A new wrapper for a reliable resolution of underdetermined nonlinear equations

Luc Jaulin

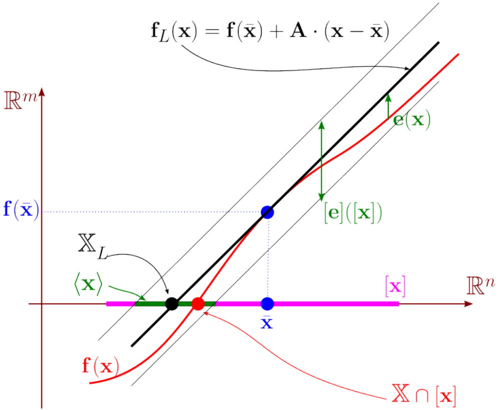
2025 September 22-26, SCAN 2025, Oldenburg

First order enclosure

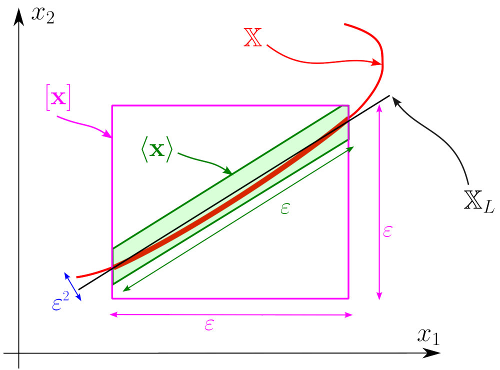
We want to characterize the set

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$, $m < n$.



Principle of the first order enclosure: $w([\mathbf{e}]([\mathbf{x}])) = o(w([\mathbf{x}]))$



$$\text{Vol}(\langle \mathbf{x} \rangle) = O(\varepsilon^n \cdot \varepsilon^{k(n-m)}), k = 1 \text{ whereas } \text{Vol}([\mathbf{x}]) = O(\varepsilon^n)$$

Polyhedron approximation of $\mathbb{X} \cap [\mathbf{x}]$

Proposition

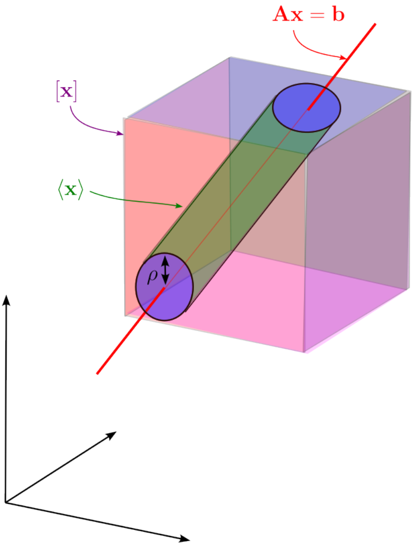
$$\left\{\begin{array}{c}\mathbf{f}(\mathbf{x})=\mathbf{0}\\ \mathbf{x}\in[\mathbf{x}]\end{array}\right. \Rightarrow \left\{\begin{array}{c}\mathbf{A}\cdot\mathbf{x}\in[\mathbf{b}]\\ \mathbf{A}=\frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}})\\ [\mathbf{b}]=\mathbf{A}\bar{\mathbf{x}}-\mathbf{f}(\bar{\mathbf{x}})+[\mathbf{e}]\\ [\mathbf{e}]=\left(\mathbf{A}-\frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])\right)\cdot([\mathbf{x}]-\bar{\mathbf{x}})\end{array}\right.$$

Buches

The *buche* (French name for *log*) associated to $[\mathbf{x}] \subset \mathbb{R}^n$, a matrix $\mathbf{A}$, a vector $\mathbf{b}$ and the radius ρ is

$$\begin{aligned} \langle \mathbf{x} \rangle &= \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle \\ &= \{ \mathbf{x} \in [\mathbf{x}], \exists \mathbf{p}, \mathbf{A}\mathbf{p} = \mathbf{b} \text{ and } \|\mathbf{x} - \mathbf{p}\| < \rho \}. \end{aligned}$$

The affine space $\mathbf{A}\mathbf{p} = \mathbf{b}$ is called a *flat*.


