

# Determining Ellipsoidal Outer and Inner Enclosures of Nonlinear Mappings of Ellipsoidal Domains

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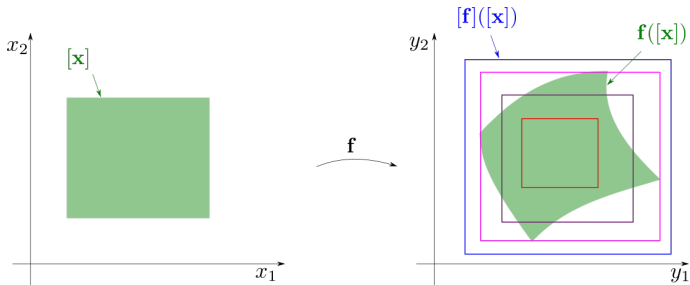
AID meeting, 11-12 October 2021  
Palaiseau, Ecole polytechnique

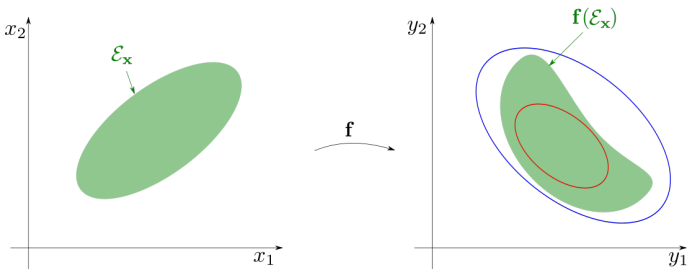


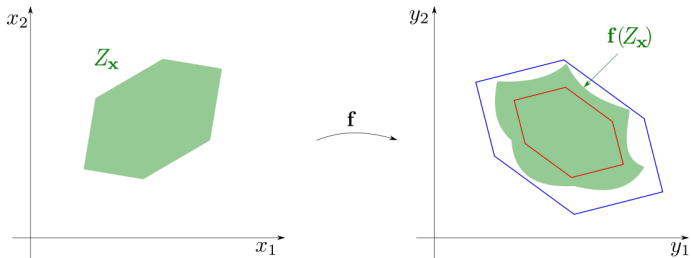
# Wrappers

Consider a function  $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ .

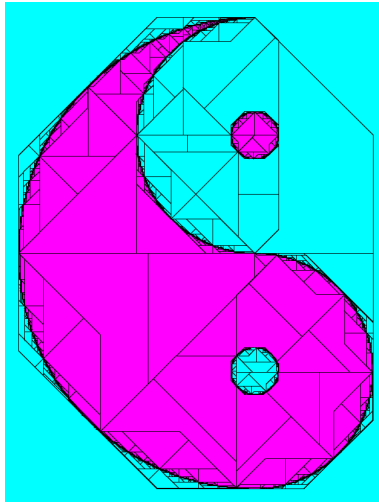
Set-membership technics propose methods to approximate  $\mathbf{f}(\mathbb{X}), \mathbb{X} \subset \mathbb{R}^n$







|                  | Boxes | Ellipsoids | octogons | polygons | LMI |
|------------------|-------|------------|----------|----------|-----|
| Closed by $\cap$ | Yes   | No         | Yes      | No       | Yes |
| Hull             | Yes   | No         | Yes      | No       | No  |
| Bisectable       | Yes   | No         | Yes      | Yes      | Yes |
| Parametric       | Yes   | <b>Yes</b> | Yes      | No       | No  |
| linear perfect   | No    | <b>Yes</b> | No       | Yes      | Yes |
| Stabilizing      | No    | <b>Yes</b> | No       | Yes      | Yes |





# Ellipsoidal enclosure: linear case

Consider the ellipsoid

$$\mathcal{E}_{\mathbf{x}} : (\mathbf{x} - \bar{\mathbf{x}})^{\top} \mathbf{Q}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) \leq 1$$

Consider a function  $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$  and  $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{c}$  where  $\mathbf{A}$  is invertible. Then

$$\mathbf{f}(\mathcal{E}_{\mathbf{x}}) : (\mathbf{y} - \bar{\mathbf{y}})^{\top} \mathbf{R}^{-1} (\mathbf{y} - \bar{\mathbf{y}}) \leq 1$$

with

$$\begin{aligned} \mathbf{R} &= \mathbf{A}\mathbf{Q}\mathbf{A}^{\top} \\ \bar{\mathbf{y}} &= \mathbf{f}(\bar{\mathbf{x}}) = \mathbf{A}\bar{\mathbf{x}} + \mathbf{c} \end{aligned}$$

