Interval analysis for the validation of autonomous vehicles properties

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Interval analysis

Problem. Given $f:\mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= ?, \\
[-1,3]\cdot[2,5] &= ?, \\
\mathsf{abs}\,([-7,1]) &= ?
\end{aligned}
$$

Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= [1,8], \\ [-1,3]\cdot[2,5] &= [-5,15], \\ \mathsf{abs}([-7,1]) &= [0,7] \end{aligned}$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned} [f]([x_1],[x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos[x_2] \\ &\quad + \sin[x_1] \cdot \sin[x_2] + 2. \end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic

If $\diamond \in \{+,-,\cdot,/,\max,\min\}$

$$[x] \diamond [y] \quad = \quad [\ \{x \diamond y \mid x \in [x], y \in [y]\}\ ].$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A} \subset \mathbb{R}$.

If $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Inclusion functions

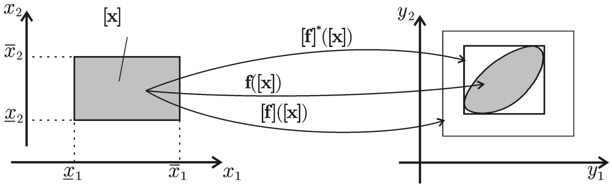
A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* for $\mathbf{f}$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{x}]).$$



Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

If $\mathbf{f}$ is given by

Algorithm $\mathbf{f}(\text{in} : \mathbf{x} = (x_1, x_2, x_3), \text{ out} : \mathbf{y} = (y_1, y_2))$
$z := x_1$
for$k := 0$ to 100
$\quad\quad z := x_2(z + k \cdot x_3)$
next
$y_1 := z$
$y_2 := \sin(z x_1)$

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in : $[\mathbf{x}] = ([x_1],[x_2],[x_3])$, out : $[\mathbf{y}] = ([y_1],[y_2])$)
$[z] := [x_1]$ for $k := 0$ to 100 $\quad [z] := [x_2] \cdot ([z] + k \cdot [x_3])$ next $[y_1] := [z]$ $[y_2] := \sin([z] \cdot [x_1])$

Is $[\mathbf{f}]$ convergent? thin? monotonic?

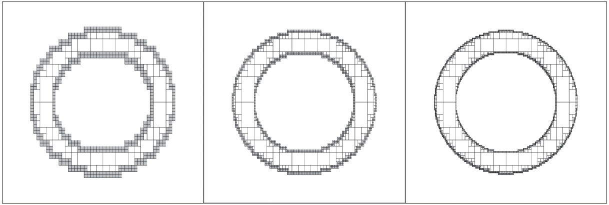
Set inversion

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$. Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1, 2]\}.$$



Let $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbb{Y}$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

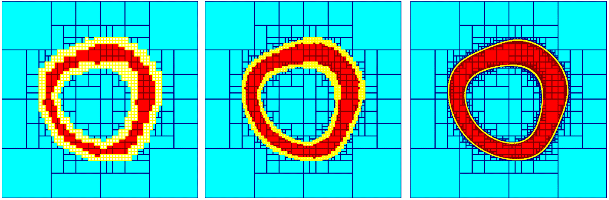
$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We shall use the following tests.