Ax = b
[x]
⟨x⟩
ρ

Motivation for using buches is:

- The box $[\mathbf{x}]$ in the buche allows to build a nonoverlapping covering of $\mathbb{X}$.
- A buche can easily be bisected
- The axis-aligned projection is easy
- Buches can easily be intersected
- A first order approximation is possible

Projection of a buche

Consider

$$\mathbb{R}^n = \left\{ \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_1 \in \mathbb{R}^m, \mathbf{x}_2 \in \mathbb{R}^{n-m} \right\}.$$

The projection of $\langle \mathbf{x} \rangle = \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$ on the $\mathbf{x}_1$-space is defined by:

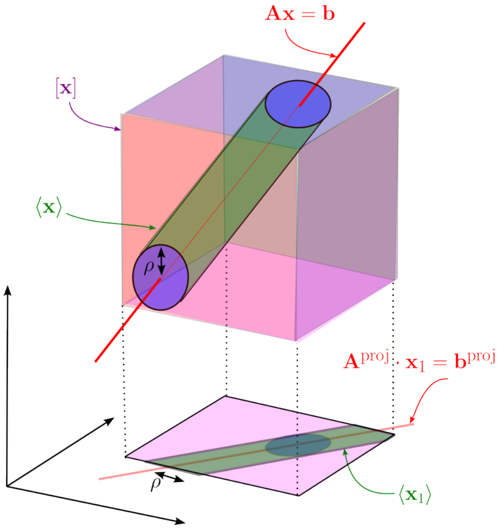
$$\mathsf{proj}_{1:m} \langle \mathbf{x} \rangle = < [\mathbf{x}_1], \mathbf{A}^{\mathsf{proj}}, \mathbf{b}^{\mathsf{proj}}, \rho >$$

where

$$\begin{array}{ccc} (\mathbf{A}_1, \mathbf{A}_2) & = & \mathbf{A} \\ \mathbf{A}^{\mathsf{proj}} & = & \left(\mathbf{A}_1^{-1} \mathbf{A}_2 \right)^{\perp} \\ \mathbf{b}^{\mathsf{proj}} & = & \mathbf{A}^{\mathsf{proj}} \mathbf{A}_1^{-1} \mathbf{b} \\ [\mathbf{x}_1] & = & \mathsf{proj}_{1:m}([\mathbf{x}]) \end{array}$$

Proposition.

$$\mathbf{x} \in \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle \Rightarrow \mathsf{proj}_{1:m}\mathbf{x} \in \mathsf{proj}_{1:m} \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$


$\mathbf{Ax} = \mathbf{b}$
$[\mathbf{x}]$
$\langle \mathbf{x} \rangle$
ρ
$\mathbf{A}^{\text{proj}} \cdot \mathbf{x}_1 = \mathbf{b}^{\text{proj}}$
ρ
$\langle \mathbf{x}_1 \rangle$

Intersection of two buches

The intersection between $\langle \mathbf{x}_1 \rangle$ and $\langle \mathbf{x}_2 \rangle$ is defined by

$$\langle [\mathbf{x}]_1, \mathbf{A}_1, \mathbf{b}_1, \rho_1 \rangle \cap \langle [\mathbf{x}]_2, \mathbf{A}_2, \mathbf{b}_2, \rho_2 \rangle = \langle [\mathbf{x}]_3, \mathbf{A}_3, \mathbf{b}_3, \rho_3 \rangle$$