# Ellipsoidal enclosure: non-linear case

**Proposition [3].** Consider  $\mathbf{x}$  inside

$$\mathcal{E}_{\mathbf{x}} : (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{Q}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) \leq 1$$

Consider a function  $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$  and  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  where  $\mathbf{A} = \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}})$  is invertible. Then

$$(\mathbf{y} - \bar{\mathbf{y}})^T ((1 + \rho)\mathbf{R})^{-1} (\mathbf{y} - \bar{\mathbf{y}}) \leq 1$$

with

$$\begin{aligned} \mathbf{R} &= \mathbf{A}\mathbf{Q}\mathbf{A}^T \\ \bar{\mathbf{y}} &= \mathbf{f}(\bar{\mathbf{x}}) \\ \rho &= \max_{\mathbf{x} \in \mathcal{E}_{\mathbf{x}}} 2\mathbf{b}(\mathbf{x})^T \mathbf{R}^{-1} (\mathbf{f}(\mathbf{x}) - \bar{\mathbf{y}}) \\ \mathbf{b}(\mathbf{x}) &= \mathbf{f}(\mathbf{x}) - (\bar{\mathbf{y}} + \mathbf{A} \cdot (\mathbf{x} - \bar{\mathbf{x}})) \end{aligned}$$

## Smallest box which encloses an ellipsoid

To enclose  $\max_{\mathbf{x} \in \mathcal{E}_{\mathbf{x}}}$ , we need first to find a box which encloses the ellipsoid  $\mathcal{E}_{\mathbf{x}}$ .

Consider the ellipsoid:

$$(\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{Q}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) \leq 1$$

The smallest box which encloses  $\mathcal{E}_{\mathbf{x}}$  is

$$\bar{\mathbf{x}} + \prod_{i \in \{1, \dots, n\}} \sqrt{\mathbf{e}_i^T \mathbf{Q} \mathbf{e}_i} \cdot [-1, 1]$$