$$\begin{array}{llll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

Contractors

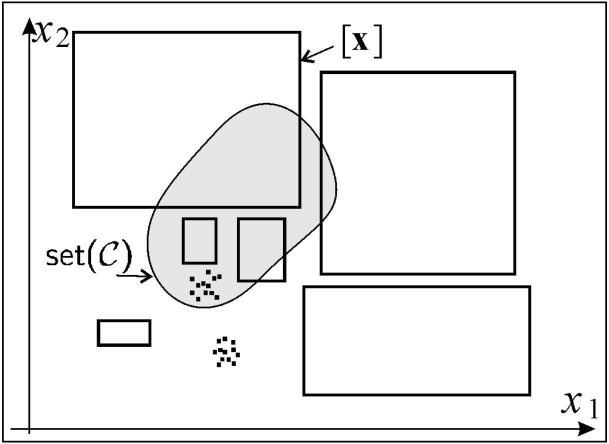


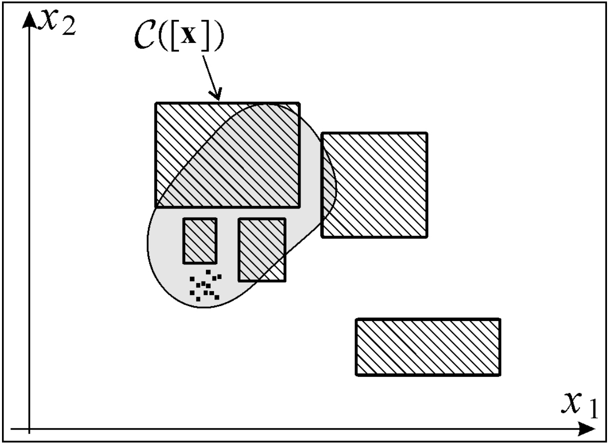
To characterize $\mathbb{X} \subset \mathbb{R}^n$, bisection algorithms bisect all boxes in all directions and become inefficient. Interval methods can still be useful if

- the solution set $\mathbb{X}$ is small (optimization problem, solving equations),
- contraction procedures are used as much as possible,
- bisections are used only as a last resort.

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance),} \\ \mathscr{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness).} \end{cases}$$


x_2
[x]
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) \end{cases}$$

Exercice. Let x, y, z be 3 variables such that

$$\begin{array}{rcl} x & \in & [-\infty, 5] \\ y & \in & [-\infty, 4] \\ z & \in & [6, \infty] \\ z & = & x + y. \end{array}$$

Contract the intervals for x, y, z.

Solution. We have

$$\begin{aligned} x &\in [x] = [2,5] \\ y &\in [y] = [1,4] \\ z &\in [z] = [6,9], \end{aligned}$$

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$\begin{array}{rl} z = x + y \Rightarrow \quad z \in & [6,\infty] \cap ([-\infty,5] + [-\infty,4]) \\ & = [6,\infty] \cap [-\infty,9] = [6,9]. \\ x = z - y \Rightarrow \quad x \in & [-\infty,5] \cap ([6,\infty] - [-\infty,4]) \\ & = [-\infty,5] \cap [2,\infty] = [2,5]. \\ y = z - x \Rightarrow \quad y \in & [-\infty,4] \cap ([6,\infty] - [-\infty,5]) \\ & = [-\infty,4] \cap [1,\infty] = [1,4]. \end{array}$$

The contractor associated with $z = x + y$ is:

Algorithm pplus(inout: $[z],[x],[y]$)

$[z] := [z] \cap ([x]+[y])$ // $z = x + y$
$[x] := [x] \cap ([z]-[y])$ // $x = z - y$
$[y] := [y] \cap ([z]-[x])$ // $y = z - x$

The contractor associated with $z = x \cdot y$ is:

Algorithm pmult (inout: $[z],[x],[y]$)

$[z] := [z] \cap ([x] \cdot [y])$	// $z = x \cdot y$
$[x] := [x] \cap ([z] \cdot 1/[y])$	$//x = z/y$
$[y] := [y] \cap ([z] \cdot 1/[x])$	$//\ y = z/x$

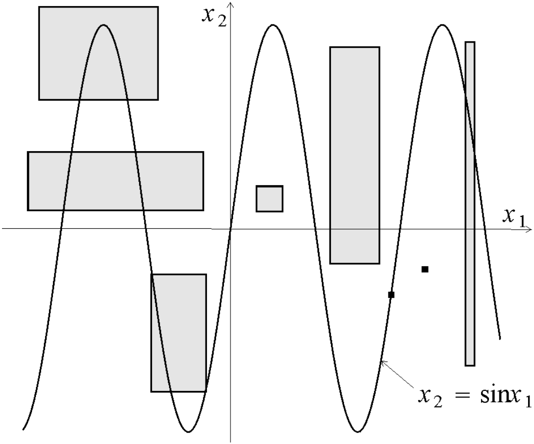
The contractor associated with $y = \exp x$ is:

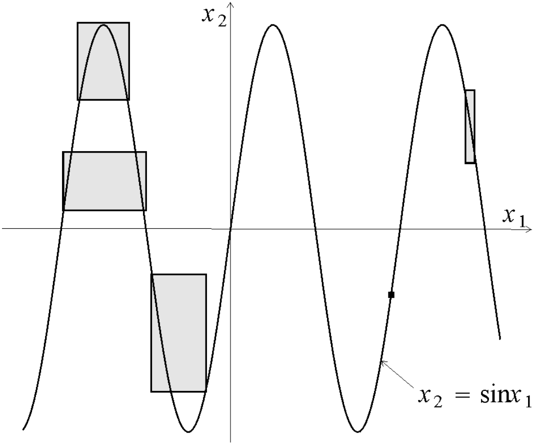
	Algorithm pexp (inout: $[y], [x]$)
1	$[y] := [y] \cap \exp([x])$
2	$[x] := [x] \cap \log([y])$

Any constraint for which such a projection procedure is available will be called a *primitive constraint*.

Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$


x_2
x_1
$x_2 = \sin x_1$


x_2
x_1
$x_2 = \sin x_1$

Decomposition

$$x + \sin(xy) \leq 0,$$
$$x \in [-1,1], y \in [-1,1]$$

Decomposition

$$x + \sin(xy) \leq 0,$$
$$x \in [-1,1], y \in [-1,1]$$

can be decomposed into

$$\left\{ \begin{array}{l} a = xy \\ b = \sin(a) \\ c = x + b \end{array} \right. , \qquad \begin{array}{ll} x \in [-1,1] & a \in [-\infty,\infty] \\ y \in [-1,1] & b \in [-\infty,\infty] \\ & c \in [-\infty,0] \end{array}$$

Forward Backward contractor

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2],$$

we decompose into

$$\begin{array}{rcl} a & = & x_1 + x_2 \\ b & = & a \cdot x_3 \\ b & \in & [1, 2] \end{array}$$

For the equation

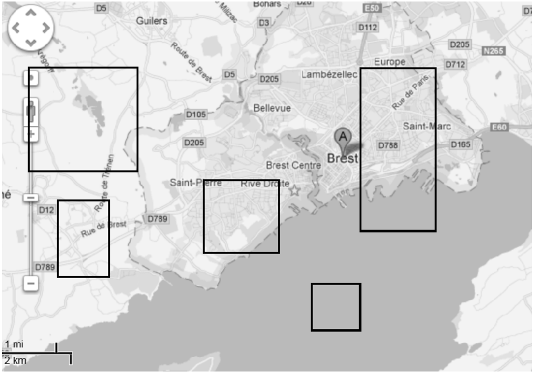
$$(x_1 + x_2) \cdot x_3 \in [1,2],$$

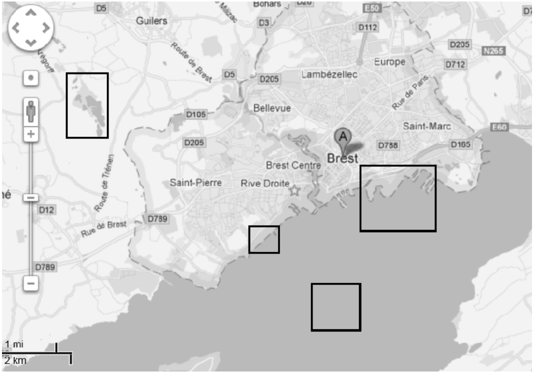
we have the following contractor:

Algorithm $\mathscr{C}$ (inout$[x_1],[x_2],[x_3]$)	
$[a] = [x_1] + [x_2]$	// $a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	// $b = a \cdot x_3$
$[b] = [b] \cap [1,2]$	// $b \in [1,2]$
$[x_3] = [x_3] \cap \frac{[b]}{[a]}$	// $x_3 = \frac{b}{a}$
$[a] = [a] \cap \frac{[b]}{[x_3]}$	// $a = \frac{b}{x_3}$
$[x_1] = [x_1] \cap [a] - [x_2]$	// $x_1 = a - x_2$
$[x_2] = [x_2] \cap [a] - [x_1]$	// $x_2 = a - x_1$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.


