Characterizing discontinuities of an hybrid system

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https://youtu.be/bNqiwW4p6WE

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Easy-boat Formalism Application to easy-boat



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Easy-boat

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$$\dot{d} = \sin u$$

 $\cos(\psi - u) + \cos \frac{\pi}{5} > 0$

Controller in:
$$(d, \psi, q)$$
; out: u
if $d^2 - 1 > 0$ then $q := \operatorname{sign}(d)$
if $\cos(\psi + \operatorname{atan}(d)) + \cos\frac{\pi}{4} \le 0 \lor (d^2 \le 1 \land \cos\psi + \cos\frac{\pi}{4} \le 0)$
then $u := \pi + \psi - q\frac{\pi}{4}$.
else $u := -\operatorname{atan}(d)$.

Simulations

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Simulation in the (t, d)-space

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Simulation in the $(\int^t \cos u, d)$ -space

Formalism

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Given \mathbb{Q}^- , \mathbb{Q}^+ disjoint, two smooth functions $\mathbf{f}_a, \mathbf{f}_b : \mathbb{R}^n \to \mathbb{R}^n$. We define [3]

$$\mathscr{S}(\mathbb{A}): \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},q) = \begin{cases} \mathbf{f}_a(\mathbf{x},q) & \text{if } \mathbf{x} \in \mathbb{A} \\ \mathbf{f}_b(\mathbf{x},q) & \text{if } \mathbf{x} \in \mathbb{B} = \overline{\mathbb{A}} \end{cases} \\ q = -1 & \text{as soon as } \mathbf{x} \in \mathbb{Q}^- \\ = +1 & \text{as soon as } \mathbf{x} \in \mathbb{Q}^+ \end{cases}$$

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The pair (x, q) always satisfies the constraint

$$egin{array}{rcl} {f x} \in {\mathbb Q}^+ & \Rightarrow & q=1 \ {f x} \in {\mathbb Q}^- & \Rightarrow & q=-1 \end{array}$$

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or equivalently, $\mathbf{x} \in \overline{\mathbb{Q}^{-q}}$.



With easy-boat

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We take
$$\mathbf{x} = (d, \psi)$$
,

Function f(x,q)
if
$$\cos(x_2 + \operatorname{atan} x_1) + \cos \frac{\pi}{4} \le 0 \lor (x_1^2 - 1 \le 0 \land \cos x_2 + \cos \frac{\pi}{4} \le 0)$$

then $u := \pi + x_2 - q \frac{\pi}{4}$
else $u := -\operatorname{atan} x_1$.
Return $\sin u$

$$\begin{aligned} \mathbf{x} &= (d, \psi) \\ \mathbf{f}_a(\mathbf{x}, q) &= \begin{pmatrix} \sin\left(\pi + x_2 - q\frac{\pi}{4}\right) \\ 0 \end{pmatrix} \\ \mathbf{f}_b(\mathbf{x}) &= \begin{pmatrix} \sin\left(-\operatorname{atan} x_1\right) \\ 0 \end{pmatrix} \\ \mathbb{A}_1 &= \left\{\mathbf{x} \mid \cos\left(x_2 + \operatorname{atan} x_1\right) + \cos\frac{\pi}{4} \le 0\right\} \\ \mathbb{A}_2 &= \left\{\mathbf{x} \mid x_1^2 - 1 \le 0\right\} \\ \mathbb{A}_3 &= \left\{\mathbf{x} \mid \cos x_2 + \cos\frac{\pi}{4} \le 0\right\} \\ \mathbb{A} &= \mathbb{A}_1 \cup (\mathbb{A}_2 \cap \mathbb{A}_3) \\ \mathbb{Q}^- &= \left\{\mathbf{x} \mid x_1 + 1 \le 0\right\} \\ \mathbb{Q}^+ &= \left\{\mathbf{x} \mid 1 - x_1 \le 0\right\} \end{aligned}$$

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Three problems

Three problems:

- Safety : the sailboat never goes upwind. [1]
- Capture : The boat will be captured by its corridor [2]
- Characterize of the sliding surface.



Show that we never have

 $h(\mathbf{x},q) \leq 0.$

Since \mathbf{f}_a and \mathbf{f}_b are continuous, we have

$$\left\{\begin{array}{rrl} h(\mathbf{x},q) &=& h_a(\mathbf{x},q) & \quad \text{if } \mathbf{x} \in \mathbb{A} \\ &=& h_b(\mathbf{x},q) & \quad \text{if } \mathbf{x} \in \mathbb{B} \end{array}\right.$$

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where h_a, h_b are continuous.

Proposition 1. If

$$\mathbb{H} = \bigcup_{q \in \{-1,1\}} \left(\{ \mathsf{x} | h_{\mathsf{a}}(\mathsf{x},q) \leq 0 \} \cap \mathbb{A} \cap \overline{\mathbb{Q}^{-q}} \right) \\ \cup \left(\{ \mathsf{x} | h_{\mathsf{b}}(\mathsf{x},q) \leq 0 \} \} \cap \mathbb{B} \cap \overline{\mathbb{Q}^{-q}} \right)$$

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is empty then we cannot have $h(\mathbf{x},q) \leq 0$.