## Algorithm.

```

In:       $\mathbf{f}, \mathcal{E}_{\mathbf{x}} : \{\bar{\mathbf{x}}, \mathbf{Q}_{\mathbf{x}}\}$ 
1   $\mathbf{A}$     =  $\frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}})$ 
2   $\bar{\mathbf{y}}$     =  $\mathbf{f}(\bar{\mathbf{x}})$ 
3   $\mathbf{R}$     =  $\mathbf{A}\mathbf{Q}_{\mathbf{x}}\mathbf{A}^T$ 
4   $[\mathbf{x}]$  =  $\bar{\mathbf{x}} + \prod_{i \in \{1, \dots, n\}} \sqrt{\mathbf{e}_i^T \mathbf{Q}_{\mathbf{x}} \mathbf{e}_i} \cdot [-1, 1]$ 
5   $[\mathbf{b}]$  =  $\left( \left[ \frac{d\mathbf{f}}{d\mathbf{x}} \right]([\mathbf{x}]) - \mathbf{A} \right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}})$ 
6   $[\rho]$  =  $2[\mathbf{b}]^T \mathbf{R}^{-1} ([\mathbf{f}]([\mathbf{x}]) - \bar{\mathbf{y}})$ 
7   $\mathbf{Q}_{\mathbf{y}}$  =  $(1 + \rho^+) \cdot \mathbf{R}$ 
8  Out:    $\mathcal{E}_{\mathbf{y}} : \{\bar{\mathbf{y}}, \mathbf{Q}_{\mathbf{y}}\}$ 
    
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An inner ellipse can also be obtained [4]

The online Python program can be found here:

<https://replit.com/@aulin/ellipse>

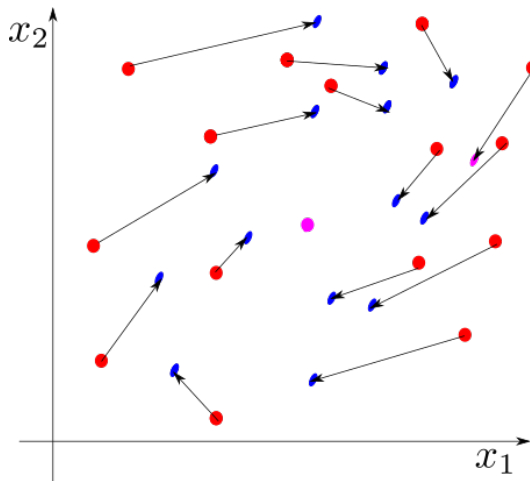


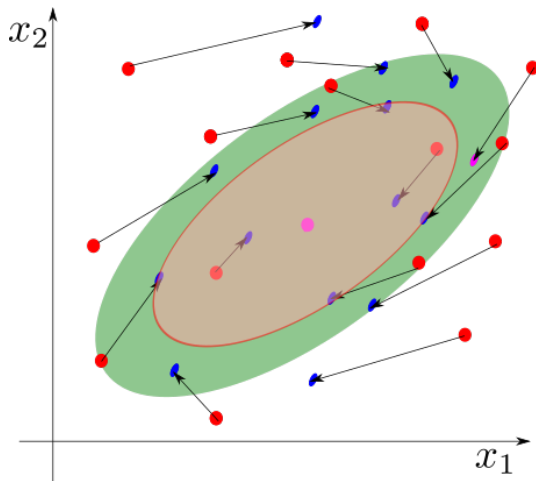
# Positive invariant sets

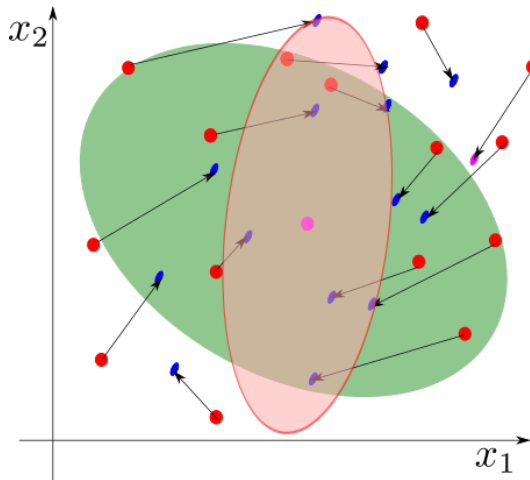
Consider the discrete time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with  $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ .







We have to find

$$\mathcal{E}_{\mathbf{x}} : \mathbf{x}^T \cdot \mathbf{P} \cdot \mathbf{x} \leq \varepsilon$$

Such that

$$\mathbf{f}(\mathcal{E}_{\mathbf{x}}) \subset \mathcal{E}_{\mathbf{x}}$$

If the system is stable and linear

$$\mathbf{x}_{k+1} = \mathbf{A} \cdot \mathbf{x}_k$$

