Interval methods for proving properties of dynamical systems; application to the validation of the control lows for the sailboat robot VAIMOS

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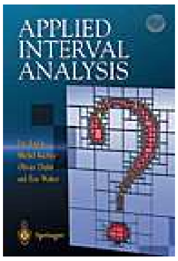
February, 18 2014, ENS Cachan.



1 Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}] \, , f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\operatorname{abs}([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$\begin{aligned} f(x_1, x_2) &= x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 \\ &\quad + \sin x_1 \cdot \sin x_2 + 2 \end{aligned}$$

is

$$\begin{aligned} [f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ &\quad + \sin [x_1] \cdot \sin [x_2] + 2. \end{aligned}$$

Theorem (Moore, 1970)

$$[f]\left([\mathbf{x}]\right) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}]\,, f\left(\mathbf{x}\right) \geq 0$$

2 Computing with sets

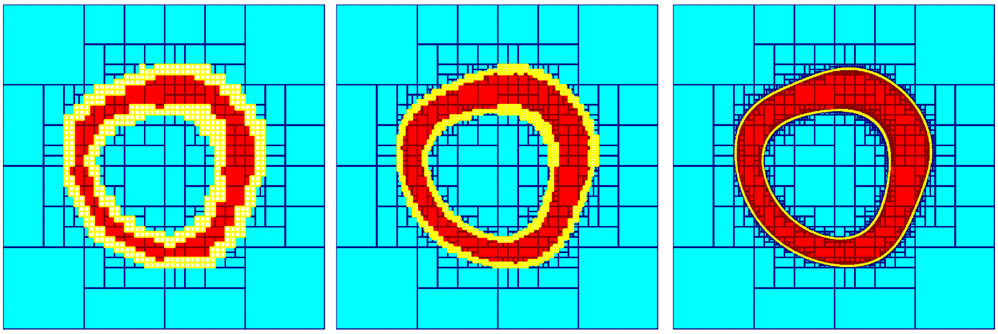
Subsets $\mathbb{X} \subset \mathbb{R}^n$ can be bracketted by subpavings :

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

which can be obtained using interval calculus

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}.$$



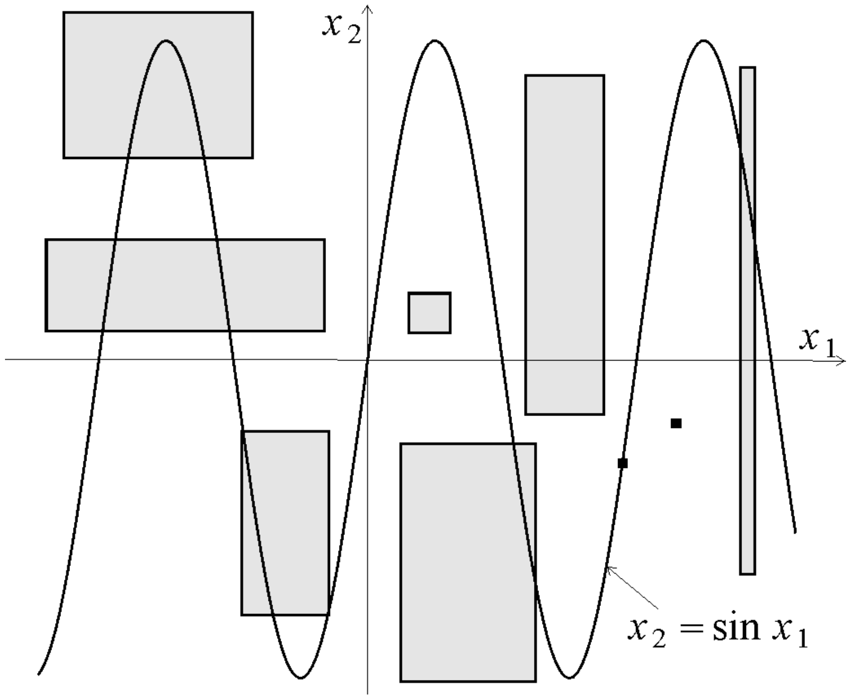
3 Contractors

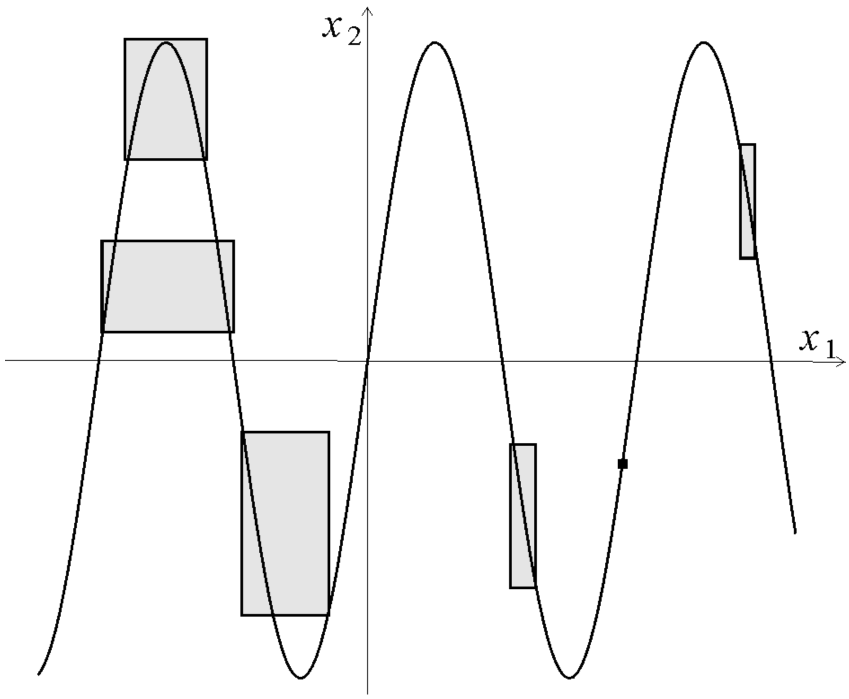
The operator $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

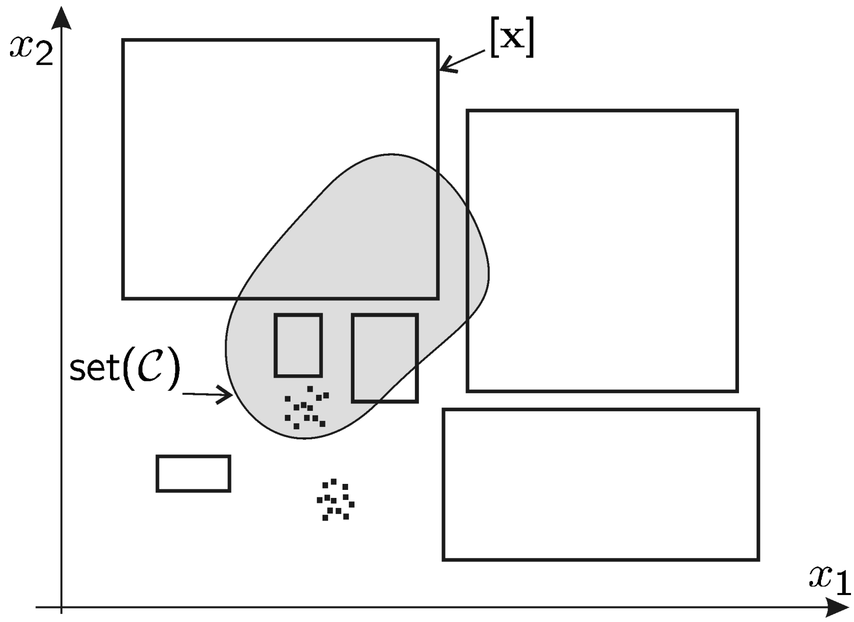
$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

Example. Consider the primitive equation:

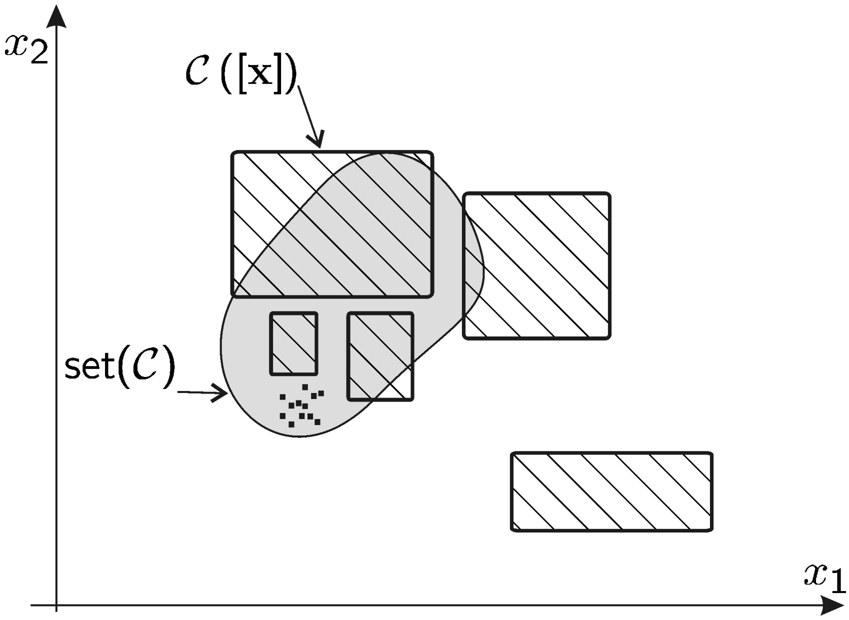
$$x_2 = \sin x_1.$$


x_2
x_1
$x_2 = \sin x_1$


x_2
x_1



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
set($\mathcal{C}$)
x_1

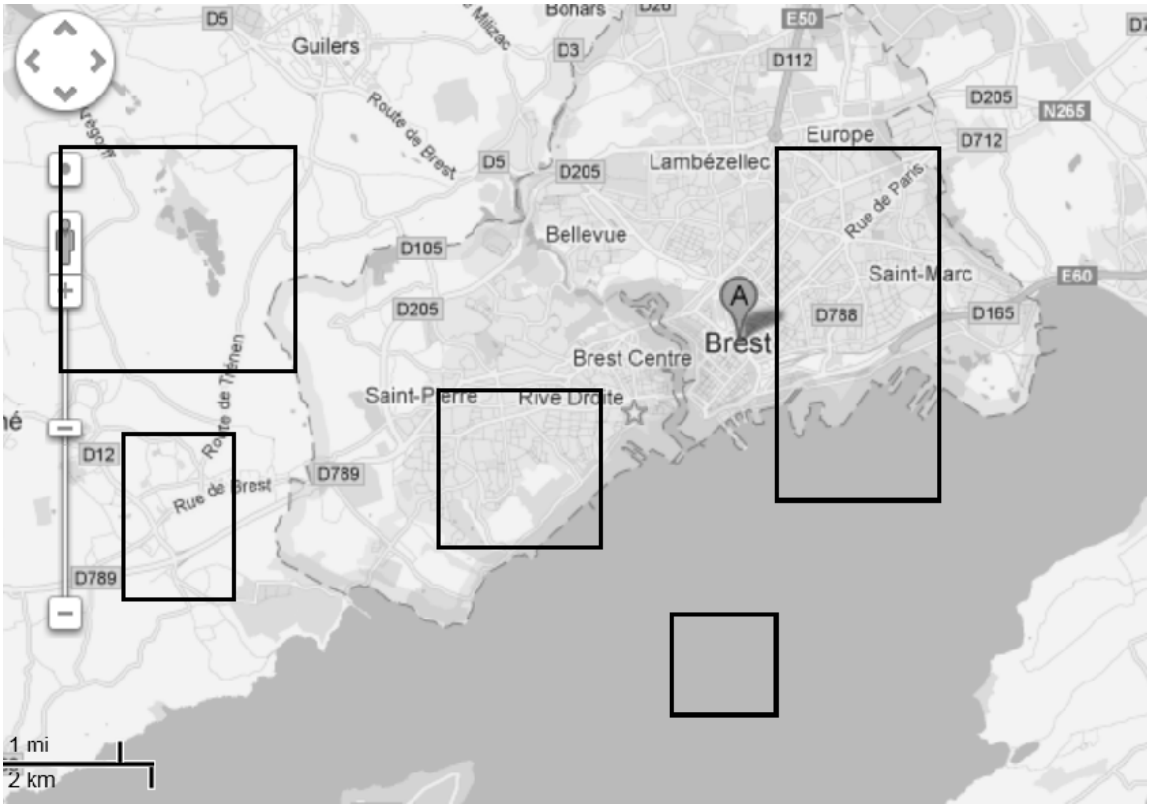
$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *idempotent* if	$\mathcal{C}\left(\mathcal{C}([\mathbf{x}])\right) = \mathcal{C}([\mathbf{x}])$

Contractor algebra

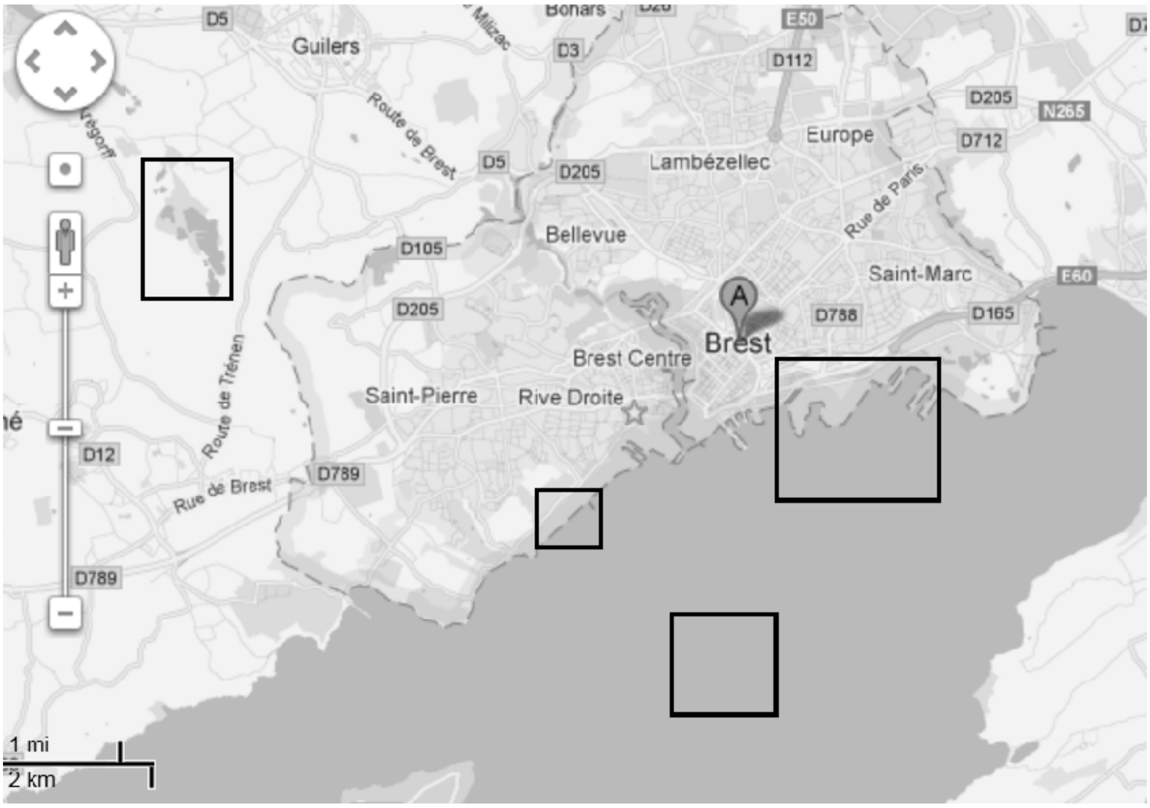
intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Contractor associated with a database

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Route de Brest
D3
D112
E50
D205
N265
D712
Europe
Lambézellec
D5
D205
Bellevue
D105
Rue de Paris
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
D12
Route de Trénen
Rue de Brest
D789
D789
1 mi
2 km


Guilers
Route de Brest
Lambézellec
Europe
Bellevue
Rue de Paris
Saint-Marc
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
Rue de Brest
D5
D3
D112
E50
D205
D712
N265
D105
E60
D788
D165
D12
D789
1 mi
2 km

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$
\begin{aligned}
x_3 = x_1 + x_2 \Rightarrow \quad & x_3 \in \quad [x_3] \cap ([x_1] + [x_2]) \quad // \text{ forward} \\
x_1 = x_3 - x_2 \Rightarrow \quad & x_1 \in \quad [x_1] \cap ([x_3] - [x_2]) \quad // \text{ backward} \\
x_2 = x_3 - x_1 \Rightarrow \quad & x_2 \in \quad [x_2] \cap ([x_3] - [x_1]) \quad // \text{ backward}
\end{aligned}
$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

4 Solver

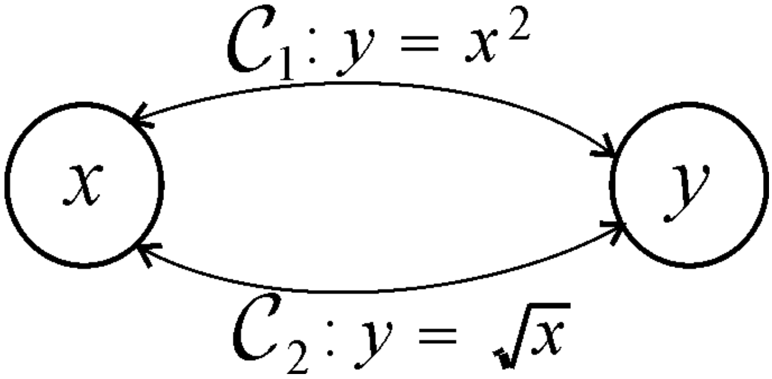
Example. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