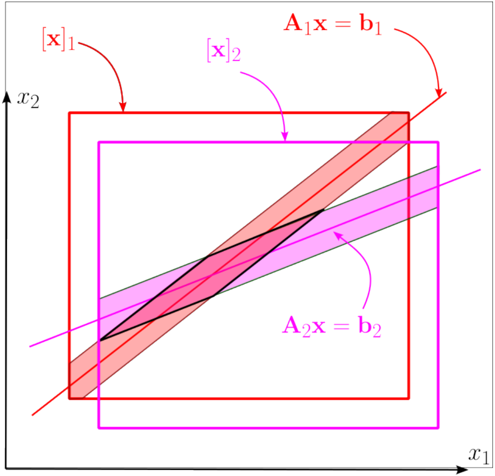
where

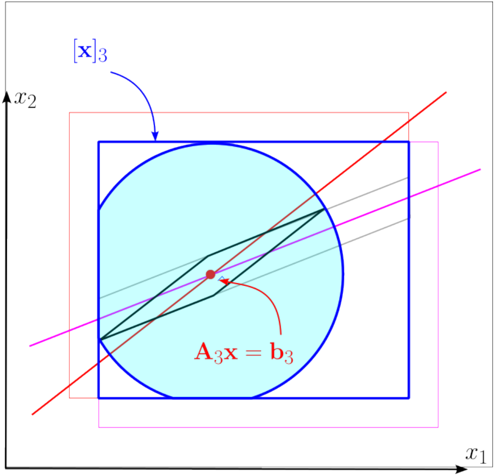
$$\begin{aligned} [\mathbf{x}]_3 &= [\mathbf{x}]_1 \cap [\mathbf{x}]_2 \\ \mathbf{A}_3 &= \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \\ \mathbf{b}_3 &= \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \\ \rho_3 &= \frac{1}{\sin\theta} \sqrt{\rho_b^2 + \rho_a^2 + 2\rho_a \rho_b \cdot \cos\theta} \end{aligned}$$

where θ is the principle angle between span$(\mathbf{A}_1)$ and span$(\mathbf{A}_2)$.

Proposition

$$\mathbf{x} \in \langle \mathbf{x}_1 \rangle \text{ and } \mathbf{x} \in \langle \mathbf{x}_2 \rangle \Rightarrow \mathbf{x} \in \langle \mathbf{x}_1 \rangle \cap \langle \mathbf{x}_2 \rangle .$$


[x]1
[x]2
A1x = b1
x2
A2x = b2
x1


$[\mathbf{x}]_3$
x_2
x_1
$\mathbf{A}_3\mathbf{x} = \mathbf{b}_3$

Buche contractors

Consider again

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$, $m < n$.

A *buche contractor* $\mathscr{B}([\mathbf{x}])$ associated to $\mathbb{X}$ is an operator

$$\mathscr{B} : [\mathbf{x}] \rightarrow \langle\mathbf{x}\rangle = \langle[\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho\rangle$$

which satisfies

$$\mathbb{X} \cap [\mathbf{x}] \subset \langle\mathbf{x}\rangle .$$

Moreover, $\mathscr{B}$ is said to have an order k if for all nested sequence of boxes converging to $\mathbf{x} \in \mathbb{X}$,

$$\frac{h\left(\langle\mathbf{x}\rangle, \mathbb{X} \cap [\mathbf{x}]\right)}{\mathrm{rad}([\mathbf{x}])^k} \rightarrow 0.$$

Proposition. For $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$, the operator

$$\mathscr{B} : [\mathbf{x}] \to \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$

where

$$\begin{array}{rcl} \mathbf{b} & = & \mathbf{A}\bar{\mathbf{x}} - \mathbf{f}(\bar{\mathbf{x}}) \\ \mathbf{A} & = & \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) \\ \rho & = & \|\mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{A}^\mathsf{T}\right)^{-1}\| \cdot \mathsf{ub}\|[\mathbf{e}]\| \\ [\mathbf{e}] & = & \left(\mathbf{A} - \frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])\right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \end{array}$$

is a buche contractor of order 1.

