Inertial control of a flat spinning disk

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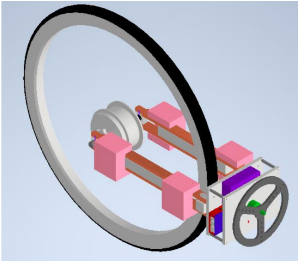
  

Roue autonome (Damien Esnault)

1. Flat disk

Consider a flat disk spinning in the space without any gravity.
Its inertia matrix

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

satisfies,

$$I_1 > 0, I_2 > 0, I_3 > 0 \qquad \text{(positivity)}$$
$$I_1 = I_2 + I_3 \qquad \text{(flatness)}$$

We assume that

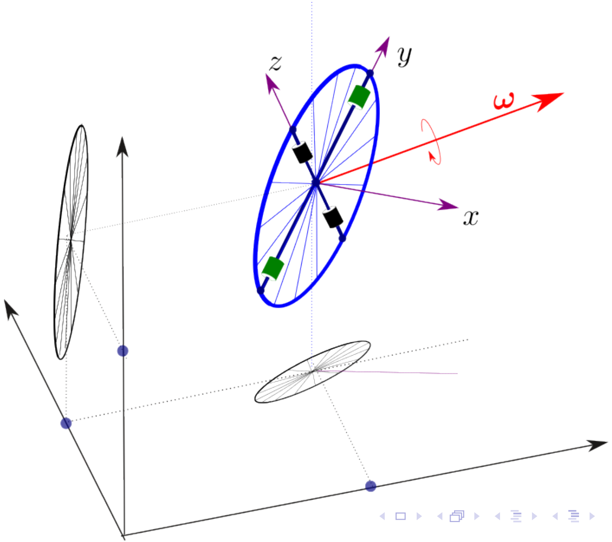
- we can control $\mathbf{I}$ in a strap down manner,
- the center of gravity of the disk remains static.

At rest,

$$\bar{\mathbf{I}} = \begin{pmatrix} \bar{I}_1 & 0 & 0 \\ 0 & \bar{I}_2 & 0 \\ 0 & 0 & \bar{I}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}mr^2 & 0 & 0 \\ 0 & \frac{1}{4}mr^2 & 0 \\ 0 & 0 & \frac{1}{4}mr^2 \end{pmatrix}$$

where r is the radius of the disk and m is its mass.

Two symmetric pairs of masses can move along the y and z axis.



Thus

$$\mathbf{I} = \begin{pmatrix} \frac{1}{2}mr^2 + \frac{1}{2}\ell_2^2 + \frac{1}{2}\ell_3^2 & 0 & 0 \\ 0 & \frac{1}{4}mr^2 + \frac{1}{2}\ell_3^2 & 0 \\ 0 & 0 & \frac{1}{4}mr^2 + \frac{1}{2}\ell_2^2 \end{pmatrix}$$

We want to control the rotation of the disk.

State equations

The angular momentum $\mathscr{L}$ is

$$\mathscr{L} = \mathbf{R} \cdot \mathbf{I} \cdot \omega_r$$

where ω_r is the rotation vector in the body frame, $\mathbf{R}$ is the orientation.

Since $\mathscr{L}$ is constant, we have

$$\begin{aligned} & \dot{\mathbf{R}}\mathbf{I}\omega_r + \mathbf{R}\cdot\dot{\mathbf{I}}\cdot\omega_r + \mathbf{R}\cdot\mathbf{I}\cdot\dot{\omega}_r = \mathbf{0} \\ \Leftrightarrow\quad & \mathbf{R}^{\mathsf{T}}\dot{\mathbf{R}}\mathbf{I}\omega_r + \dot{\mathbf{I}}\cdot\omega_r + \mathbf{I}\cdot\dot{\omega}_r = \mathbf{0} \end{aligned}$$

We get the Euler's rotation equation

$$\dot{\omega}_r = \mathbf{I}^{-1}\cdot\left(-\dot{\mathbf{I}}\cdot\omega_r - \omega_r \wedge \mathbf{I}\cdot\omega_r\right)$$