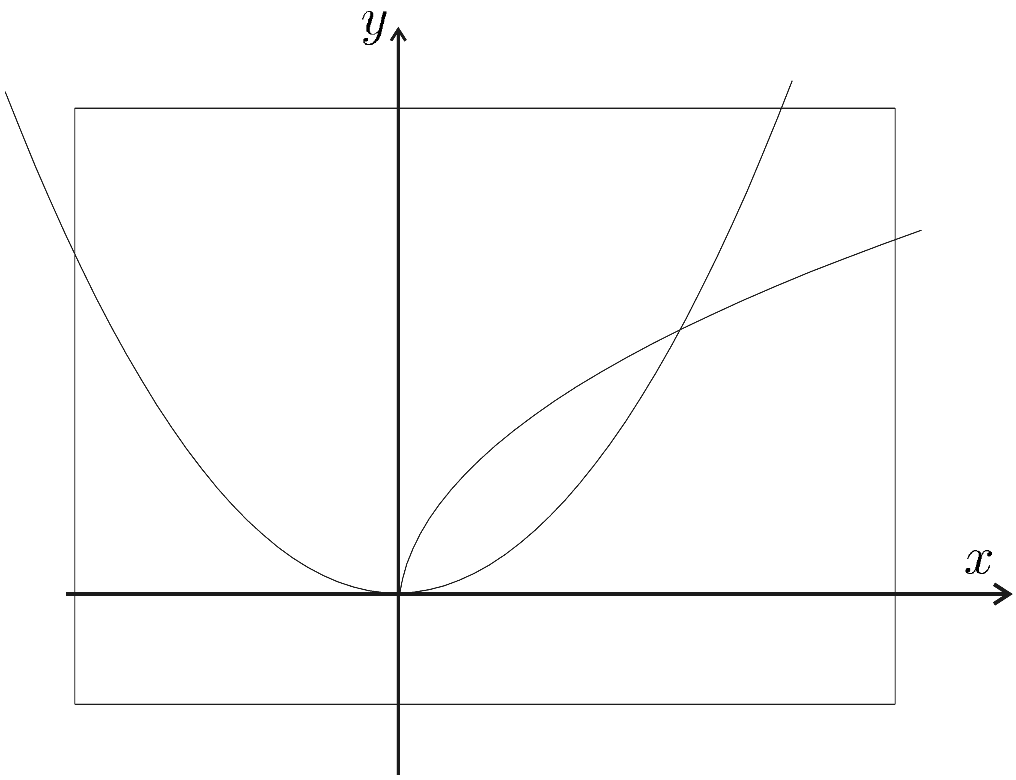
We build two contractors

$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated with } y = x^2$$

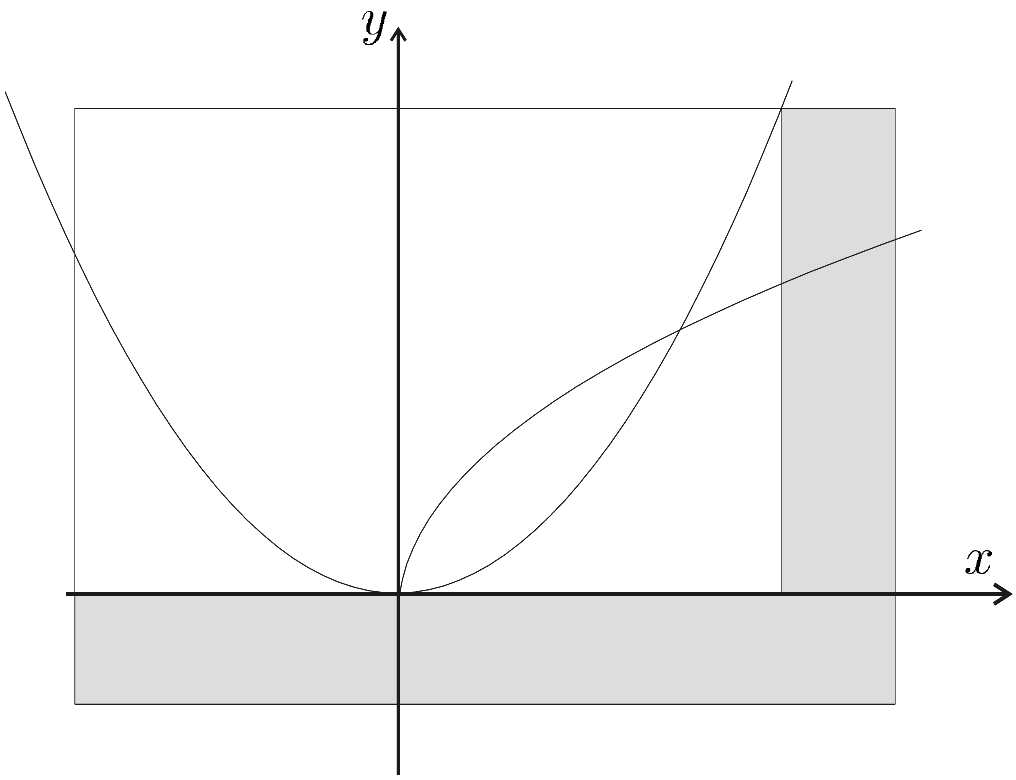
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated with } y = \sqrt{x}$$