Test-case

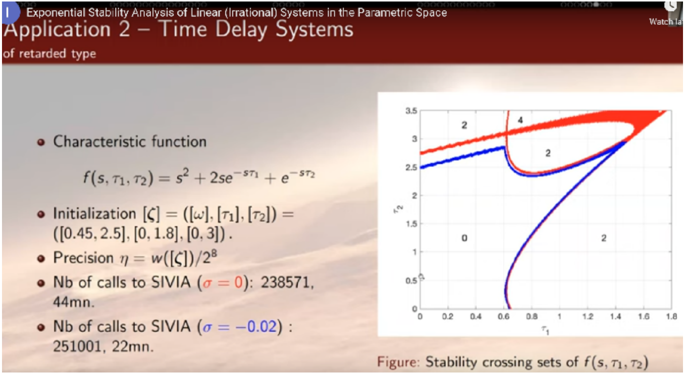
Consider the system :

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_3^2 + 2x_3 \sin(x_3 x_1) + \cos(x_3 x_2) \\ 2x_3 \cos(x_3 x_1) - \sin(x_3 x_2) \end{pmatrix} = \mathbf{0}$$

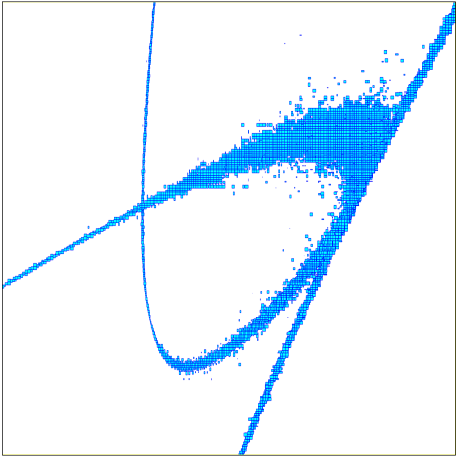
We want to characterize

$$\mathbb{P} = \{(x_1, x_2) | \exists x_3 \in [0, 10], \mathbf{f}(x_1, x_2, x_3) = \mathbf{0}\}.$$

With $[x_1] = [0, 2.5]$, $[x_2] = [1, 4]$, $[x_3] = [0, 10]$, a Matlab implementation, with a forward-backward contractor, and $\varepsilon = 2^{-8}$, yields:



https://youtu.be/DaR2NZZIV10?t=2453



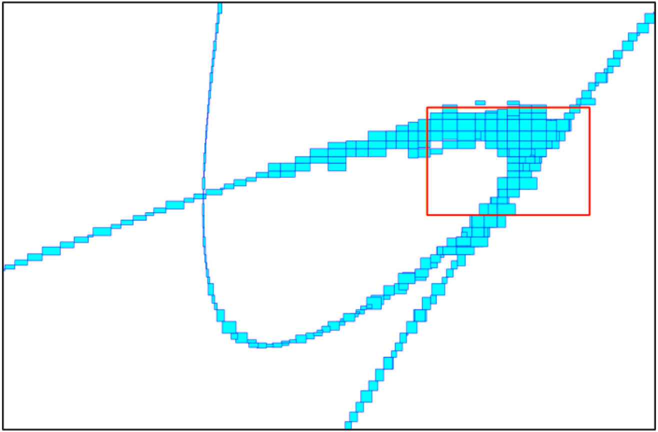
$\varepsilon = 2^{-8}$, Codac [16] generated 43173 boxes.

We still have a *Clustering effect*

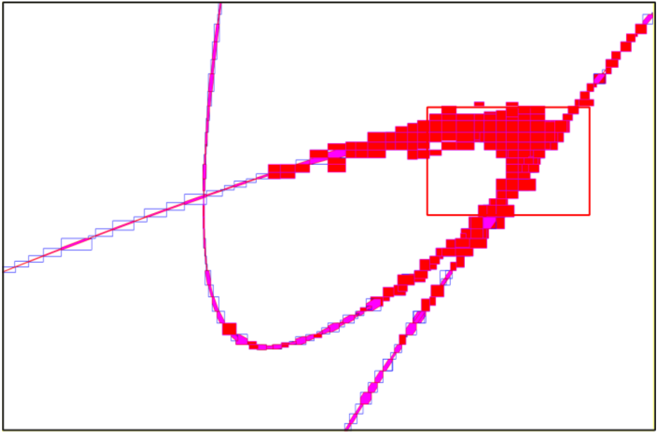
Algorithm with buches

Initialization: $\mathscr{L} = \{[\mathbf{x}]\}$; $\mathbb{X}^+ = \{\}$.
Resolution.