The state equations are thus

$$\begin{cases} \dot{\mathbf{R}} & = & \mathbf{R}\cdot(\omega_r \wedge) \\ \dot{\omega}_r & = & \mathbf{I}^{-1}\cdot\left(-\dot{\mathbf{I}}\cdot\omega_r - \omega_r \wedge \mathbf{I}\cdot\omega_r\right) \end{cases}$$

Set

$$\omega_r = (\omega_1, \omega_2, \omega_3)^\mathsf{T}.$$

The Euler equation $\dot{\omega}_r = \mathbf{I}^{-1} \cdot \left(-\dot{\mathbf{I}} \cdot \omega_r - \omega_r \wedge \mathbf{I} \cdot \omega_r\right)$ rewrites into

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}^{-1} \left(\begin{pmatrix} -\dot{I}_1 \omega_1 \\ -\dot{I}_2 \omega_2 \\ -\dot{I}_3 \omega_3 \end{pmatrix} - \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \wedge \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix} \right)$$

$$\begin{cases} \dot{\omega}_1 &= I_1^{-1} \cdot \left(-\dot{I}_1 \omega_1 - (I_3 - I_2)\omega_2\omega_3\right) \\ \dot{\omega}_2 &= I_2^{-1} \cdot \left(-\dot{I}_2 \omega_2 - (I_1 - I_3)\omega_3\omega_1\right) \\ \dot{\omega}_3 &= I_3^{-1} \cdot \left(-\dot{I}_3 \omega_3 - (I_2 - I_1)\omega_1\omega_2\right) \end{cases}$$

Since

$$\begin{aligned} I_1 &= \tfrac{1}{2}mr^2 + \tfrac{1}{2}\ell_2^2 + \tfrac{1}{2}\ell_3^2 \\ I_2 &= \tfrac{1}{4}mr^2 + \tfrac{1}{2}\ell_2^2 \\ I_3 &= \tfrac{1}{4}mr^2 + \tfrac{1}{2}\ell_3^2 \end{aligned}$$

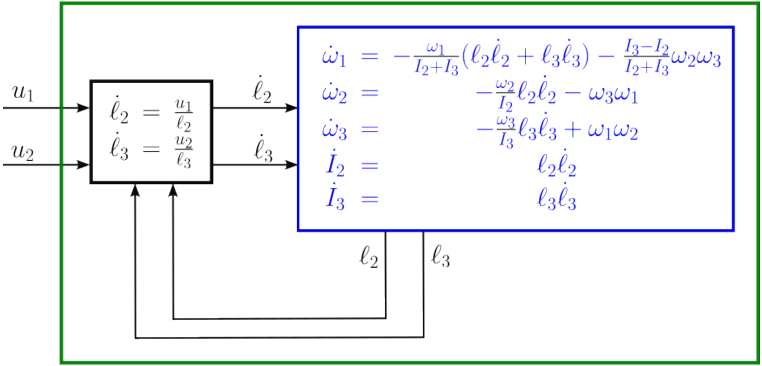
We have

$$\begin{aligned} \dot{I}_1 &= \ell_2\dot{\ell}_2 + \ell_3\dot{\ell}_3 \\ \dot{I}_2 &= \ell_2\dot{\ell}_2 \\ \dot{I}_3 &= \ell_3\dot{\ell}_3 \end{aligned}$$

The state equations of the system are thus

$$\begin{cases} \dot{\omega}_1 &= -\frac{\omega_1}{I_2+I_3}(\ell_2\dot{\ell}_2+\ell_3\dot{\ell}_3)-\frac{I_3-I_2}{I_2+I_3}\omega_2\omega_3 \\ \dot{\omega}_2 &= -\frac{\omega_2}{I_2}\ell_2\dot{\ell}_2-\omega_3\omega_1 \\ \dot{\omega}_3 &= -\frac{\omega_3}{I_3}\ell_3\dot{\ell}_3+\omega_1\omega_2 \\ \dot{I}_2 &= \ell_2\dot{\ell}_2 \\ \dot{I}_3 &= \ell_3\dot{\ell}_3 \end{cases}$$