Guilers
Lambézellec
Europe
Bellevue
Saint-Marc
Brest Centre
Brest
Saint-Pierre
Rive Droite
Route de Brest
Rue de Paris
Rue de Brest
D5
D3
D112
E50
D205
N265
D712
D105
D788
D165
E60
D12
D789
1 mi
2 km


Guilers
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D112
E50
D205
N265
Europe
D712
Lambézellec
Route de Brest
D205
Bellevue
Rue de Paris
D105
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Route de Trénen
Saint-Pierre
Rive Droite
D12
D789
Rue de Brest
D789
1 mi
2 km

Solving equations

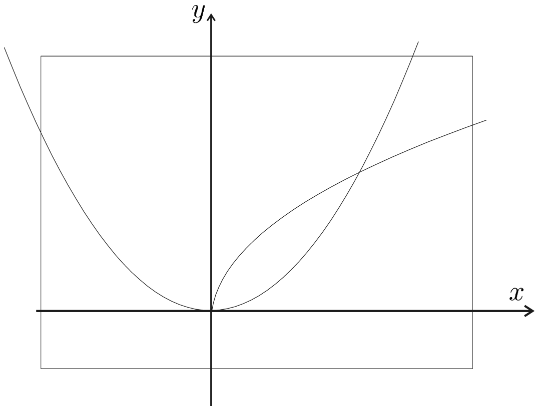
Consider the system of two equations.

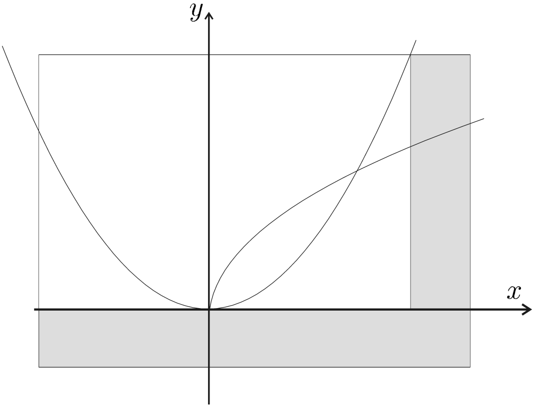
$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

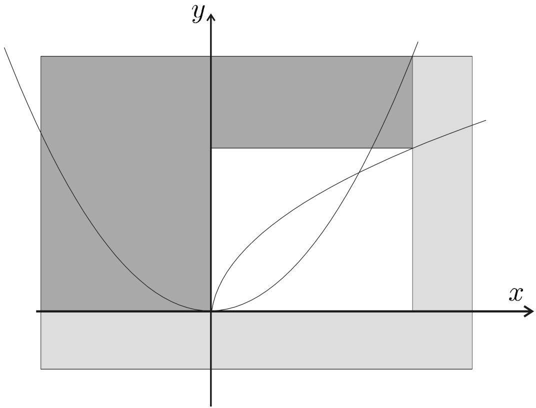
We can build two contractors

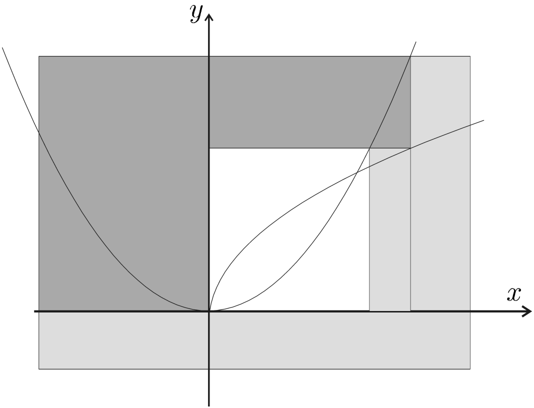
$$\mathscr{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

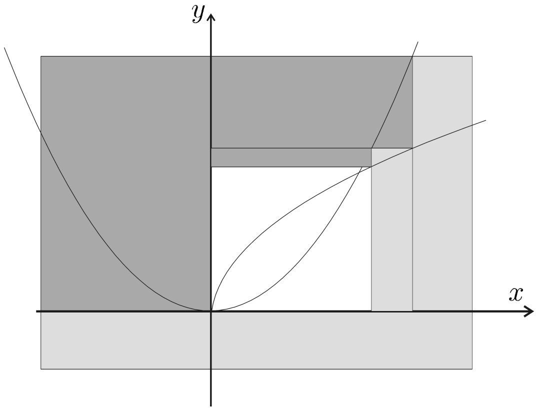
$$\mathscr{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$