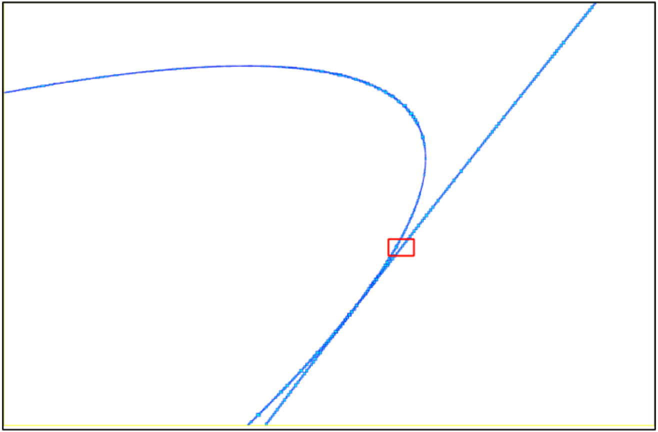
- **Contraction step**. Replace each $[\mathbf{x}] \in \mathscr{L}$, by the smallest box enclosing the buche $\langle \mathbf{x} \rangle = \mathscr{B}([\mathbf{x}])$.
- **Bisection step**. For each $[\mathbf{x}] \in \mathscr{L}$, if Vol$(\langle \mathbf{x} \rangle) < \varepsilon^n$ then push $\langle \mathbf{x} \rangle$ on $\mathbb{X}^+$, otherwise, bisect $[\mathbf{x}]$ and push the two resulting boxes in $\mathscr{L}$.



