

Calibrating magnetometers and accelerometers

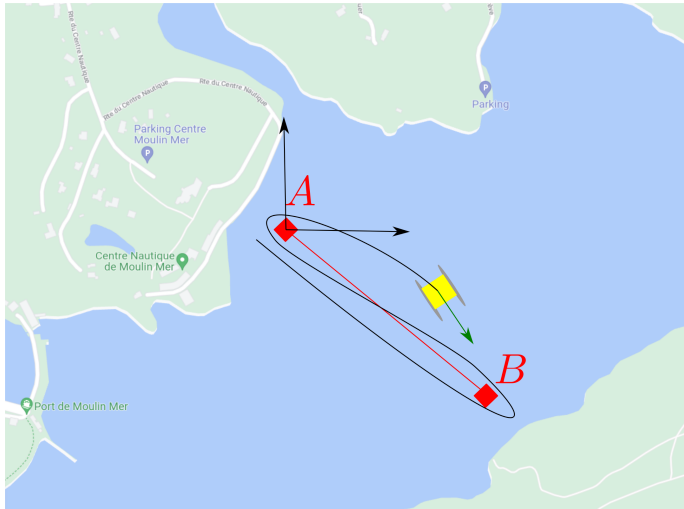
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1. Calibration of the magnetometers





In Brest, the inclination of the magnetic field is: $I = 64^\circ$.
The declination is 1.4° and its intensity is : $\beta = 46.0\mu T$.
See: franceipgp_Jussieu.html

For a dipole,

$$I = \arctan(2\tan\lambda)$$

For simplicity, we assume that the declination is 0, *i.e.*, the magnetic field is pointing toward the North.

We assume that

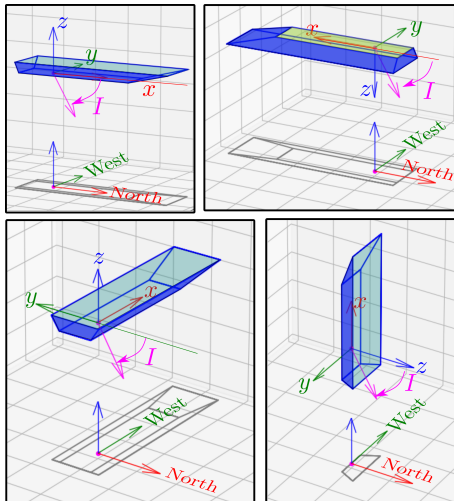
$$\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x} + \mathbf{b})$$

where $\mathbf{x} \in \mathbb{R}^3$ is the vector of magnetic measurements, and $\mathbf{y}$ is the corrected vector.

The calibration step searches for an estimation of $\mathbf{A}$ and $\mathbf{b}$.

We take four measurements $\mathbf{x}_N, \mathbf{x}_S, \mathbf{x}_W, \mathbf{x}_U$, with the robot

- Direction N : The robot is laid flat toward the North
- Direction S : The robot is laid toward the South, upside down
- Direction W : The robot is laid flat toward the West
- Direction U : The robot is set vertically with its z axis toward the North



The four corresponding transformation matrices are

$$\mathbf{R}_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{R}_W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}_U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The corrected magnetic field should be

$$\mathbf{y}_N = \begin{pmatrix} \beta \cdot \cos I \\ 0 \\ -\beta \cdot \sin I \end{pmatrix}, \quad \mathbf{y}_S = \mathbf{R}_S^T \cdot \mathbf{y}_N = \begin{pmatrix} -\beta \cdot \cos I \\ 0 \\ \beta \cdot \sin I \end{pmatrix}$$
$$\mathbf{y}_W = \mathbf{R}_W^T \cdot \mathbf{y}_N = \begin{pmatrix} 0 \\ -\beta \cdot \cos I \\ -\beta \cdot \sin I \end{pmatrix}, \quad \mathbf{y}_U = \mathbf{R}_U^T \cdot \mathbf{y}_N = \begin{pmatrix} -\beta \cdot \sin I \\ 0 \\ \beta \cdot \cos I \end{pmatrix}$$

where $\beta = 46\mu T$.