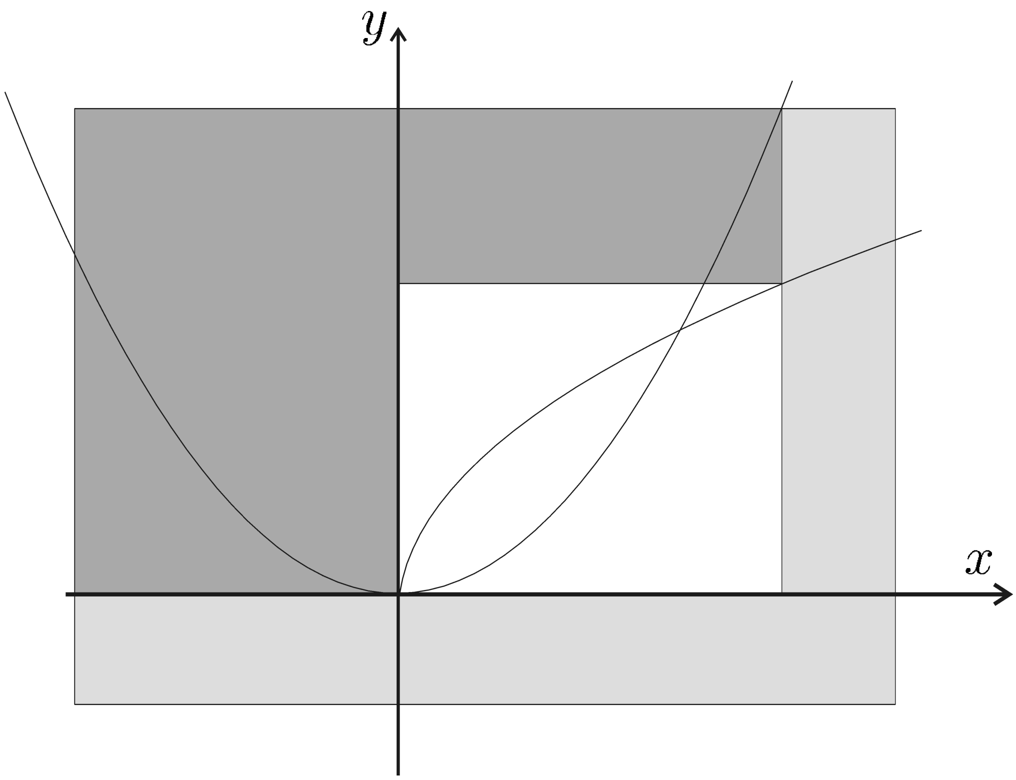


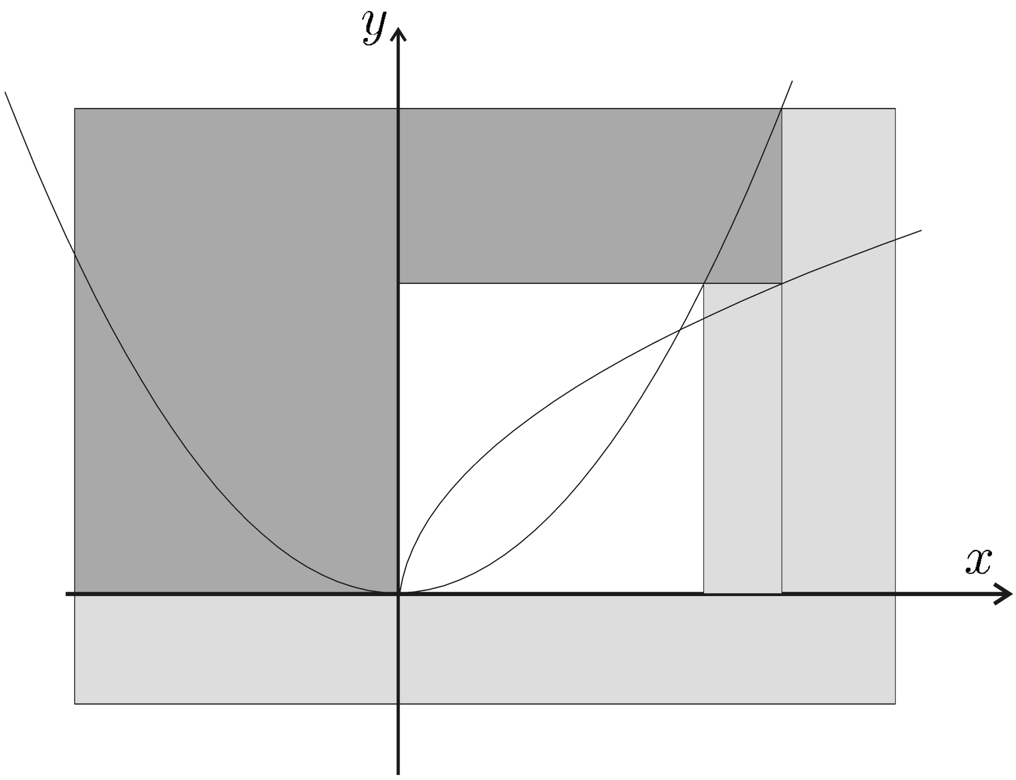
Contractor graph



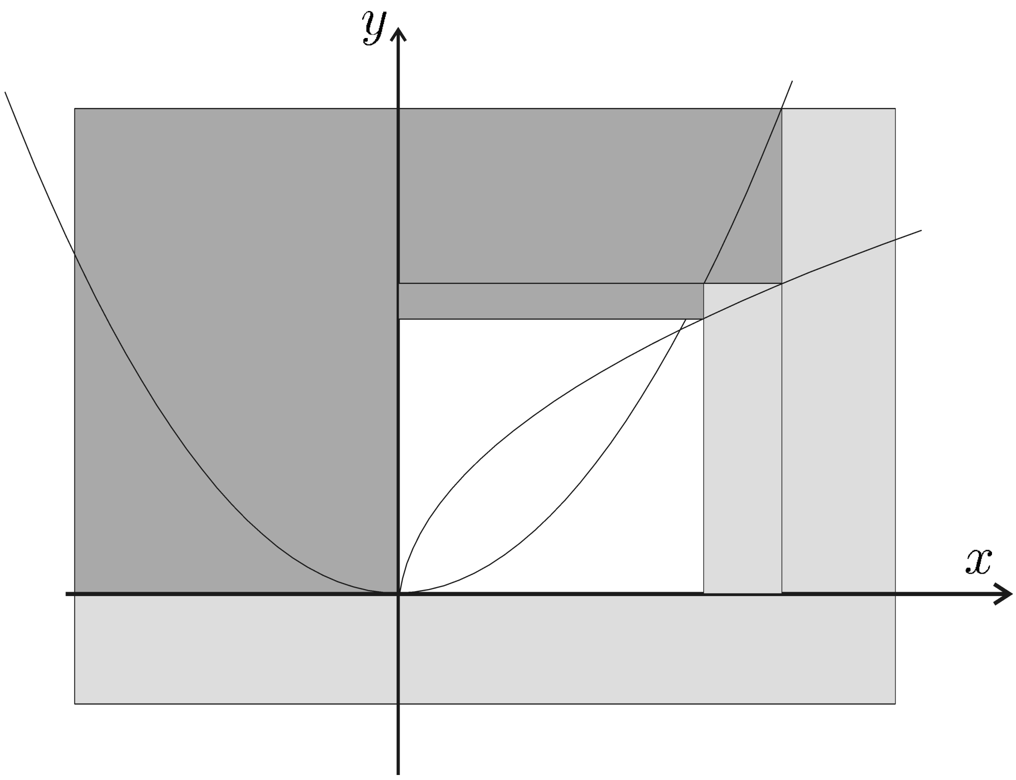
y
x


y
x

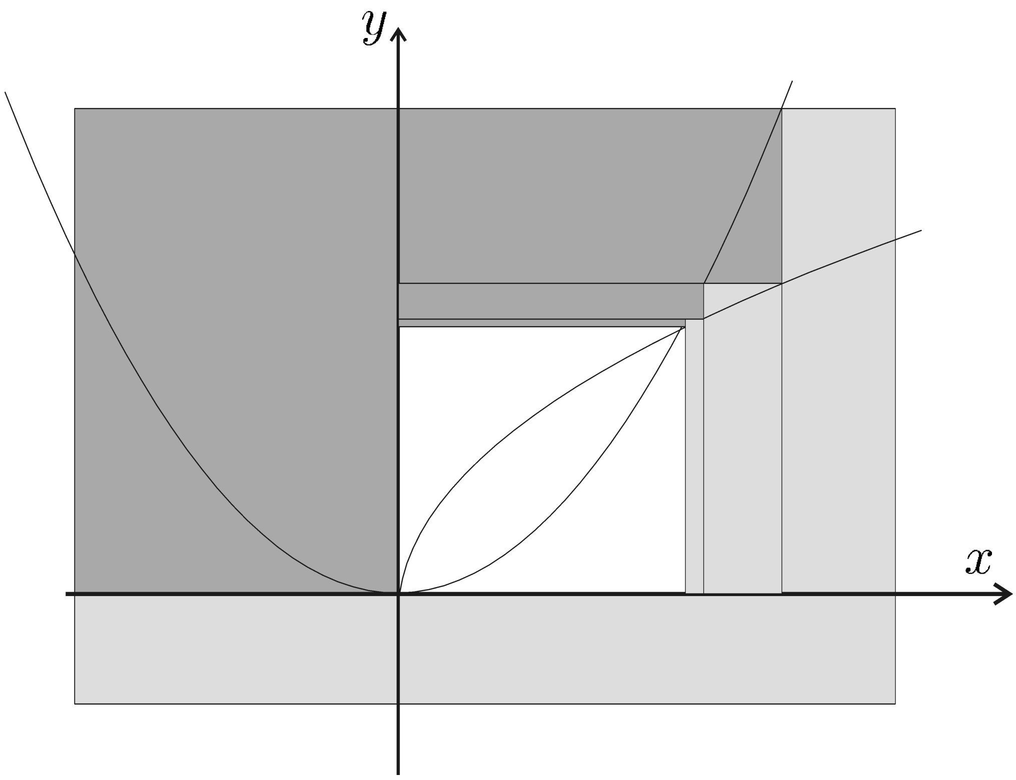

y
x



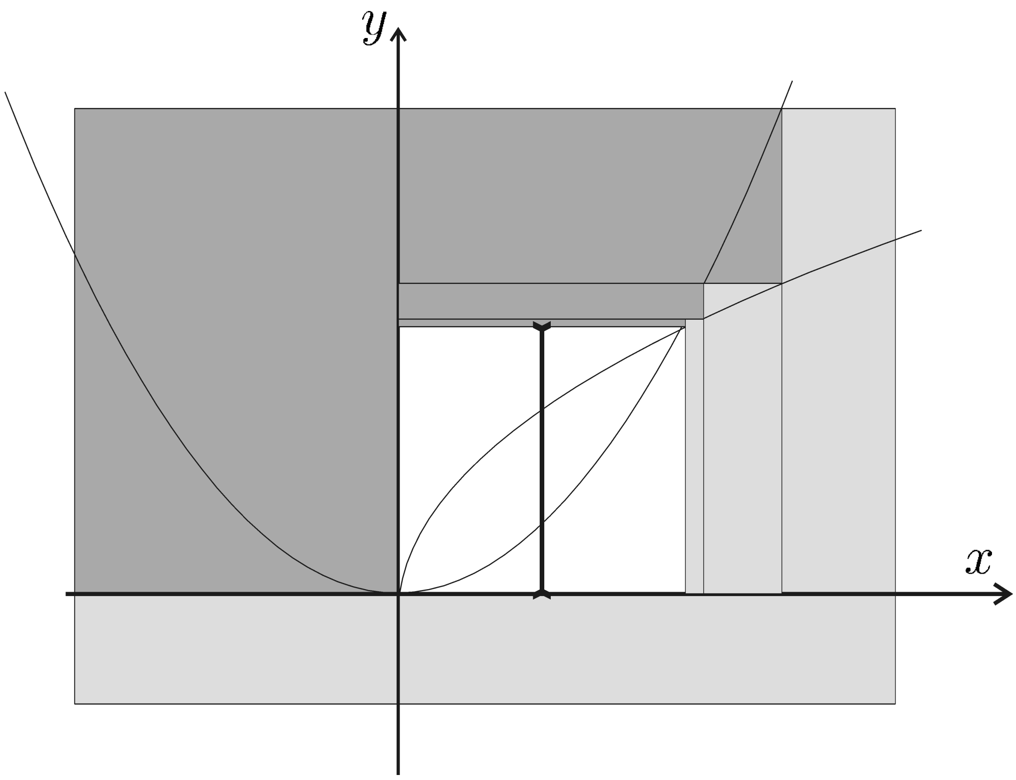
y
x

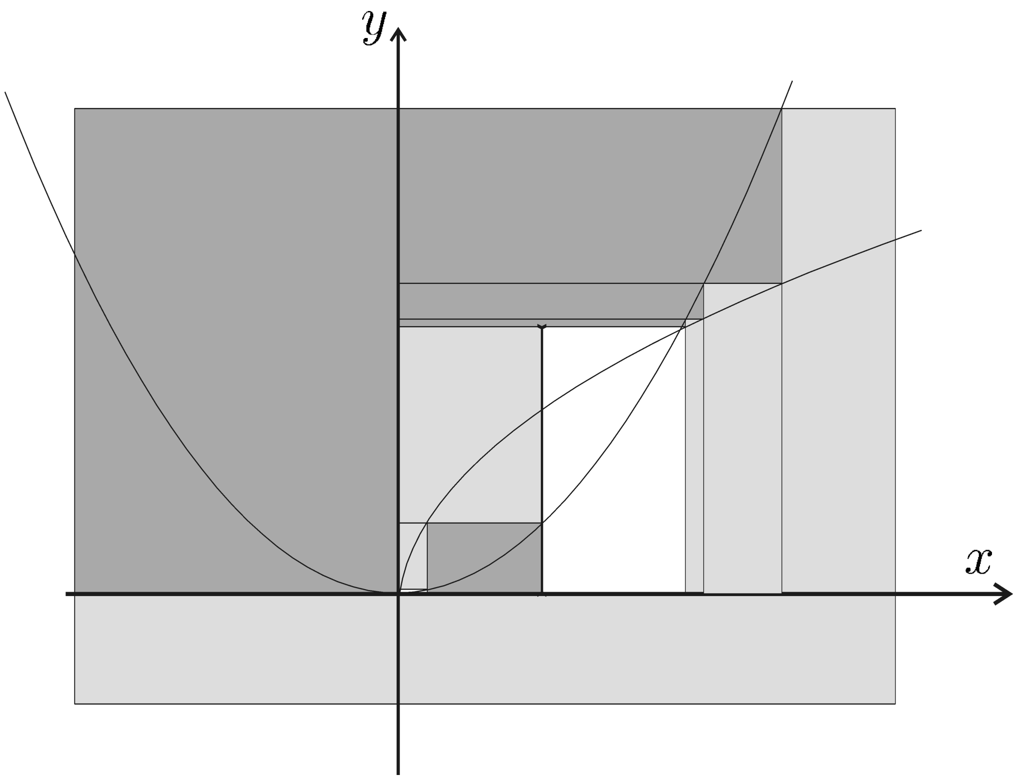


y
x

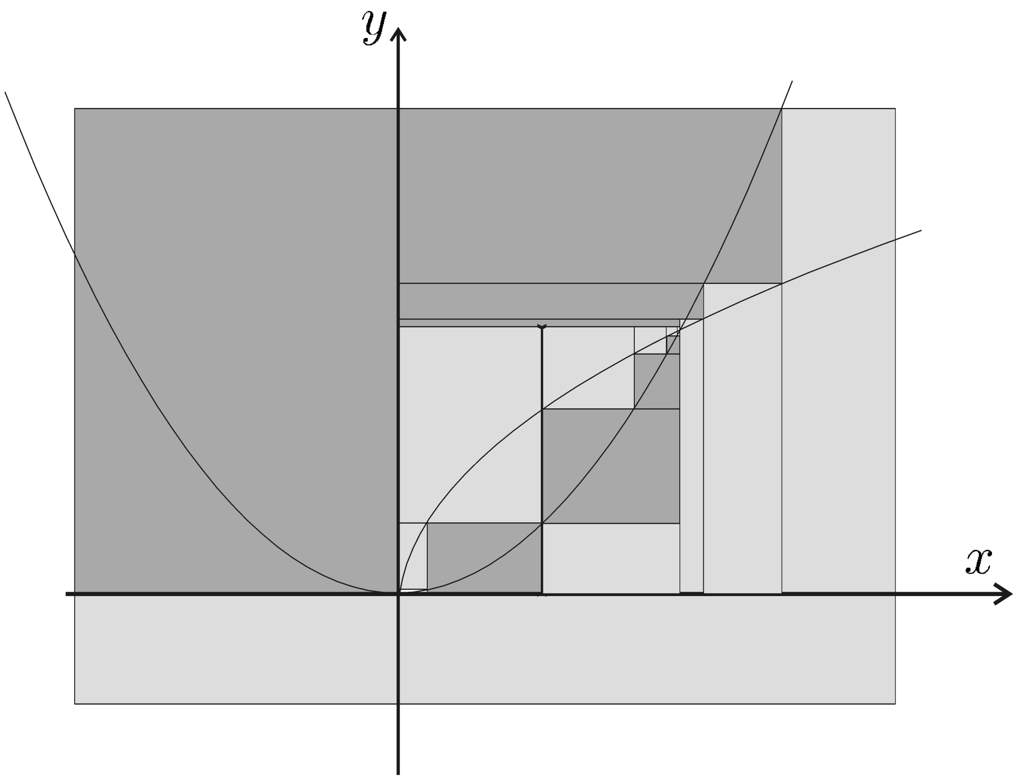


y
x


y
x



y
x


y
x

5 Sailboat robotics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SPIRIT
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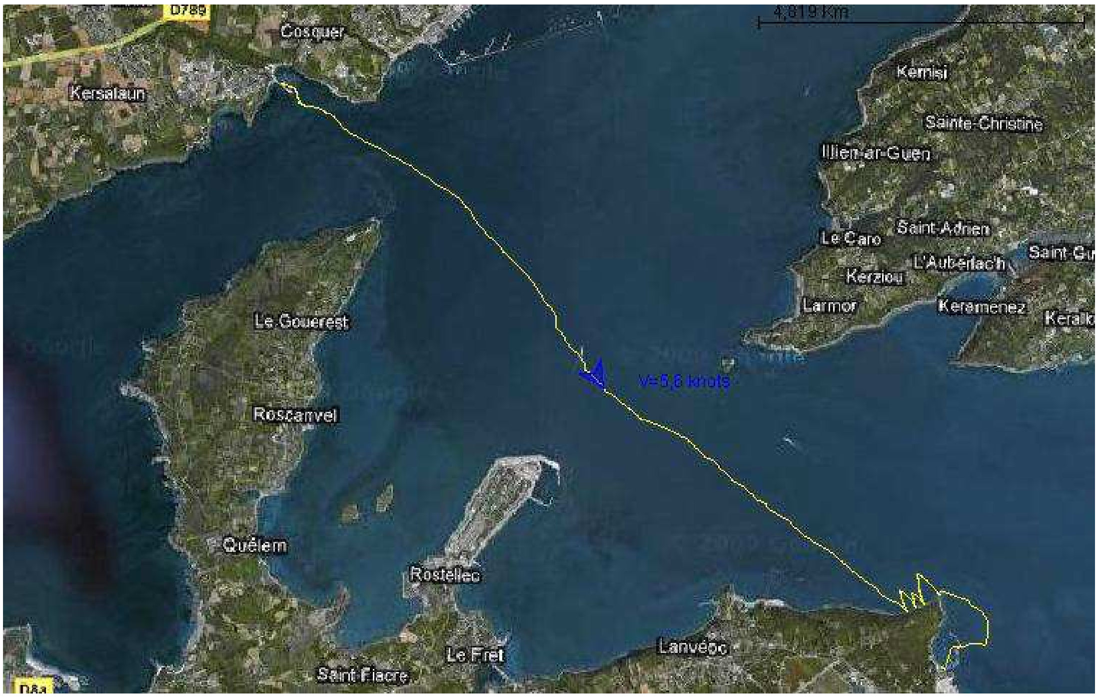
D789
Cosquer
Kersalaun
4,819 Km
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Sainte-Christine
Illien-ar-Guen
Le Caro
Saint-Adrien
L'Auberlac'h
Kerziou
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Roscanvel
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Le Fret
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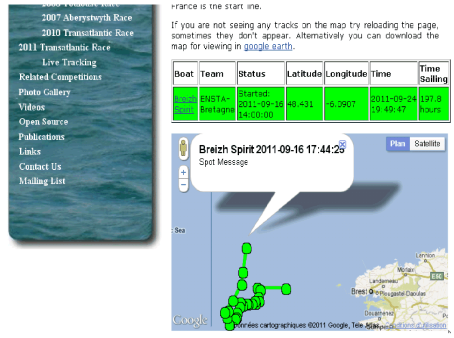
Saint Fiacre


2007 Aberystwyth Race
2010 Transatlantic Race
2011 Transatlantic Race
Live Tracking
Related Competitions
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Videos
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Publications
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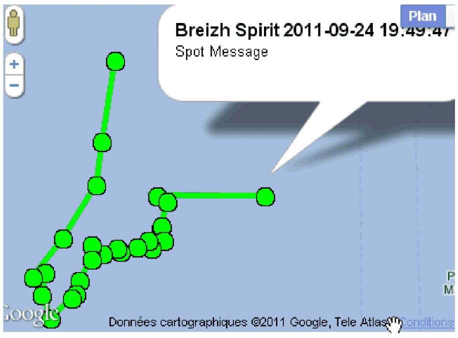
Mailing List
France is the start line.