WIth the inner loop



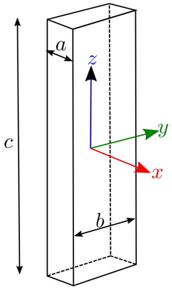
we get

$$\begin{cases} \dot{\omega}_1 & = & -\frac{\omega_1}{I_2+I_3}(u_1+u_2) - \frac{I_3-I_2}{I_2+I_3}\omega_2\omega_3 \\ \dot{\omega}_2 & = & -\frac{\omega_2}{I_2}u_1 - \omega_3\omega_1 \\ \dot{\omega}_3 & = & -\frac{\omega_3}{I_3}u_2 + \omega_1\omega_2 \\ \dot{I}_2 & = & u_1 \\ \dot{I}_3 & = & u_2 \end{cases}$$

The state vector is $\mathbf{x} = (\omega_1, \omega_2, \omega_3, I_2, I_3)$ and the input vector is $\mathbf{u} = (u_1, u_2)$.

2. Dzhanibekov

The Dzhanibekov effect, discovered by Poinsot (1834):
The second principal axes is unstable



https://youtu.be/EA1Rh5MgGKI
(take t=0:22 and t=48:19)

The inertia matrix of the is

$$\mathbf{I} = \int_V \rho(x,y,z) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dx\,dy\,dz$$

i.e.,

$$\mathbf{I} = \frac{m}{12} \cdot \begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

The Euler equation is

$$\begin{cases} \dot{\omega}_1 = -\underbrace{I_1^{-1} \cdot (I_3 - I_2)}_{\alpha_1} \omega_2 \omega_3 \\ \dot{\omega}_2 = -\underbrace{I_2^{-1} \cdot (I_1 - I_3)}_{\alpha_2} \omega_3 \omega_1 \\ \dot{\omega}_3 = -\underbrace{I_3^{-1} \cdot (I_2 - I_1)}_{\alpha_3} \omega_1 \omega_2 \end{cases}$$

Stability

The equilibrium points satisfy $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. We get

$$\begin{aligned} \omega_2\omega_3 &= 0 \\ \omega_1\omega_3 &= 0 \\ \omega_1\omega_2 &= 0 \end{aligned}$$

i.e., among $\omega_1, \omega_2, \omega_3$ two at least are equal to zero.

Since

$$\begin{aligned} \dot{\omega}_1 &= \alpha_1 \omega_2 \omega_3 \\ \dot{\omega}_2 &= \alpha_2 \omega_1 \omega_3 \\ \dot{\omega}_3 &= \alpha_3 \omega_1 \omega_2 \end{aligned}$$

at the neighborhood of $\bar{\omega}$, we have,

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} \simeq \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \\ \bar{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \bar{\omega}_3 & \alpha_1 \bar{\omega}_2 \\ \alpha_2 \bar{\omega}_3 & 0 & \alpha_2 \bar{\omega}_1 \\ \alpha_3 \bar{\omega}_2 & \alpha_3 \bar{\omega}_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 - \bar{\omega}_1 \\ \omega_2 - \bar{\omega}_2 \\ \omega_3 - \bar{\omega}_3 \end{pmatrix}.$$

Let us study the equilibrium at $\bar{\omega} = (\bar{\omega}_1, 0, 0)$:

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} \simeq \begin{pmatrix} \bar{\omega}_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\omega}_1 \alpha_2 \\ 0 & \bar{\omega}_1 \alpha_3 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 - \bar{\omega}_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

The characteristic polynomial is

$$P(s) = s(s^2 - \bar{\omega}_1^2 \alpha_2 \alpha_3).$$

The system is unstable if $\alpha_2 \alpha_3 > 0$.

Now,

$$\begin{aligned}\alpha_2\alpha_3 > 0 \quad &\Leftrightarrow \quad \tfrac{I_3-I_1}{I_2}\cdot\tfrac{I_1-I_2}{I_3} > 0\\ &\Leftrightarrow \quad (I_3-I_1)(I_1-I_2) > 0\end{aligned}$$

This means that I_1 is between I_2 and I_3.

Intermediate axis theorem: the rotation of a rigid body is stable around its first and third principal axes and unstable around its second principal axis.