Since

$$\begin{aligned}\mathbf{0} &= \mathbf{y}_N + \mathbf{y}_S \\ &= \mathbf{A}^{-1}(\mathbf{x}_N + \mathbf{b}) + \mathbf{A}^{-1}(\mathbf{x}_S + \mathbf{b})\end{aligned}$$

we get

$$(\mathbf{x}_N + \mathbf{b}) + (\mathbf{x}_S + \mathbf{b}) = \mathbf{0}$$

i.e.,

$$\mathbf{b} = -\frac{1}{2}(\mathbf{x}_N + \mathbf{x}_S).$$

Moreover

$$\mathbf{A}\mathbf{y}_N = \mathbf{x}_N + \mathbf{b}$$

$$\mathbf{A}\mathbf{y}_W = \mathbf{x}_W + \mathbf{b}$$

$$\mathbf{A}\mathbf{y}_U = \mathbf{x}_U + \mathbf{b}$$

yields

$$\mathbf{A} \cdot \underbrace{\begin{pmatrix} \mathbf{y}_N & \mathbf{y}_W & \mathbf{y}_U \end{pmatrix}}_{\mathbf{Y}} = \underbrace{\begin{pmatrix} \mathbf{x}_N + \mathbf{b} & \mathbf{x}_W + \mathbf{b} & \mathbf{x}_U + \mathbf{b} \end{pmatrix}}_{\mathbf{X}}$$

i.e.,

$$\mathbf{A} = \mathbf{X} \cdot \mathbf{Y}^{-1}$$

Algorithm MagCalib(In: $\mathbf{x}_N, \mathbf{x}_S, \mathbf{x}_W, \mathbf{x}_U$; Out : $\mathbf{A}, \mathbf{b}$)

$$1 \quad I = 64 \cdot \frac{\pi}{180}; \beta = 46.0 \cdot 10^{-6} T.$$

$$2 \quad \mathbf{b} = -\frac{1}{2} (\mathbf{x}_N + \mathbf{x}_S)$$

$$3 \quad \mathbf{Y} = \beta \cdot \begin{pmatrix} \cos I & 0 & -\sin I \\ 0 & -\cos I & 0 \\ -\sin I & -\sin I & \cos I \end{pmatrix}$$

$$4 \quad \mathbf{X} = \begin{pmatrix} \mathbf{x}_N + \mathbf{b} & \mathbf{x}_W + \mathbf{b} & \mathbf{x}_U + \mathbf{b} \end{pmatrix}$$

$$5 \quad \mathbf{A} = \mathbf{X} \cdot \mathbf{Y}^{-1}$$

$$6 \quad \text{Return } \mathbf{A}, \mathbf{b}$$

2. Calibration of the accelerometers

We assume that

$$\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x} + \mathbf{b})$$

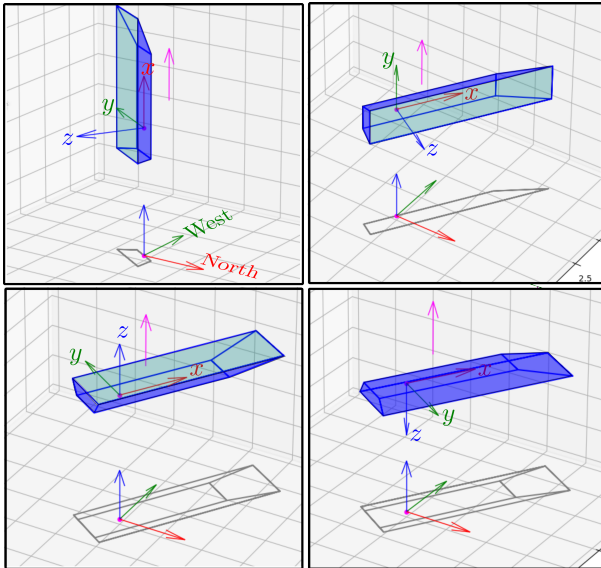
where $\mathbf{x} \in \mathbb{R}^3$ is the acceleration vector which is measured, and $\mathbf{y}$ is gravity vector (oriented upward).

We assume the boat static so that the acceleration corresponds to the gravity.

The calibration step searches for an estimation of $\mathbf{A}$ and $\mathbf{b}$.

We take four measurements $\mathbf{x}_x$, $\mathbf{x}_y$, $\mathbf{x}_z$, $\mathbf{x}_{-z}$, with the robot. We do not care where the North is.

- Direction x : The robot is set vertically with its x axis looking upward
- Direction y : The robot is set with roll its y axis upward
- Direction z : The robot is laid flat with its z axis upward
- Direction $-z$: The robot is laid flat upside down with its z axis downward



The corresponding corrected acceleration vectors should be

$$\mathbf{y}_x = \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix}, \mathbf{y}_y = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \mathbf{y}_z = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \mathbf{y}_{-z} = \begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix}$$

where $\beta = 9.8m \cdot s^{-2}$.