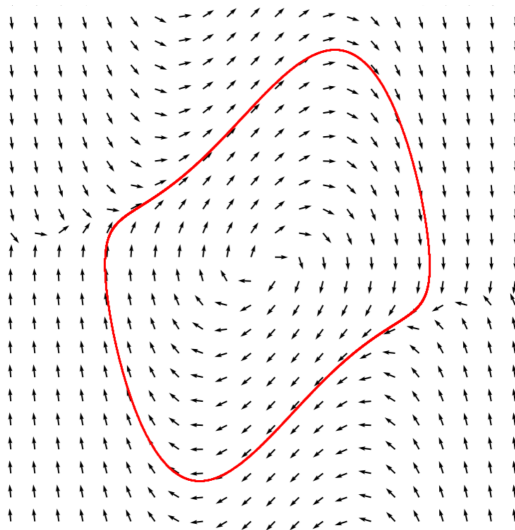
we can find  $\mathbf{P} \succ 0$  such that  $V(\mathbf{x}) = \mathbf{x}^T \cdot \mathbf{P} \cdot \mathbf{x}$  is a Lyapunov function

$$\begin{aligned} V(\mathbf{x}_{k+1}) &= V(\mathbf{x}_k) - \mathbf{x}_k^T \mathbf{x}_k \\ \Leftrightarrow \mathbf{x}_{k+1}^T \cdot \mathbf{P} \cdot \mathbf{x}_{k+1} &= \mathbf{x}_k^T \cdot \mathbf{P} \cdot \mathbf{x}_k - \mathbf{x}_k^T \mathbf{x}_k \\ \Leftrightarrow \mathbf{x}_k^T \cdot \mathbf{A}^T \cdot \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{x}_k - \mathbf{x}_k^T \cdot \mathbf{P} \cdot \mathbf{x}_k &= -\mathbf{x}_k^T \mathbf{x}_k \end{aligned}$$

We have to solve the Lyapunov equation

$$\mathbf{A}^T \cdot \mathbf{P} \cdot \mathbf{A} - \mathbf{P} = -\mathbf{I}$$

# Stability of cycles



System:  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

Poincaré section  $\mathcal{G}$ :  $g(\mathbf{x}) = 0$

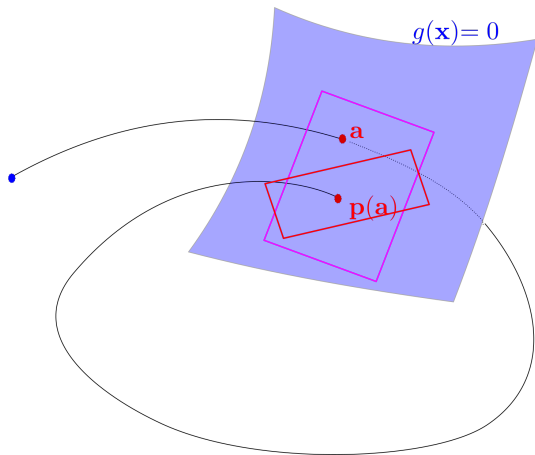
We define

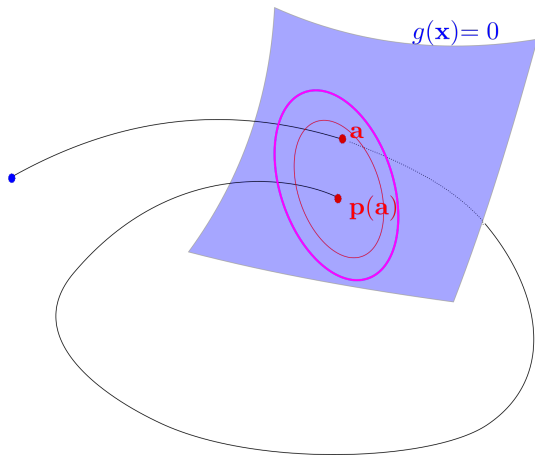
$$\mathbf{p}: \begin{array}{ccc} \mathcal{G} & \rightarrow & \mathcal{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where  $\mathbf{p}(\mathbf{a})$  is the point of  $\mathcal{G}$  such that the trajectory initialized at  $\mathbf{a}$  intersects  $\mathcal{G}$  for the first time.

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$





See [2]

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

The flow is  $\Phi(\mathbf{x}_0, t)$ .

Define  $\mathbf{A}(\mathbf{x}_0, t) = \frac{\partial \Phi(\mathbf{x}_0, t)}{\partial \mathbf{x}_0}$ . We have the *variational equation*

$$\dot{\mathbf{A}} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{A}$$

with  $\mathbf{A}(0) = \mathbf{I}$ .