Simulation

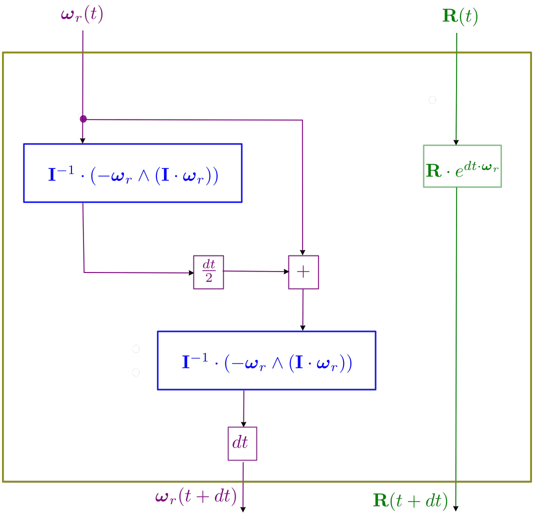
For the simulation,

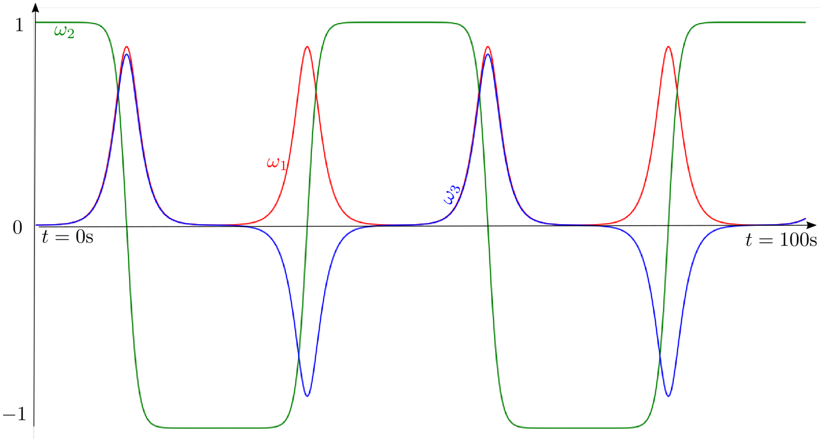
$$\begin{aligned} \dot{\omega}_r &= \mathbf{I}^{-1} \cdot (-\omega_r \wedge (\mathbf{I} \cdot \omega_r)) \\ \dot{\mathbf{R}} &= \mathbf{R} \cdot (\omega_r \wedge) \end{aligned}$$

we take

$$\begin{aligned} \omega_r(t+dt) &:= \omega_r(t) + dt \cdot \mathbf{f}\left(\omega_r(t) + \tfrac{dt}{2} \cdot \mathbf{f}(\omega_r(t))\right) \\ \mathbf{R}(t+dt) &:= \mathbf{R}(t) \cdot e^{dt \cdot \omega_r(t)} \end{aligned}$$

where $\mathbf{f}(\omega_r) = \mathbf{I}^{-1} \cdot (-\omega_r \wedge (\mathbf{I} \cdot \omega_r)$.


$\boldsymbol{\omega}_r(t)$
$\mathbf{R}(t)$
$\mathbf{I}^{-1} \cdot (-\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r))$
$\mathbf{R} \cdot e^{dt \cdot \boldsymbol{\omega}_r}$
$\frac{dt}{2}$
$+$
$\mathbf{I}^{-1} \cdot (-\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r))$
dt
$\boldsymbol{\omega}_r(t + dt)$
$\mathbf{R}(t + dt)$


1
0
−1
ω_2
ω_1
ω_3
$t = 0$s
$t = 100$s

Poinsot ellipsoid

Take again

$$\dot{\omega}_r = -\mathbf{I}^{-1} \cdot (\omega_r \wedge (\mathbf{I} \cdot \omega_r))$$