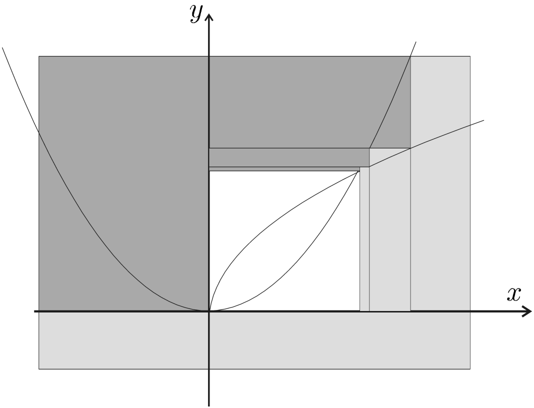

y
x

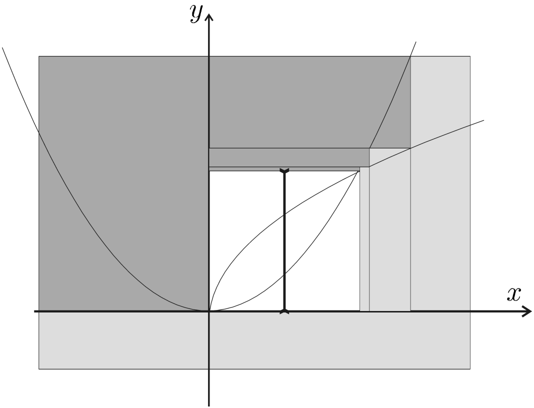

y
x

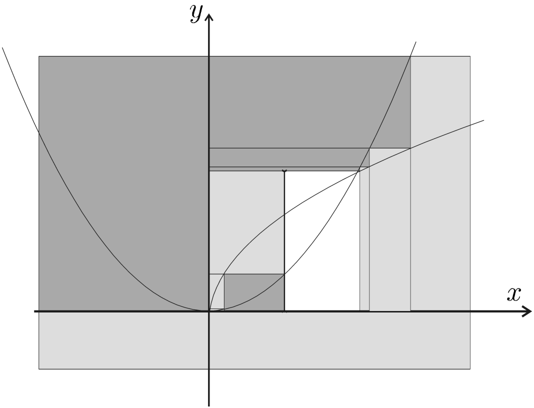

y
x

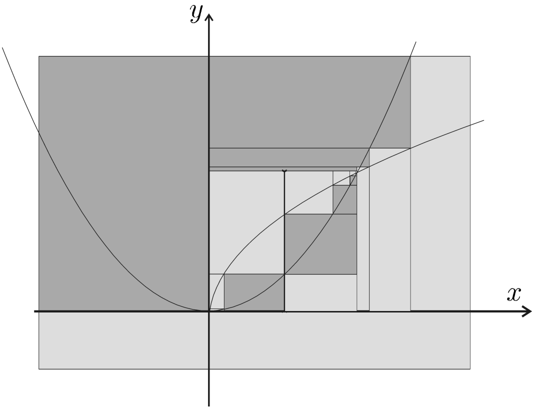

y
x


y
x


y
x

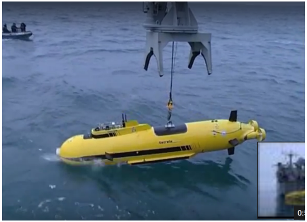

y
x


y
x


y
x

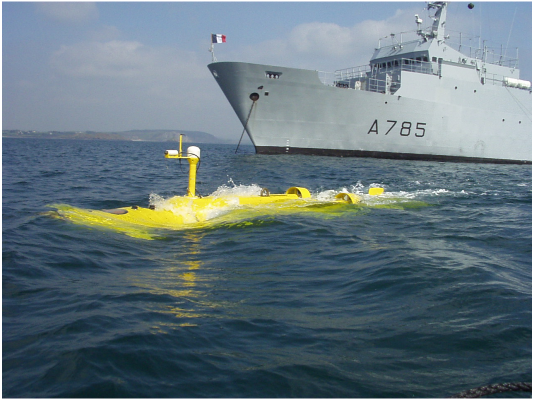
SLAM

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},\mathbf{u}) & \text{(evolution equation)} \\ \mathbf{y} & = & \mathbf{g}(\mathbf{x},\mathbf{u}) & \text{(observation equation)} \\ \mathbf{z}_i & = & \mathbf{h}(\mathbf{x},\mathbf{u},\mathbf{m}_i) & \text{(mark equation)} \end{cases}$$