If you are not seeing any tracks on the map try reloading the page, sometimes they don't appear. Alternatively you can download the map for viewing in google earth.
Boat | Team | Status | Latitude | Longitude | Time | Time Sailing
Breizh Spirit | ENSTA-Bretagne | Started: 2011-09-16 14:00:00 | 48.431 | -6.0907 | 2011-09-24 19:49:47 | 197.8 hours
Breizh Spirit 2011-09-16 17:44:25
Spot Message
Plan
Satellite
Sea
Lannion
Morlaix
Landerneau
Brest
Plougastel-Daoulas
Douarnenez
Données cartographiques ©2011 Google, Tele Atlas - Conditions d'utilisation


Breizh Spirit 2011-09-24 19:49:47
Spot Message
Plan
Données cartographiques ©2011 Google, Tele Atlas

6 Vaimos

Collaboration ENSTA/IFREMER

Vaimos à la WRSC (ENSTA-IFREMER-Ecole Navale).

$$
\left\{
\begin{array}{lcl}
\dot{x} & = & v\cos\theta + p_1 a\cos\psi \\
\dot{y} & = & v\sin\theta + p_1 a\sin\psi \\
\dot{\theta} & = & \omega \\
\dot{v} & = & \frac{f_s \sin\delta_s - f_r \sin u_1 - p_2 v^2}{p_9} \\
\dot{\omega} & = & \frac{f_s(p_6 - p_7\cos\delta_s) - p_8 f_r \cos u_1 - p_3\omega}{p_{10}} \\
f_s & = & p_4 a\sin\left(\theta - \psi + \delta_s\right) \\
f_r & = & p_5 v\sin u_1 \\
\sigma & = & \cos\left(\theta - \psi\right) + \cos\left(u_2\right) \\
\delta_s & = & \left\{
\begin{array}{ll}
\pi - \theta + \psi & \text{si } \sigma \leq 0 \\
sign\left(\sin\left(\theta - \psi\right)\right).u_2 & \text{sinon.}
\end{array}
\right.
\end{array}
\right.
$$