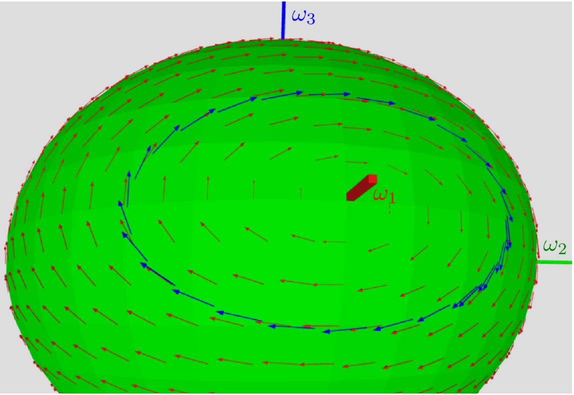
The kinetic energy given by

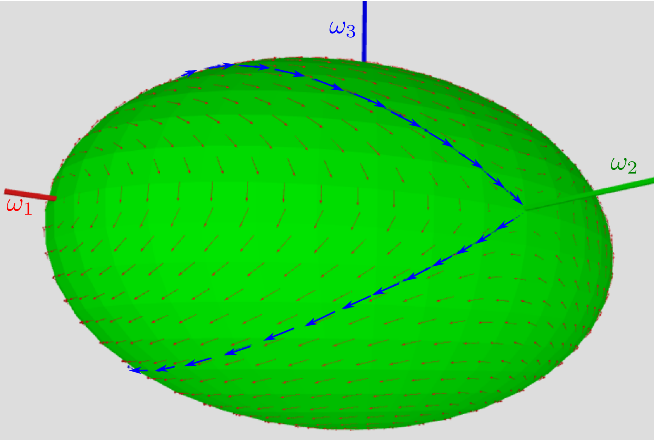
$$E_K = \frac{1}{2} \omega_r^{\mathsf{T}} \mathbf{I} \omega_r$$

Since

$$\begin{aligned}
\dot{E}_K &= \tfrac{d}{dt}\left(\tfrac{1}{2}\omega_r^{\mathsf{T}}\mathbf{I}\omega_r\right) \\
&= \tfrac{1}{2}\left(2\omega_r^{\mathsf{T}}\mathbf{I}\right)\dot{\omega}_r \\
&= -\omega_r^{\mathsf{T}}\cdot\mathbf{I}\cdot\mathbf{I}^{-1}\cdot(\omega_r\wedge(\mathbf{I}\cdot\omega_r)) \\
&= -\omega_r^{\mathsf{T}}(\omega_r\wedge(\mathbf{I}\cdot\omega_r)) = 0
\end{aligned}$$

we get that E_K is constant.


ω_3
ω_1
ω_2



Dzhanibekov effect ; heteroclinic orbit

3. Controller

Recall that

$$\begin{cases} \dot{\omega}_1 &= -\frac{\omega_1}{I_2+I_3}(u_1+u_2) - \frac{I_3-I_2}{I_2+I_3}\omega_2\omega_3 \\ \dot{\omega}_2 &= -\frac{\omega_2}{I_2}u_1 - \omega_3\omega_1 \\ \dot{\omega}_3 &= -\frac{\omega_3}{I_3}u_2 + \omega_1\omega_2 \\ \dot{I}_2 &= u_1 \\ \dot{I}_3 &= u_2 \end{cases}$$

i.e.,

$$\underbrace{\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix}}_{=\dot{\mathbf{x}}} = \underbrace{\begin{pmatrix} -\frac{I_3-I_2}{I_2+I_3}\omega_2\omega_3 \\ -\omega_3\omega_1 \\ \omega_1\omega_2 \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} -\frac{\omega_1}{I_2+I_3} \\ -\frac{\omega_2}{I_2} \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=\mathbf{g}_1(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} -\frac{\omega_1}{I_2+I_3} \\ 0 \\ -\frac{\omega_3}{I_3} \\ 0 \\ 1 \end{pmatrix}}_{=\mathbf{g}_2(\mathbf{x})} \cdot u_2.$$

Choose $V(\mathbf{x}) \geq 0$ such that $V(\mathbf{x}) = 0$ when the objective is reached.