Since

$$\begin{aligned} \mathbf{0} &= \mathbf{y}_z + \mathbf{y}_{-z} \\ &= \mathbf{A}^{-1}(\mathbf{x}_z + \mathbf{b}) + \mathbf{A}^{-1}(\mathbf{x}_{-z} + \mathbf{b}) \end{aligned}$$

we get

$$(\mathbf{x}_z + \mathbf{b}) + (\mathbf{x}_{-z} + \mathbf{b}) = \mathbf{0}$$

i.e.,

$$\mathbf{b} = -\frac{1}{2}(\mathbf{x}_z + \mathbf{x}_{-z}).$$

Moreover

$$\mathbf{A}\mathbf{y}_x = \mathbf{x}_z + \mathbf{b}$$

$$\mathbf{A}\mathbf{y}_y = \mathbf{x}_y + \mathbf{b}$$

$$\mathbf{A}\mathbf{y}_z = \mathbf{x}_z + \mathbf{b}$$

yields

$$\mathbf{A} \cdot \underbrace{\begin{pmatrix} \mathbf{y}_x & \mathbf{y}_y & \mathbf{y}_z \end{pmatrix}}_{\beta \mathbf{I}} = \underbrace{\begin{pmatrix} \mathbf{x}_x + \mathbf{b} & \mathbf{x}_y + \mathbf{b} & \mathbf{x}_z + \mathbf{b} \end{pmatrix}}_{\mathbf{X}}$$

Thus

$$\mathbf{A} = \frac{1}{\beta} \mathbf{X}$$

Algorithm MagAccelero(In: $\mathbf{x}_x, \mathbf{x}_y, \mathbf{x}_z, \mathbf{x}_{-z}$; Out: $\mathbf{A}, \mathbf{b}$)

$$1 \quad \beta = 9.8m \cdot s^{-2}$$

$$2 \quad \mathbf{b} = -\frac{1}{2}(\mathbf{x}_z + \mathbf{x}_{-z})$$

$$3 \quad \mathbf{X} = \begin{pmatrix} \mathbf{x}_x + \mathbf{b} & \mathbf{x}_y + \mathbf{b} & \mathbf{x}_z + \mathbf{b} \end{pmatrix}$$

$$4 \quad \mathbf{A} = \frac{1}{\beta} \mathbf{X}$$

5 Return $\mathbf{A}, \mathbf{b}$

3. IMU

The boat is moving in the waves.

An IMU estimates the Euler angles from a magnetometer and an accelerometer.

These Euler angles φ, θ, ψ have to be estimated from

- 1 the magnetic field $\mathbf{y}_1$ and
- 2 its acceleration $\mathbf{a}_1$, both given in the boat frame.

Estimate the vertical

Assumption. In average the accelerometers returns the gravity vector $\mathbf{g}_1$ is the robot frame, i.e.,

$$\mathbf{a}_1 = \mathbf{g}_1 + \beta$$

where β is a centered noise.

The assumption is false if the boat turns around a buoy for long.

The sensors return $\mathbf{a}_1$ and $\boldsymbol{\omega}_1$ in the robot frame.
The gravity vector $\mathbf{g}_1$ returns the vertical.

We have

$$\mathbf{g}_0 = \mathbf{R}\mathbf{g}_1$$

Differentiate with respect to t :

$$\mathbf{0} = \dot{\mathbf{R}}\mathbf{g}_1 + \mathbf{R}\dot{\mathbf{g}}_1$$

Thus

$$\begin{aligned}\dot{\mathbf{g}}_1 &= -\mathbf{R}^\top \dot{\mathbf{R}}\mathbf{g}_1 \\ &= -\boldsymbol{\omega}_1 \wedge \mathbf{g}_1\end{aligned}$$

We have

$$\mathbf{g}_1(k+1) = \underbrace{(\mathbf{I} - dt\boldsymbol{\omega}_1(k)\wedge)}_{=\mathbf{A}(k)} \cdot \mathbf{g}_1(k) \quad (\text{prediction})$$

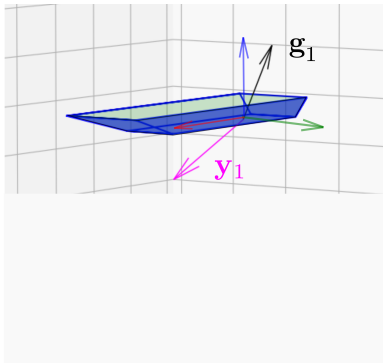
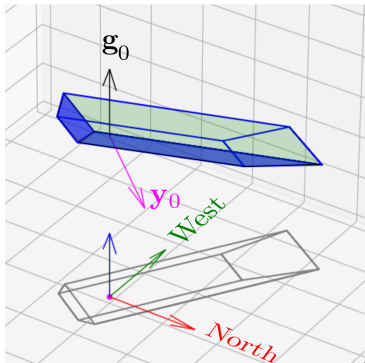
$$\mathbf{a}_1(k) = \mathbf{g}_1(k) + \boldsymbol{\beta}(k) \quad (\text{observation})$$

with a large variance for $\boldsymbol{\beta}(k)$, which means that we measure $\mathbf{g}_1$ (through the acceleration $\mathbf{a}_1$), only in average.