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

With the controller $\mathbf{u} = \mathbf{g}(\mathbf{x})$, the robot satisfies

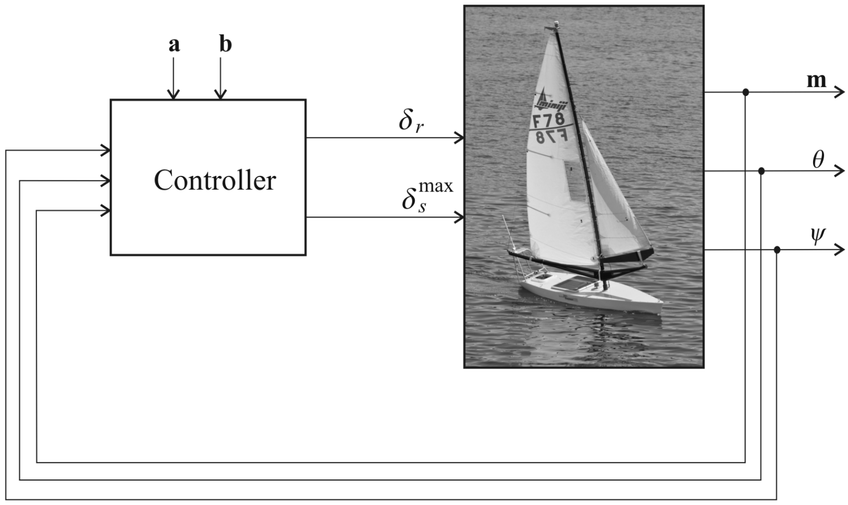
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

With all uncertainties, the robot satisfies.

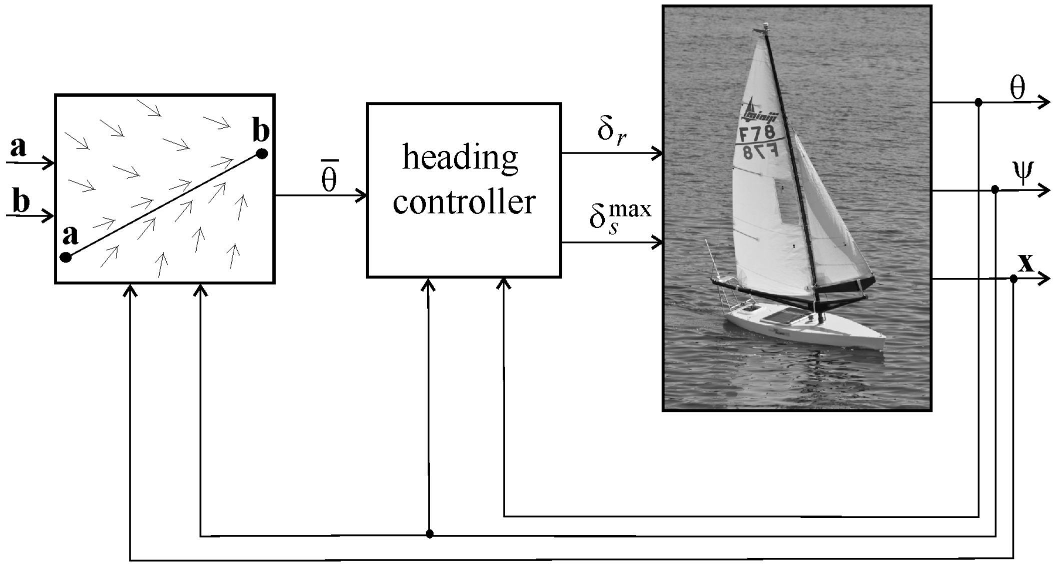
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

7 Line following



Controller of a sailboat robot



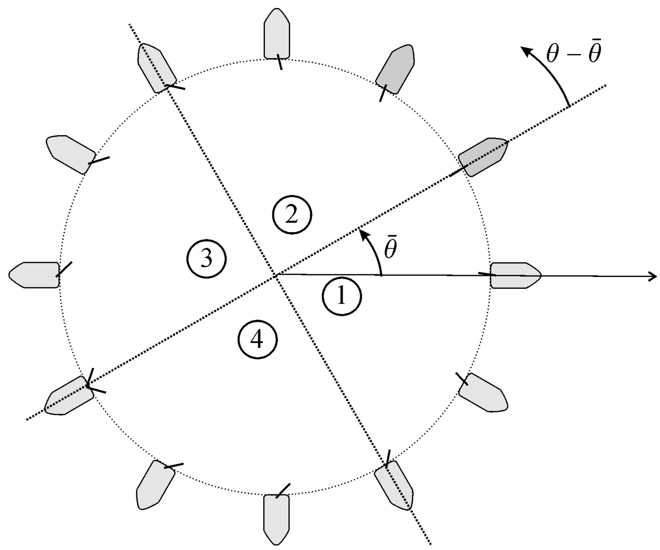
a
b
a
b
$\bar{\theta}$
heading
controller
δ_r
δ_s^{max}
θ
ψ
x

Heading controller

$$\begin{cases} \delta_r & = & \frac{\delta_r^{\mathsf{max}}}{\pi}.\mathsf{atan}(\mathsf{tan}\,\frac{\theta-\bar{\theta}}{2}) \\ \delta_s^{\mathsf{max}} & = & \frac{\pi}{2}.\left(\frac{\mathsf{cos}(\psi-\bar{\theta})+1}{2}\right). \end{cases}$$

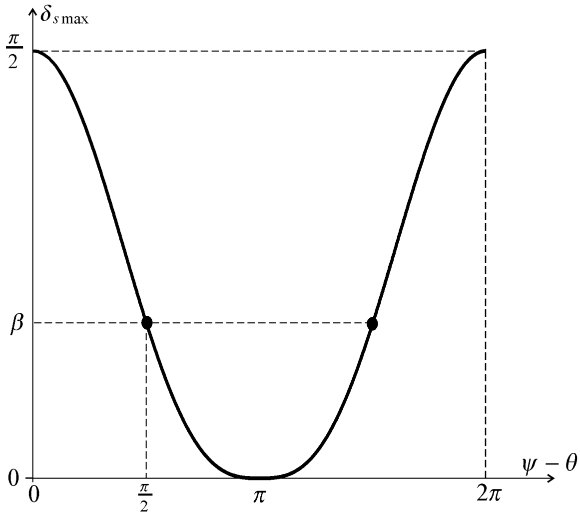
Rudder

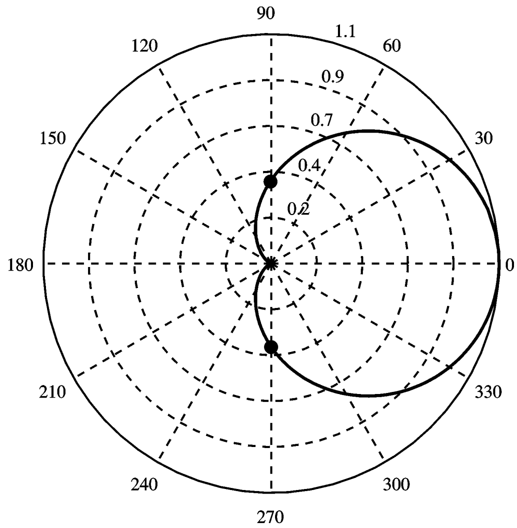
$$\left\{ \delta_r = \frac{\delta_r^{\max}}{\pi}.\mathrm{atan}(\tan\frac{\theta-\bar{\theta}}{2}) \right.$$



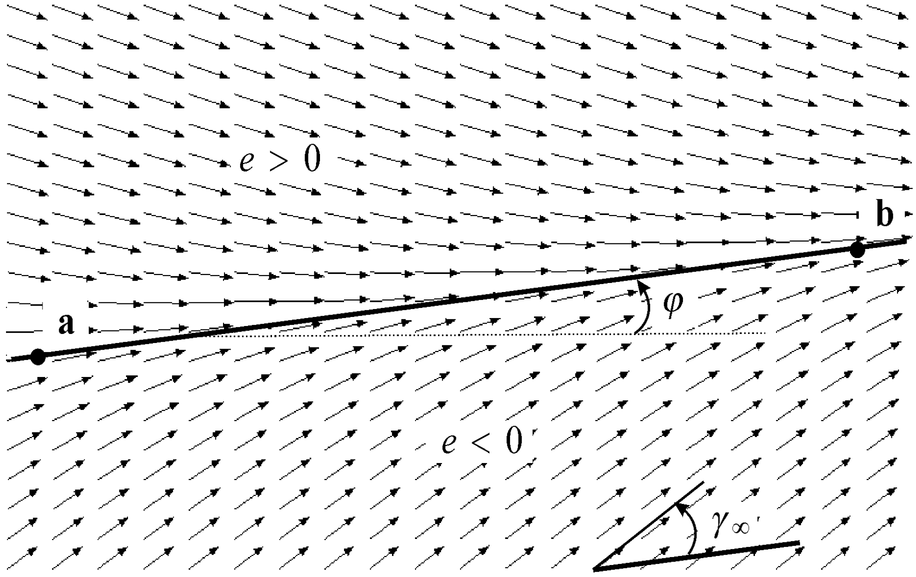
Sail

$$\delta_s^{\max} = \frac{\pi}{2} \cdot \left(\frac{\cos\left(\psi - \bar{\theta}\right) + 1}{2} \right)$$

