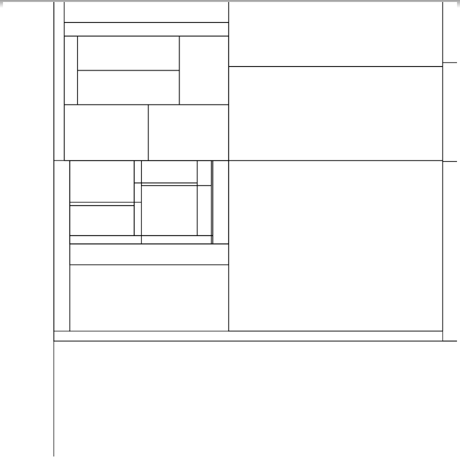
Resolution. We used the solver Bubbibex.
The tube is proved to be unsafe.
Bubbibex is able to compute the margin (*i.e.*, width($[\mathbb{G}^-(t), \mathbb{G}^+(t)]$)).

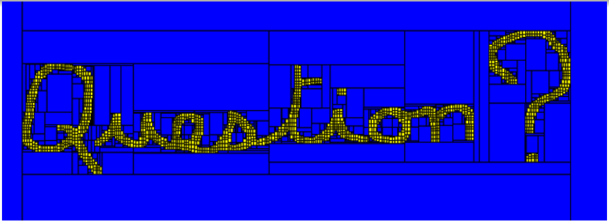











Question?

Reference.

L. Jaulin and F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE Transaction on Robotics, Volume 27, Issue 5.

L. Jaulin, D. Lopez, Le Doze, S. Le Menec, J. Ninin, G. Chabert, M. S. Ibnseddik, A. Stancu (2015), Computing capture tubes, Reliable Computing.