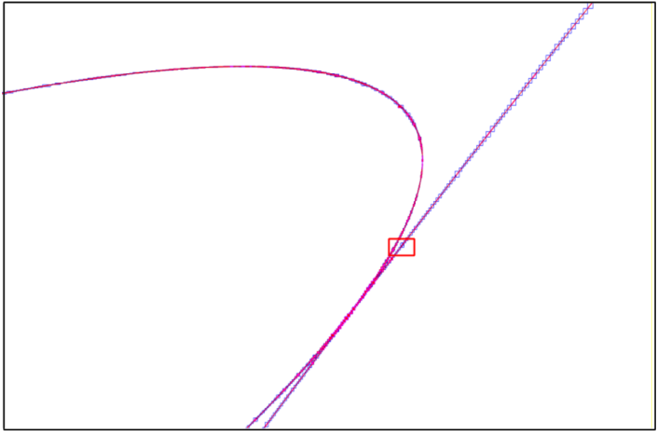
(a) Asymptotically minimal contractor



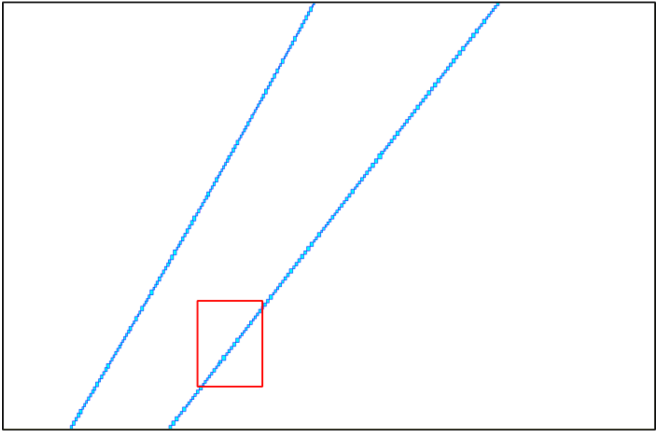
(a) Buche contractor



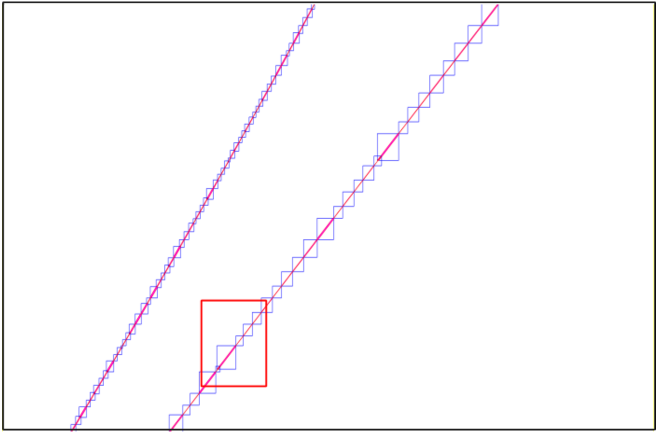
(b) Asymptotically minimal contractor



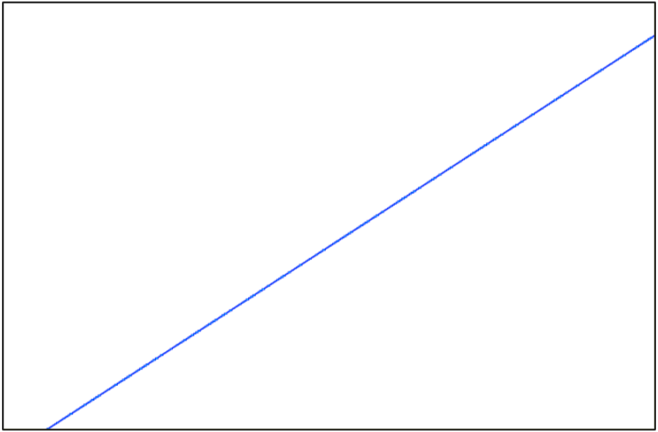
(b) Buche contractor



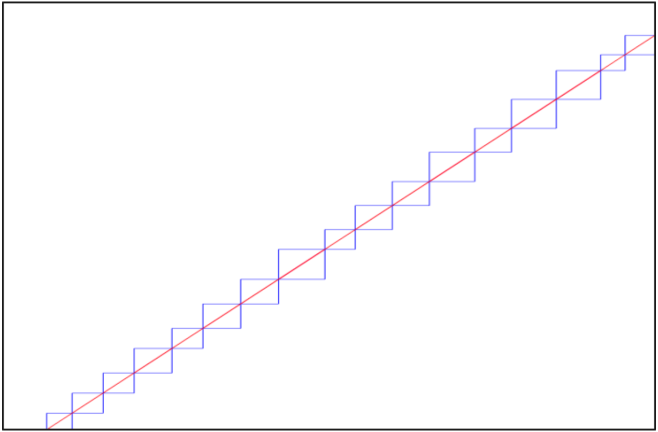
(c) Asymptotically minimal contractor



(c) Buche contractor



(d) Asymptotically minimal contractor



(d) Buche contractor

	Case (a)	Case (b)	Case (c)	Case (d)
$[\mathbf{x}]$	$\begin{pmatrix} [0,2] \\ [2,4] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.3,1.8] \\ [3,3.5] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.595,1.615] \\ [3.2,3.22] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.601,1.603] \\ [3.202,3.206] \\ [0,10] \end{pmatrix}$
ε	2^{-4}	2^{-8}	2^{-12}	2^{-16}
$T_1(s)$	0.51	0.84	0.11	0.19
$T_2(s)$	0.86	1.46	0.17	0.05
V_1	0.092	$2.05 \cdot 10^{-3}$	$3.06 \cdot 10^{-6}$	$4.18 \cdot 10^{-7}$
V_2	0.02	$1.93 \cdot 10^{-3}$	$7.2 \cdot 10^{-7}$	$5.78 \cdot 10^{-9}$
$\frac{V_1}{V_2}$	3.8	4.6	15.3	72.4