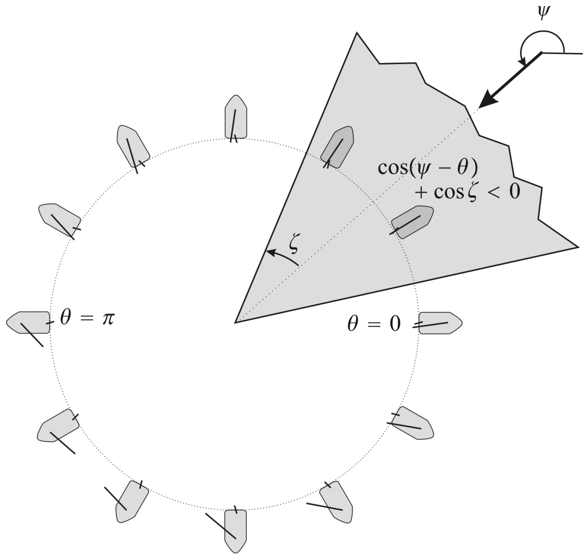




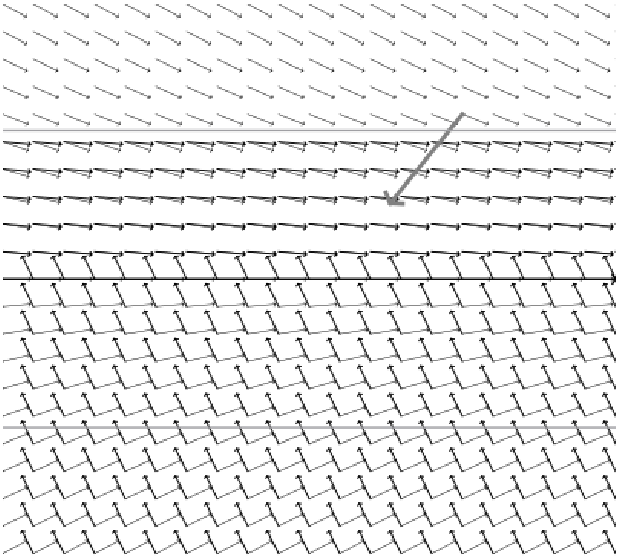
7.1 Vector field



Nominal vector field: $\theta^* = \varphi - \frac{1}{2}.\mathrm{atan}\left(\frac{e}{r}\right)$



A course θ^* may be unfeasible



Keep close hauled strategy.

7.2 Controller

Controlleur : in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\max}$; inout: q

$$e = \frac{\det(\mathbf{b}-\mathbf{a}, \mathbf{m}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|}$$

if $|e| > \frac{r}{2}$ then $q = \mathrm{sign}(e)$

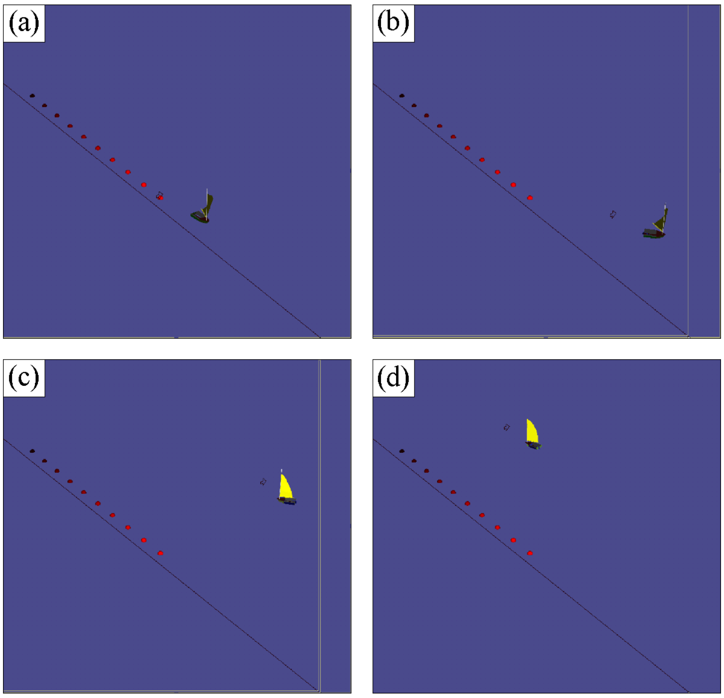
$$\bar{\theta} = \mathrm{atan2}(\mathbf{b}-\mathbf{a}) - \frac{1}{2}.\mathrm{atan}\left(\frac{e}{r}\right)$$

if $\cos\left(\psi - \bar{\theta}\right) + \cos\zeta < 0$ then $\bar{\theta} = \pi + \psi - q.\zeta$.

$$\delta_r = \frac{\delta_r^{\max}}{\pi}.\mathrm{atan}\left(\tan\frac{\theta-\bar{\theta}}{2}\right)$$

$$\delta_s^{\max} = \frac{\pi}{2}.\left(\frac{\cos\left(\psi-\bar{\theta}\right)+1}{2}\right).$$

8 Validation by simulation


(a)
(b)
(c)
(d)

9 Theoretical validation

*Jaulin, Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE TRO.

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

The system

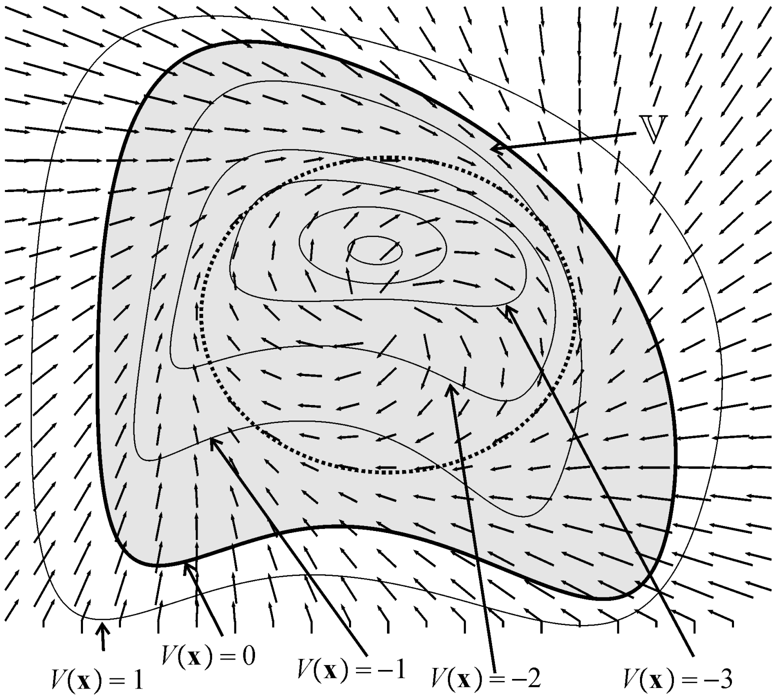
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}, \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0}. \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) \leq \varepsilon < 0\right).$$



𝕍
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$
\begin{aligned}
& \Big(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0 \Big) \\
\Leftrightarrow\; & \Big(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \,.\mathbf{f}(\mathbf{x}) < 0 \Big) \\
\Leftrightarrow\; & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \,.\mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow\; & \neg \Big(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \,.\mathbf{f}(\mathbf{x}) \geq 0 \text{ and } V(\mathbf{x}) \geq 0 \Big)
\end{aligned}
$$

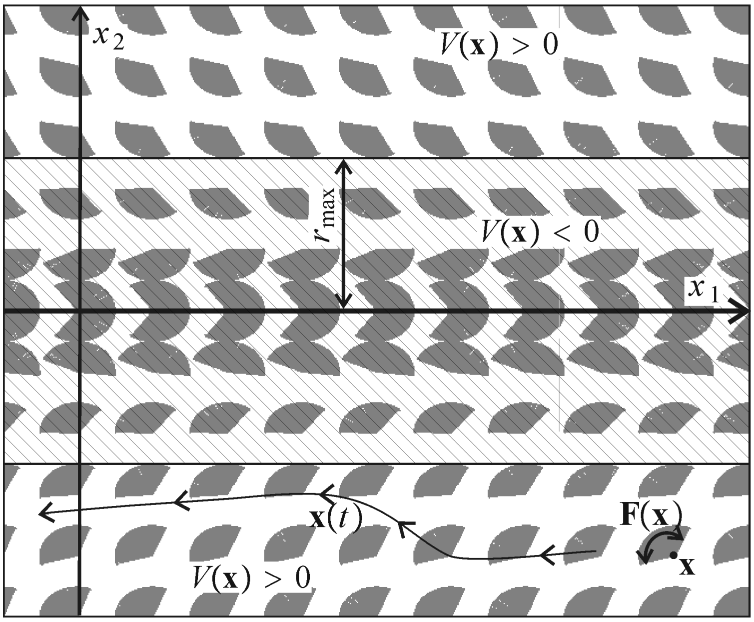
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