Redermor (GESMA, Brest)

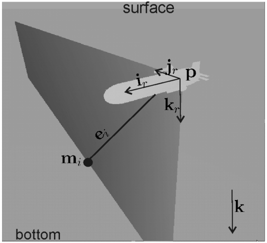
https://youtu.be/X0IqZxb-tFs

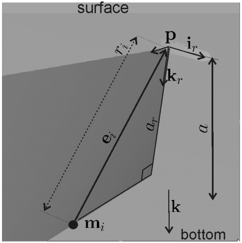

A785

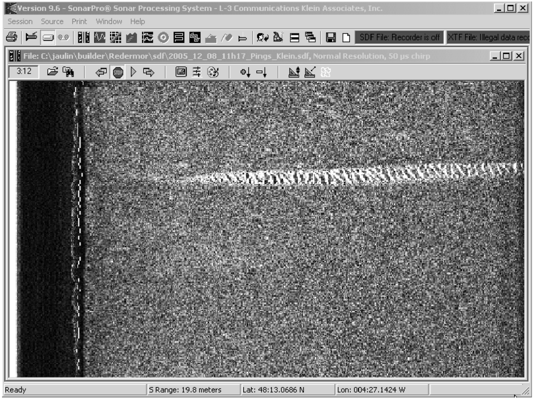
GPS (Global positioning system), only at the surface.

$$t_0 = 0000 \text{ s}, \quad \ell^0 = (-4.4582279°, 48.2129206°) \pm 2.5m$$
$$t_f = 6000 \text{ s}, \quad \ell^f = (-4.4546607°, 48.2191297°) \pm 2.5m$$

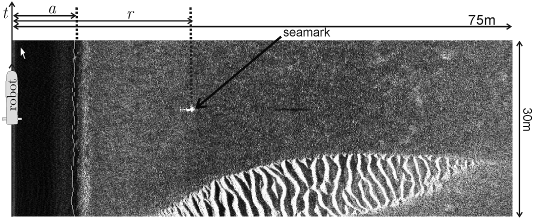
Sonar (KLEIN 5400 side scan sonar).


surface
$\mathbf{j}_r$
$\mathbf{p}$
$\mathbf{i}_r$
$\mathbf{k}_r$
$\mathbf{e}_i$
$\mathbf{m}_i$
$\mathbf{k}$
bottom


surface
$\mathbf{p}$
r_i
$\mathbf{i}_r$
$\mathbf{k}_r$
a_r
a
$\mathbf{e}_i$
$\mathbf{k}$
$\mathbf{m}_i$
bottom



Screenshot of SonarPro



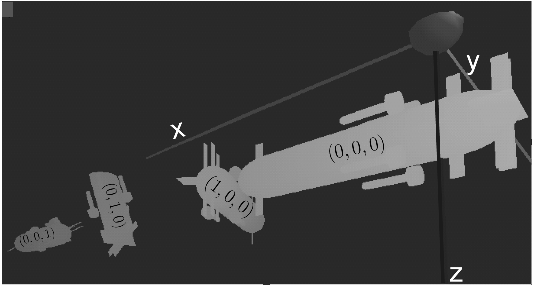
Mine detection with SonarPro

Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \tilde{\mathbf{v}}_r + 0.004 * [-1,1] . \tilde{\mathbf{v}}_r + 0.004 * [-1,1]$$

Inertial unit (Octans III).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.[-1,1] \\ 1.75 \times 10^{-4}.[-1,1] \\ 5.27 \times 10^{-3}.[-1,1] \end{pmatrix}.$$


x
y
z
(0, 0, 0)
(1, 0, 0)
(0, 1, 0)
(0, 0, 1)

Six landmarks have been detected.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	1054	1092	1374	1748	3038	3688	4024	4817	5172	5232	5279	5688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

Contraint network

$$t \in \{.0, 0.1, 0.2, \ldots, 5999.9\}$$

$$i \in \{0, 1, \ldots, 11\}$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) * \frac{\pi}{180}\right) & 0 \end{pmatrix} \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix}$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t))$$

$$\mathbf{R}_{\psi}(t)=\begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_{\theta}(t)=\begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix}$$

$$\mathbf{R}_{\varphi}(t)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{pmatrix}$$