Since

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x}) \cdot u_1 + \mathbf{g}_2(\mathbf{x}) \cdot u_2$$

we have

$$\dot{V}(\mathbf{x}, \mathbf{u}) = \mathscr{L}_{\mathbf{f}} V(\mathbf{x}) + \mathscr{L}_{\mathbf{g}_1} V(\mathbf{x}) \cdot u_1 + \mathscr{L}_{\mathbf{g}_2} V(\mathbf{x}) \cdot u_2$$

Since the solution of

$$\begin{array}{ll} \min & a_1 u_1 + a_2 u_2 \\ \text{s.t.} & u_1^2 + u_2^2 = 1 \end{array}$$

where $a_1 = \mathscr{L}_{\mathbf{g}_1} V(\mathbf{x}), a_2 = \mathscr{L}_{\mathbf{g}_2} V(\mathbf{x})$, is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\frac{1}{\sqrt{a_1^2 + a_2^2}} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

we conclude that a Lyapunov like controller to reach our objective is

$$\mathbf{u} = -\frac{1}{\sqrt{(\mathscr{L}_{\mathbf{g}_1} V(\mathbf{x}))^2 + (\mathscr{L}_{\mathbf{g}_2} V(\mathbf{x}))^2}} \cdot \begin{pmatrix} \mathscr{L}_{\mathbf{g}_1} V(\mathbf{x}) \\ \mathscr{L}_{\mathbf{g}_2} V(\mathbf{x}) \end{pmatrix}.$$

Alignment control

We want the disk spins around one principal axis of the body.
For instance its first axis, *i.e.*, the x-axis.

We define

$$V(\mathbf{x}) = \frac{1}{2}\left(\omega_2^2 + \omega_3^2 + (I_2 - \bar{I}_2)^2 + (I_3 - \bar{I}_3)^2\right).$$

The quantity $\omega_2^2 + \omega_3^2$ corresponds to the *alignment error.*
When $V(\mathbf{x}) = 0$, we have $\omega_2 = \omega_3 = 0$ and $\mathbf{I} = \bar{\mathbf{I}}$.

We have

$$
\begin{aligned}
\dot{V}(\mathbf{x}) &= \omega_2\dot{\omega}_2 + \omega_3\dot{\omega}_3 + (I_2 - \bar{I}_2)\dot{I}_2 + (I_3 - \bar{I}_3)\dot{I}_3 \\
&= -\tfrac{\omega_2^2 u_1}{I_2} - \omega_3\omega_1\omega_2 - \tfrac{\omega_3^2 u_2}{I_3} + \omega_1\omega_2\omega_3 + (I_2 - \bar{I}_2)u_1 + (I_3 - \bar{I}_3)u_2 \\
&= \underbrace{0\cdot\omega_1\omega_2\omega_3}_{\mathscr{L}_{\mathbf{f}}V} + \underbrace{\left(-\frac{\omega_2^2}{I_2} + I_2 - \bar{I}_2\right)}_{\mathscr{L}_{\mathbf{g}_1}V}\cdot u_1 + \underbrace{\left(-\frac{\omega_3^2}{I_3} + I_3 - \bar{I}_3\right)}_{\mathscr{L}_{\mathbf{g}_2}V}\cdot u_2
\end{aligned}
$$

The same result is obtained using Sympy

```
from sympy import *
from sympy.diffgeom import *
w1,w2,w3,I2,I3 = C.coord_functions()
E = C.base_vectors()
F=-((I3-I2)/(I2+I3))*w2*w3*E[0]-w3*w1*E[1]+w1*w2*E[2]
G1=-(w1/(I2+I3))*E[0]-(w2/I2)*E[1]+E[3]
G2=-(w1/(I2+I3))*E[0]-(w3/I3)*E[2]+E[4]
V = 1/2*(w2)**2+1/2*(w3)**2+1/2*(I2-I20)**2
    +1/2*(I3-I30)**2
LfV=LieDerivative(F,V)
Lg1V=LieDerivative(G1,V)
Lg2V=LieDerivative(G2,V)
```