**Example :** Van der Pol system.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} \\ \dot{a}_{21} & \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2x_1x_2 - 1 & 1 - x_1^2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

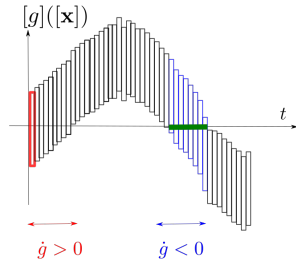
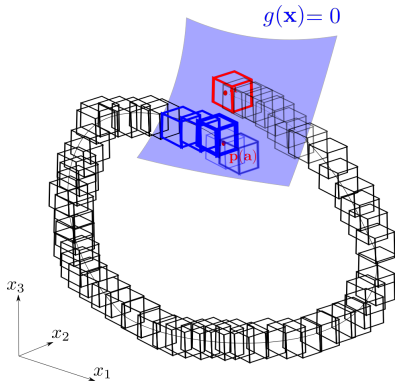
with

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{a}_{11} \\ \dot{a}_{12} \\ \dot{a}_{21} \\ \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 \\ a_{21} \\ a_{22} \\ (-2x_1x_2 - 1)a_{11} + (1 - x_1^2)a_{21} \\ (-2x_1x_2 - 1)a_{12} + (1 - x_1^2)a_{22} \end{pmatrix}$$

Using an interval ODE solver, we get an enclosure for  $\mathbf{x}(t), \mathbf{A}(t)$ , for a given initial box  $[\mathbf{x}_0]$ .

We can also get the time at which the system crosses the surface  $g(\mathbf{x}) = 0$ .

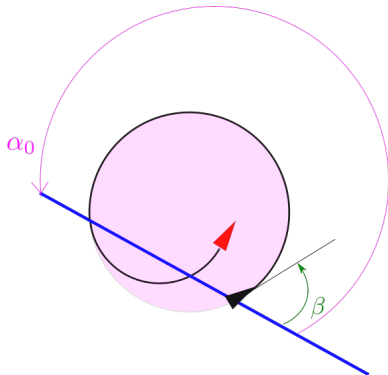
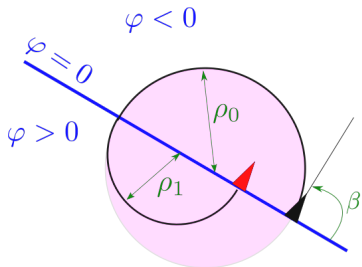


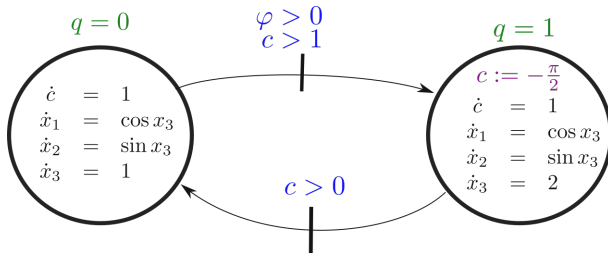
# Rolling stability problem

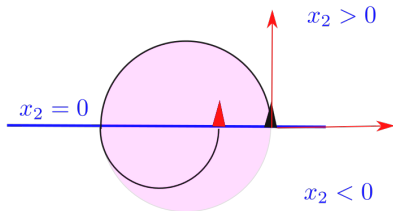
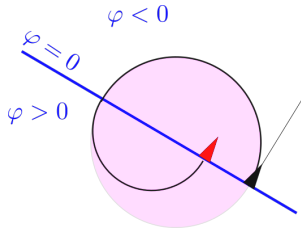
Robot moving on a plane described by

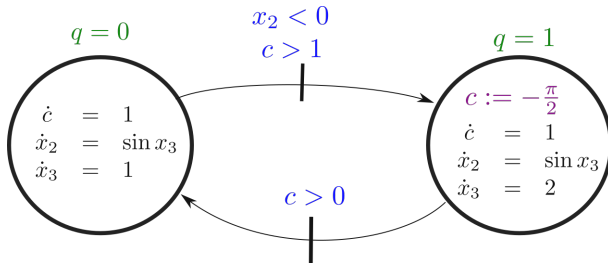
$$\begin{cases} \dot{x}_1 &= \cos x_3 \\ \dot{x}_2 &= \sin x_3 \\ \dot{x}_3 &= u \end{cases}$$

The robot is able to measure a function  $\varphi(x_1, x_2)$  has to moves along  $\varphi(x_1, x_2) = 0$ . [1]



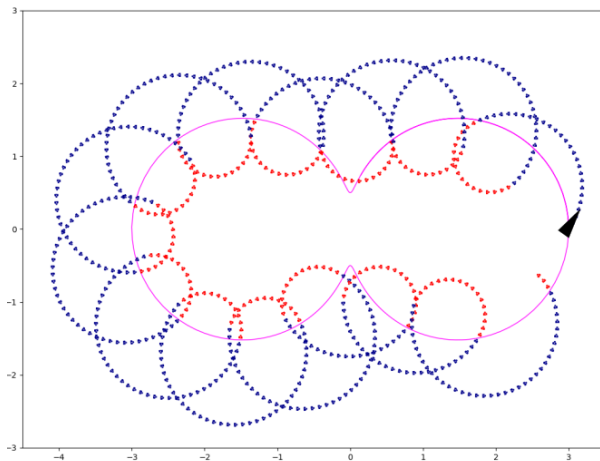






We consider the Hippopede of Proclus given by  $\varphi(x_1, x_2) = 0$  where

$$\varphi(x_1, x_2) = 9x_1^2 + x_2^2 - (x_1^2 + y_2^2)^2.$$



The online Python program can be found here:  
<https://replit.com/@aulin/rolling>



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