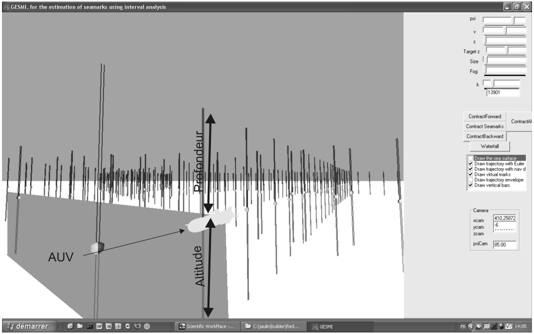
$$\mathbf{R}(t) = \mathbf{R}_{\psi}(t)\mathbf{R}_{\theta}(t)\mathbf{R}_{\varphi}(t)$$

$$\dot{\mathbf{p}}(t) = \mathbf{R}(t).\mathbf{v}_r(t)$$

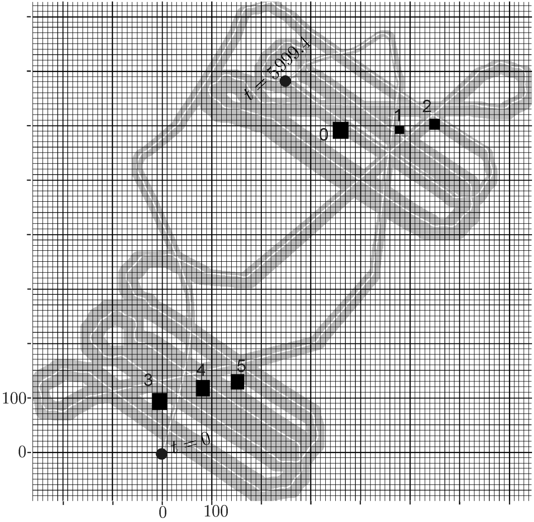
$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| = r(i)$$

$$\mathbf{R}^{\mathsf{T}}(\tau(i))\left(\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))\right) \in [0] \times [0,\infty]^{\times 2}$$

$$m_z(\sigma(i)) - p_z(\tau(i)) - a(\tau(i)) \in [-0.5, 0.5]$$



youtu.be/lzJtAfAT7h4


t = 5999.4
0
1
2
3
4
5
t = 0
100
0
0
100

tubex-lib
1.0
Search docs

The following is an extract from the paper Guaranteed computation of robot trajectories.

Definition

A tube $[\mathbf{x}](\cdot)$ is defined as an envelope enclosing an uncertain trajectory $\mathbf{x}(\cdot):\mathbb{R}\rightarrow\mathbb{R}^n$. It is built as an interval of two functions $[\mathbf{x}^-(\cdot),\mathbf{x}^+(\cdot)]$ such that $\forall t,\ \mathbf{x}^-(t)\leqslant\mathbf{x}^+(t)$. A trajectory $\mathbf{x}(\cdot)$ belongs to the tube $[\mathbf{x}](\cdot)$ if $\forall t,\ \mathbf{x}(t)\in[\mathbf{x}](t)$. Fig. 1 illustrates a tube implemented with a set of boxes. This sliced implementation is detailed hereinafter.

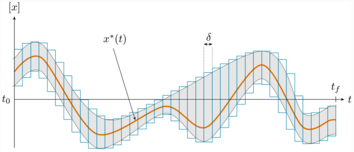
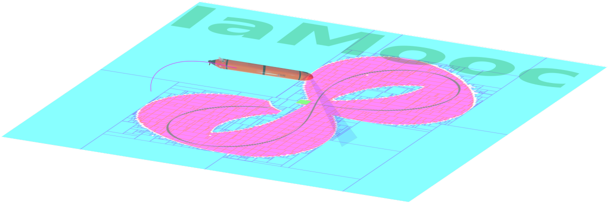


Fig. 1 A tube $[x](\cdot)$ represented by a set of slices. This representation can be used to enclose signals such as $x^*(\cdot)$.

Code example:

```
float timestep = 0.1;
Interval domain(0,10);
Tube x(domain, timestep, Function("t", "(t-5)^2 + [-0.5,0.5]"));
```

http://www.codac.io/



https://www.ensta-bretagne.fr/iamooc/

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