Stability analysis. With our controller, we have $\dot{V}(\mathbf{x}) = 0$ if

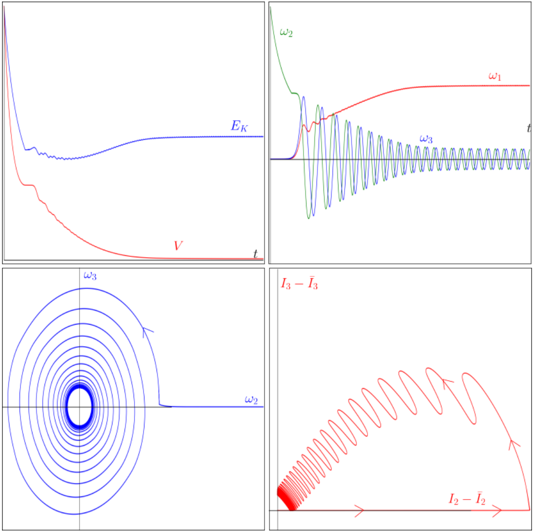
$$\begin{cases} -\frac{\omega_2^2}{I_2} + I_2 - \bar{I}_2 & = & 0 \\ -\frac{\omega_3^2}{I_3} + I_3 - \bar{I}_3 & = & 0 \end{cases}$$

Thus

$$\dot{V}(\mathbf{x}) = 0 \not\Rightarrow \begin{cases} \bar{\omega}_2 = 0 \\ \bar{\omega}_3 = 0 \end{cases}$$

Take $\omega(0) = (10^{-5}, 10, 0)$.
Our controller

- first increases I_2 to create a Dzhanibekov effect.
- generates an oscillation between I_2 and I_3.


E_K
V
t
ω_2
ω_1
ω_3
t
ω_3
ω_2
$I_3 - \bar{I}_3$
$I_2 - \bar{I}_2$

Passive control

If the disk losses energy it may limit the precession.

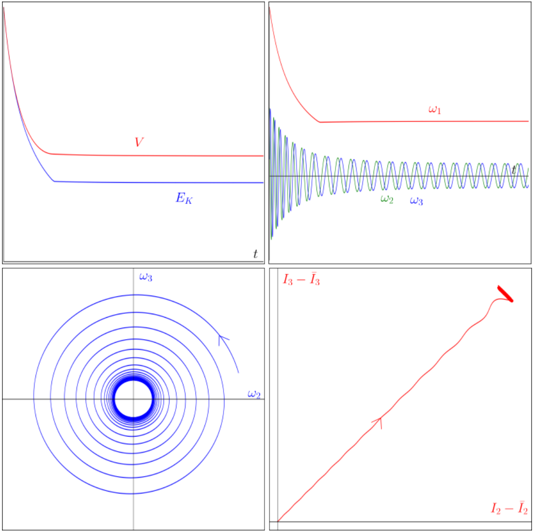
We define the *mechanical energy* as

$$V(\mathbf{x}) \quad = \quad \tfrac{1}{2}I_1\omega_1^2+\tfrac{1}{2}I_2\omega_2^2+\tfrac{1}{2}I_3\omega_3^2+\tfrac{1}{2}(I_2-\bar{I}_2)^2+\tfrac{1}{2}(I_3-\bar{I}_3)^2 \ .$$

- $\frac{1}{2}\omega_r^{\mathsf{T}}\mathbf{I}\omega_r$ is the kinetic energy
- $\frac{1}{2}(I_2-\bar{I}_2)^2+\frac{1}{2}(I_3-\bar{I}_3)^2$ is the artificial potential energy.

We have

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 + \tfrac{1}{2}\omega_1^2\dot{I}_1 + \tfrac{1}{2}\omega_2^2\dot{I}_2 + \tfrac{1}{2}\omega_3^2\dot{I}_3 \\
&\quad +(I_2-\bar{I}_2)\dot{I}_2 + (I_3-\bar{I}_3)\dot{I}_3 \\
&= \underbrace{0}_{\mathscr{L}_{\mathbf{f}}V} + \underbrace{\left(-\frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_2^2 + I_2 - \bar{I}_2\right)}_{\mathscr{L}_{\mathbf{g}_1}V}\cdot u_1 \\
&\quad + \underbrace{\left(-\frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_3^2 + I_3 - \bar{I}_3\right)}_{\mathscr{L}_{\mathbf{g}_2}V}\cdot u_2
\end{aligned}$$