A Luenberger observer could be

$$\hat{\mathbf{g}}_1(k+1) = \lambda \cdot (\mathbf{I} - dt\boldsymbol{\omega}_1(k)\wedge) \cdot \hat{\mathbf{g}}_1(k) + (1 - \lambda) \cdot \mathbf{a}_1(k+1)$$

with $\lambda = 0.99$.



We now assume that we are able to have a good estimation of $\mathbf{g}_1$

For simplicity, we assume that $\mathbf{y}$ and $\mathbf{g}$ have been normalized, *i.e.*, their norm are both equal to 1. Moreover, we assume that sensors are well calibrated.

In $\mathcal{R}_0$, we have

$$\mathbf{y}_0 = \begin{pmatrix} \cos I \\ 0 \\ -\sin I \end{pmatrix}, \quad \mathbf{g}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Step 1. From $\mathbf{g}_1$, we get an estimation of the bank φ (roll angle) and the elevation θ (pitch angle).

We have

$$\hat{\phi} = \arcsin((0, 1, 0) \cdot \mathbf{g}_1)$$

and

$$\hat{\theta} = -\arcsin((1, 0, 0) \cdot \mathbf{g}_1)$$

Step 2. For the heading, we first compute the rotation which leads our boat to the nearest horizontal orientation.

If $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors, the rotation $\mathbf{R}$ which minimizes $\|\mathbf{R} - \mathbf{I}\|$ and transforms $\mathbf{u}$ to $\mathbf{v}$ is given by

$$\mathbf{R}_{uv}(\mathbf{u}, \mathbf{v}) = \text{Exp} \left(\arccos(\mathbf{u}^T \cdot \mathbf{v}) \cdot \frac{\mathbf{u} \wedge \mathbf{v}}{\|\mathbf{u} \wedge \mathbf{v}\|} \right)$$

The rotation which transforms $\mathbf{g}_1$ to $\mathbf{g}_0$ is given by

$$\mathbf{R}_h = \mathbf{R}_{uv}(\mathbf{g}_1, \mathbf{g}_0).$$

Step 3. Provide an estimation $\hat{\psi}$ of the heading angle ψ .

We form the vector

$$\mathbf{y}_h = \mathbf{R}_h \mathbf{y}_1$$

which corresponds to the magnetic field measured by the horizontal avatar.

The heading is then obtained by

$$\hat{\psi} = -\text{atan2}(y_{h2}, y_{h1}).$$

Algorithm EulerAngles(In: $\mathbf{a}_1, \mathbf{y}_1$; out: $\hat{\phi}, \hat{\theta}, \hat{\psi}$)

- 1 $\mathbf{g}_0 = (0, 0, 1)^T, \mathbf{g}_1^n = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}, \mathbf{y}_1^n = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|}$
- 2 $\hat{\phi} = \arcsin((0, 1, 0) \cdot \mathbf{g}_1^n)$
- 3 $\hat{\theta} = -\arcsin((1, 0, 0) \cdot \mathbf{g}_1^n)$
- 4 $\mathbf{R}_h = \mathbf{R}_{uv}(\mathbf{g}_1^n, \mathbf{g}_0)$
- 5 $\mathbf{y}_h = \mathbf{R}_h \mathbf{y}_1^n$
- 6 $\hat{\psi} = -\text{atan2}(y_{h2}, y_{h1})$

With magnetometers, accelerometers and without gyro

Algorithm EulerAngles(In: $\mathbf{a}_1, \mathbf{y}_1, \omega_1, \hat{\mathbf{g}}_1$; out: $\hat{\phi}, \hat{\theta}, \hat{\psi}, \hat{\mathbf{g}}_1$)

- 1 $\mathbf{g}_0 = (0, 0, 1)^T, \mathbf{y}_1^n = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|}$
- 2 $\hat{\mathbf{g}}_1 = \lambda \cdot (\mathbf{I} - dt\omega_1 \wedge) \cdot \hat{\mathbf{g}}_1 + (1 - \lambda) \cdot \mathbf{a}_1$, with $\lambda \simeq 1$
- 3 $\mathbf{g}_1^n = \frac{\hat{\mathbf{g}}_1}{\|\hat{\mathbf{g}}_1\|}$
- 4 $\hat{\phi} = \arcsin((0, 1, 0) \cdot \mathbf{g}_1^n)$
- 5 $\hat{\theta} = -\arcsin((1, 0, 0) \cdot \mathbf{g}_1^n)$
- 6 $\mathbf{R}_h = \mathbf{R}_{uv}(\mathbf{g}_1^n, \mathbf{g}_0)$
- 7 $\mathbf{y}_h = \mathbf{R}_h \mathbf{y}_1^n$
- 8 $\hat{\psi} = -\text{atan2}(y_{h2}, y_{h1})$

With magnetometers, accelerometers and gyro