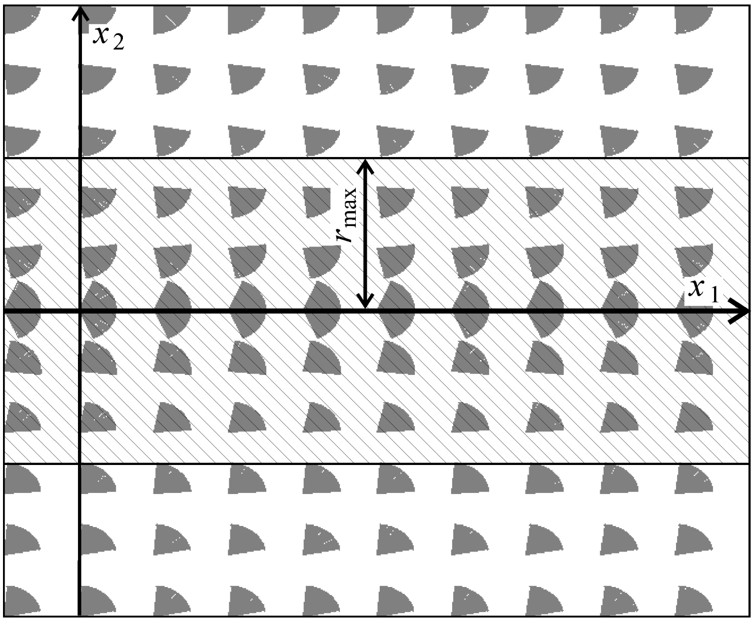
Theorem. We have

$$\left\{\begin{array}{c}\frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{a}\geq 0\\ \mathbf{a}\in\mathbf{F}\left(\mathbf{x}\right)\\ V(\mathbf{x})\geq 0\end{array}\right.\quad \text{inconsistent} \;\Leftrightarrow\; \dot{\mathbf{x}}\in\mathbf{F}\left(\mathbf{x}\right)\text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\text{max}}^2.$$



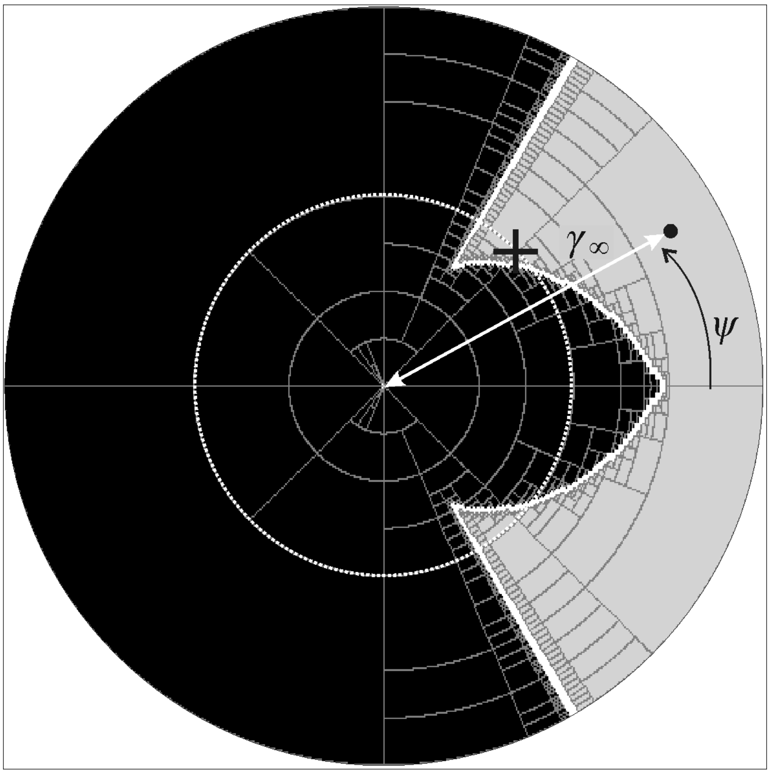
x_2
r_{max}
x_1

10 Parametric case

Consider the differential inclusion

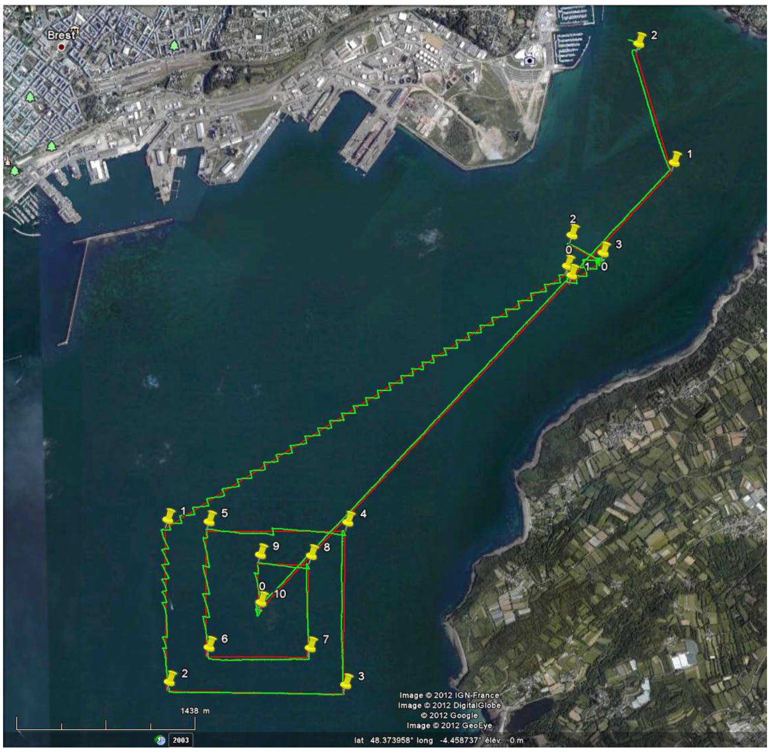
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, \mathbf{p}).$$

We characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V-stable.



+
γ_∞
ψ

11 Experimental validation

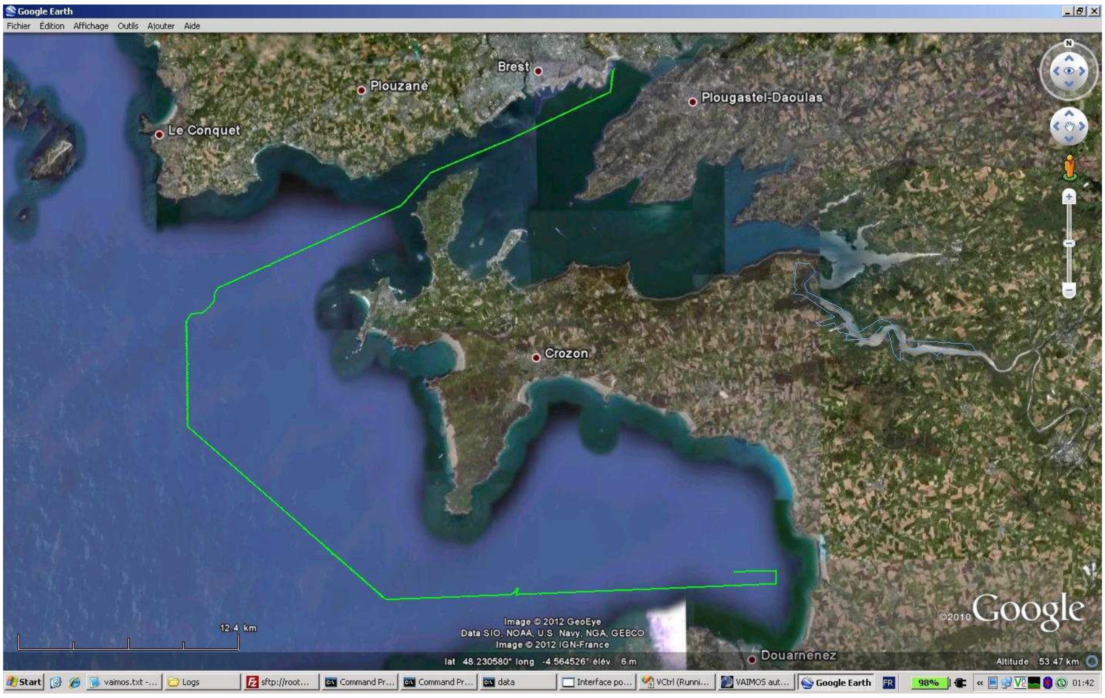


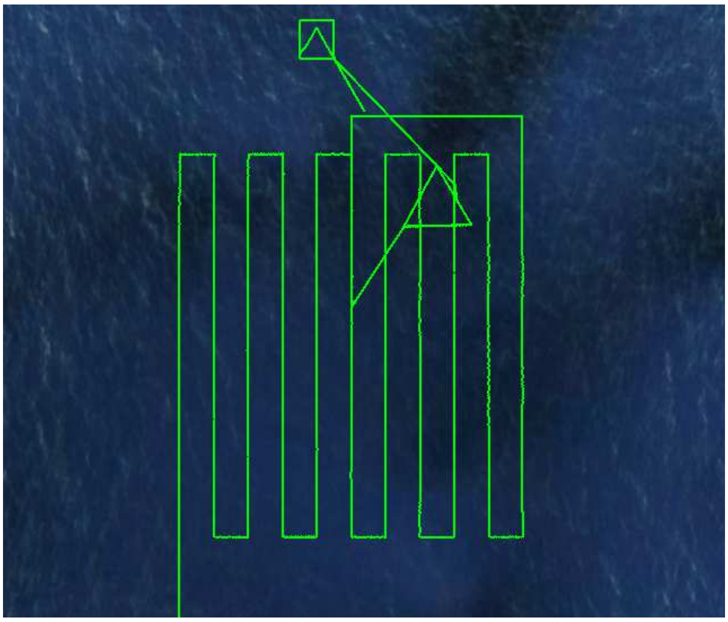
Rade de Brest

Brest-Douarnenez. January 17, 2012, 8am


BR D98 505
YAMAHA

F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google



Middle of Atlantic ocean, 350 km made by Vaimos in 53h, September 6-9, 2012.

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.