The passivity approach does not cancel the precession

Precession control

Recall the Euler's rotation equation

$$\dot{\omega}_r = \mathbf{I}^{-1} \cdot \left(-\dot{\mathbf{I}} \cdot \omega_r - \omega_r \wedge \mathbf{I} \cdot \omega_r\right).$$

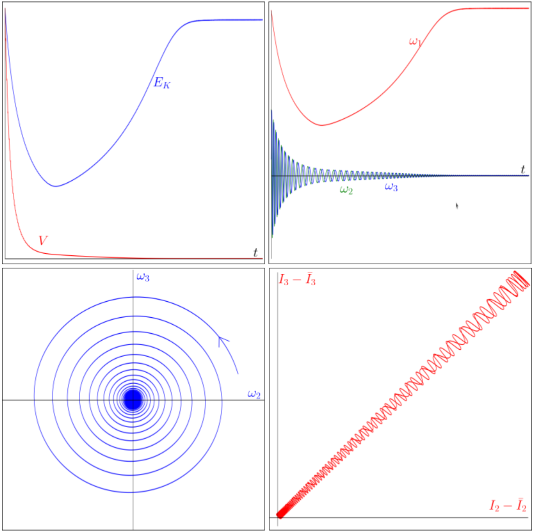
The precession energy is

$$\begin{aligned}
\|\omega_r \wedge \mathbf{I} \cdot \omega_r\|^2 &= \left\| \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \wedge \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix} \right\| \\
&= ((I_3 - I_2)\omega_2 \omega_3)^2 + ((I_1 - I_3)\omega_3 \omega_1)^2 \\
&\quad + ((I_2 - I_1)\omega_1 \omega_2)^2
\end{aligned}$$

Take

$$V(\mathbf{x}) = \tfrac{1}{2}\left((I_3 - I_2)\omega_2\omega_3\right)^2 + \tfrac{1}{2}\left((I_1 - I_3)\omega_3\omega_1\right)^2 + \tfrac{1}{2}\left((I_2 - I_1)\omega_1\omega_2\right)^2 + \tfrac{1}{2}(I_2 - \bar{I}_2)^2 + \tfrac{1}{2}(I_3 - \bar{I}_3)^2.$$

With $\omega(0) = (10, 4, 1)$, we get


E_K
V
t
ω_1
ω_2
ω_3
t
ω_3
ω_2
$I_3 - \bar{I}_3$
$I_2 - \bar{I}_2$

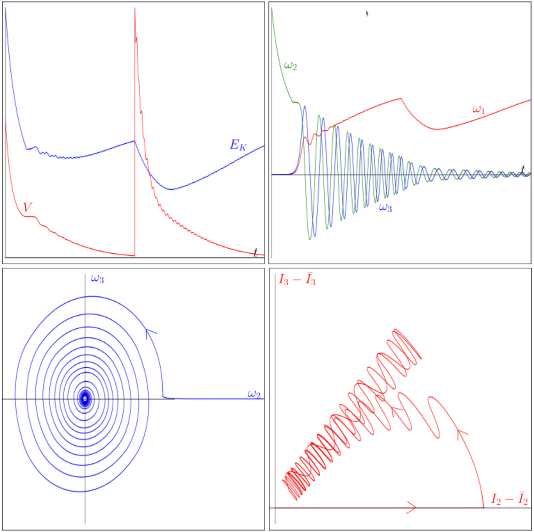
Combination

At time $t = 0$, we spin around the second axis:

$$\omega_r(0) = (10^{-5}, 10, 0)$$

and we want to spin around the second axis.

- We apply first the alignment controller
- Then we switch to the precession controller so that ω_r converges to the nearest principal axis.


E_K
V
t
ω_2
ω_1
ω_3
t
ω_3
ω_2
$I_3 - \bar{I}_3$
$I_2 - \bar{I}_2$

An illustrative video : https://youtu.be/gRzlDYFuMts

Perspective

Find a unique objective function which would allow the controller to switch from one principal axes to another.

References

1. Flat disk problem [4, 3]
2. Roue autonome (Damien Esnault) [2]
3. Lagrangian mechanics [8]
4. Stability of rotating object [6]
5. Lyapunov control [5]
6. Dzhanibekov effect [7]
7. Passivity control [1]

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