We want to prove that we never have

$$h(\mathbf{x},q) = \cos(x_2 - u) + \cos\frac{\pi}{5} \le 0.$$

with

$$\begin{cases} u = \pi + x_2 - q\frac{\pi}{4} & \text{if } \mathbf{x} \in \mathbb{A} \\ = -\operatorname{atan} x_1 & \text{otherwise} \end{cases}$$

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The set \mathbb{H} has no solution. We conclude that the forbidden constraint $\cos(x_2 - u) + \cos\frac{\pi}{5} \le 0$ is never reached.

```
c1=Function("x1", "x2", "cos(atan(x1)+x2)+sqrt(2)/2")
c2=Function("x1","x2","x1^2-1")
c3=Function("x1","x2","cos(x2)+sqrt(2)/2")
A1=SepFwdBwd(c1, [-oo, 0])
A2=SepFwdBwd(c2, [-oo, 0])
A3=SepFwdBwd(c3, [-oo, 0])
A=A1 | A2&A3
B=~A
Hb=SepFwdBwd(Function("x1", "x2",
"cos(x2+atan(x1))+cos(2*asin(1)/5)"),[-oo,0])
H=Hb&B
pySIVIA(H)
```



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The set $\mathbb{C} = \{\mathbf{x} | V(\mathbf{x}) \leq 0\}$, with $V : \mathbb{R}^n \to \mathbb{R}$ is a *capture set* if all trajectories that enter inside \mathbb{C} stays inside forever.

The Lie derivative of V with respect to \mathbf{f} is

$$\mathscr{L}_{\mathbf{f}}^{V}(\mathbf{x}) = \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

The Lie set as

$$\mathbb{L}_{\mathbf{f}}^{V} = \left\{ \mathbf{x} | \mathscr{L}_{\mathbf{f}}^{V}(\mathbf{x}) \leq \mathbf{0} \right\}.$$

Proposition 2. Define the set

$$\mathbb{V} = \bigcup_{q \in \{-1,1\}} \left(\overline{\mathbb{L}_{a}^{V}(q)} \cap \mathbb{A} \cap \overline{\mathbb{Q}^{-q}} \right) \cup \left(\overline{\mathbb{L}_{b}^{V}(q)} \cap \mathbb{B} \cap \overline{\mathbb{Q}^{-q}} \right)$$

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If $\mathbb{V} \cap \overline{\mathbb{C}} = \emptyset$ then \mathbb{C} is a capture set.

Take $V(\mathbf{x}) = x_1^2 - 4$. We have

$$\mathcal{L}_{a}^{V}(\mathbf{x},q) = \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{a}(\mathbf{x},q) = 2x_{1} \cdot \sin(\frac{q\pi}{4} - x_{2})$$
$$\mathcal{L}_{b}^{V}(\mathbf{x}) = \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{b}(\mathbf{x},q) = \frac{-2x_{1}^{2}}{\sqrt{x_{1}^{2}+1}}$$

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Fields $\mathbf{f}_{a}(\mathbf{x},q)$, $\mathbf{f}_{b}(\mathbf{x})$, the sets \mathbb{C} (green) and \mathbb{V} (red)

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Sliding surface

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The *sliding surface* $\mathbb{S}(\mathbb{A})$ for $\mathscr{S}(\mathbb{A})$ is largest subset of $\partial \mathbb{A}$ such that the state can slide inside for a non degenerated interval of time.

If \mathbb{A} : $c(\mathbf{x}) \leq 0$, then

$$\begin{split} \mathbb{S}(\mathbb{A}) &= \partial \mathbb{A} \cap \left\{ \mathbf{x} | \exists q, \mathbf{x} \in \overline{\mathbb{Q}^{-q}}, \mathscr{L}^{c}_{a}(\mathbf{x}, q) \geq 0 \land \mathscr{L}^{c}_{b}(\mathbf{x}, q) \leq 0 \right\} \\ &= \partial \mathbb{A} \cap \bigcup_{q \in \{-1,1\}} \overline{\mathbb{Q}^{-q}} \cap \overline{\mathbb{L}^{V}_{a}(q)} \cap \mathbb{L}^{V}_{b}(q) \end{split}$$