The code source of the test-case is based on the codac library [16] and can be found at:

https://www.ensta-bretagne.fr/jaulin/buche.html

References

1. Interval analysis : [11][9] [13][15][14]
2. Solving equations : [5][12][18]
3. Contractor programming : [3][2]
4. Applications of interval analysis : [1][17][8]
5. Asymptotically minimal contractor [7].
6. Order one approximation : [4][6]
7. Test-case : [19][10]

J. Alexandre dit Sandretto, G. Trombettoni, D. Daney, and G. Chabert.
Certified calibration of a cable-driven robot using interval contractor programming.
In F. Thomas and A. P. Gracia, editors, *Computational Kinematics, Mechanisms and Machine Science*, Spinger, 2014.

M. Cébério and L. Granvilliers.
Solving nonlinear systems by constraint inversion and interval arithmetic.
In J. A. Campbell and E. Roanes-Lozano, editors, *Artificial Intelligence and Symbolic Computation, International Conference AISC 2000 Madrid, Spain, July 17-19, 2000*, volume 1930 of *Lecture Notes in Computer Science*, pages 127–141. Springer, 2000.

G. Chabert and L. Jaulin.

Contractor Programming.
Artificial Intelligence, 173:1079–1100, 2009.

M. Godard, L. Jaulin, and D. Masse.
Inner and outer approximation of the image of a set by a nonlinear function.
International Journal of Advanced Research (IJAR), ??(??):??–??, 2025.

E. R. Hansen.
Global Optimization using Interval Analysis.
Marcel Dekker, New York, NY, 1992.

M. HladÃk.
Enclosures for the solution set of parametric interval linear systems.
International Journal of Applied Mathematics and Computer Science, 22(3):561–574, 2012.

L. Jaulin.
Asymptotically minimal interval contractors based on the centered form.
Acta Cybernetica, 26(4):933–954, 2024.

L. Jaulin, M. Kieffer, O. Didrit, and E. Walter.
Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics.
Springer-Verlag, London, 2001.

V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl.
Computational complexity and feasibility of data processing and interval computations.
Reliable Computing, 4(4):405–409, 1998.

R. Malti, M. Rapaić, and V. Turkulov.
A unified framework for robust stability analysis of linear irrational systems in the parametric space.

Annual Reviews in Control, 57, 2024.
https://hal.archives-ouvertes.fr/hal-03646956.

R. Moore.
Methods and Applications of Interval Analysis.
Society for Industrial and Applied Mathematics, jan 1979.

H. Ratschek and J. Rokne.
New Computer Methods for Global Optimization.
Ellis Horwood, Chichester, UK, 1988.

A. Rauh and E. Auer.
Modeling, Design, and Simulation of Systems with Uncertainties.
Springer, 2011.

N. Revol.
Introduction to the IEEE 1788-2015 Standard for Interval Arithmetic.

10th International Workshop on Numerical Software Verification - NSV 2017, 2017.

N. Revol, L. Benet, L. Ferranti, and S. Zhilin.
Testing interval arithmetic libraries, including their ieee-1788 compliance.
arXiv:2205.11837, math.NA, 2022.

S. Rohou, B. Desrochers, and F. L. Bars.
The codac library.
Acta Cybernetica, 26(4):871–887, 2024.

S. Rohou, L. Jaulin, L. Mihaylova, F. L. Bars, and S. Veres.
Reliable Robot Localization.
Wiley, dec 2019.

S. M. Rump.
INTLAB - INTerval LABoratory.

In T. Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer, Dordrecht, the Netherlands, 1999.

V. Turkulov, M. Rapaić, and R. Malti.
Stability analysis of time-delay systems in the parametric space.

Automatica, 157, 2023.