Without the discrete variable q.

```
\mathbb{S}(\mathbb{A}) = \partial \mathbb{A} \cap \{ \mathbf{x} \, | \, \mathscr{L}_a^c(\mathbf{x}) \ge \mathbf{0} \land \mathscr{L}_b^c(\mathbf{x}) \le \mathbf{0} \} \,.
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Sliding set $\mathbb{S}(\mathbb{A})$ (magenta) for $\mathbb{A} = \{\mathbf{x} | c(\mathbf{x}) \leq 0\}$

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Proposition 3. If we have two closed sets \mathbb{A}_1 and \mathbb{A}_2 . We have

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 $\mathbb{S}(\mathbb{A}_1 \cup (\mathbb{A}_2 \cap \mathbb{A}_3))$

For our boat, the sliding surface for $A_i : c_i(\mathbf{x}) \leq 0$ is

$$\begin{split} \mathbb{S}(\mathbb{A}_{i}) &= \partial \mathbb{A}_{i} \cap \bigcup_{q \in \{-1,1\}} \overline{\mathbb{Q}^{-q}} \cap \overline{\mathbb{L}_{a}^{i}(q)} \cap \mathbb{L}_{b}^{i} \\ &= \partial \mathbb{A}_{i} \cap \mathbb{L}_{b}^{i} \cap \left(\overline{\mathbb{L}_{a}^{i}(1)} \cap \overline{\mathbb{Q}^{-}} \cup \overline{\mathbb{L}_{a}^{i}(-1)} \cap \overline{\mathbb{Q}^{+}}\right) \end{aligned}$$

where

$$\begin{array}{rcl} \mathbb{L}_a^i(q) &=& \{\mathbf{x} | \mathscr{L}_a^{c_i}(\mathbf{x},q) \leq 0\} \\ \mathbb{L}_b^i &=& \left\{\mathbf{x} | \mathscr{L}_b^{c_i}(\mathbf{x}) \leq 0\right\} \end{array}$$

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$$\begin{array}{rcl} \mathcal{L}_{a}^{c_{1}}(\mathbf{x},q) & = & \frac{dc_{1}}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{a}(\mathbf{x},q) & = & \frac{-\sin(\frac{q\pi}{4} - x_{2}) \cdot \sin(\operatorname{atan}(x_{1}) + x_{2})}{x_{1}^{2} + 1} \\ \mathcal{L}_{b}^{c_{1}}(\mathbf{x}) & = & \frac{dc_{1}}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{b}(\mathbf{x},q) & = & \frac{\sin(\operatorname{atan}(x_{1}) + x_{2}) \cdot x_{1}}{\sqrt{x_{1}^{2} + 1}^{3}} \\ \mathcal{L}_{a}^{c_{2}}(\mathbf{x},q) & = & \frac{dc_{2}}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{a}(\mathbf{x}) & = & 2\sin(\frac{q\pi}{4} - x_{2}) \cdot x_{1} \\ \mathcal{L}_{b}^{c_{2}}(\mathbf{x}) & = & \frac{dc_{2}}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{b}(\mathbf{x},q) & = & \frac{-2x_{1}^{2}}{\sqrt{x_{1}^{2} + 1}} \\ \mathcal{L}_{a}^{c_{3}}(\mathbf{x},q) & = & \frac{dc_{3}}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{a}(\mathbf{x},q) & = & 0 \\ \mathcal{L}_{b}^{c_{3}}(\mathbf{x}) & = & \frac{dc_{3}}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_{b}(\mathbf{x},q) & = & 0 \end{array}$$

$$\begin{split} \mathbb{S}(\mathbb{A}_{1}) &= \partial \mathbb{A}_{1} \cap \mathbb{L}_{b}^{1} \cap \left(\overline{\mathbb{L}_{a}^{1}(1)} \cap \overline{\mathbb{Q}^{-}} \cup \overline{\mathbb{L}_{a}^{1}(-1)} \cap \overline{\mathbb{Q}^{+}}\right) \\ \mathbb{S}(\mathbb{A}_{2}) &= \partial \mathbb{A}_{2} \cap \mathbb{L}_{b}^{2} \cap \left(\overline{\mathbb{L}_{a}^{2}(1)} \cap \overline{\mathbb{Q}^{-}} \cup \overline{\mathbb{L}_{a}^{2}(-1)} \cap \overline{\mathbb{Q}^{+}}\right) \\ \mathbb{S}(\mathbb{A}_{3}) &= \partial \mathbb{A}_{3} \end{split}$$

```
La1Q=Function("x1", "x2", "-sin(asin(1)/2-x2)*sin(atan(x1))
La1R=Function("x1", "x2", "sin(asin(1)/2+x2)*sin(atan(x1)-x2))
Lb1=Function("x1", "x2", "sin(atan(x1)+x2)*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt((x1^2+x2))*x1/sqrt(
La2Q=Function("x1", "x2", "2*sin(asin(1)/2-x2)*x1")
La2R=Function("x1", "x2", "-2*sin(asin(1)/2-x2)*x1")
Lb2=Function("x1", "x2", "-2*x1^2/sqrt(x1^2+1)")
dA1 = A1\&^{\sim}A1
SLa1Q=SepFwdBwd(La1Q, [-oo, 0])
SLa1R=SepFwdBwd(La1R, [-oo, 0])
SLb1=SepFwdBwd(Lb1, [-oo, 0])
S1=dA1 & SLb1 & ((~SLa1Q)&~R | (~SLa1R)&~Q)
dA2=A2\&^{A2}
SLa2Q=SepFwdBwd(La2Q,[-oo,0])
SLa2R=SepFwdBwd(La2R, [-oo, 0])
SLb2=SepFwdBwd(Lb2,[-oo,0])
S2=dA2 & SLb2 & ((~SLa2Q)&~R | (~SLa2R)&~Q)
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