Explore and return for an underwater robot in a minimalist environment and with few computation

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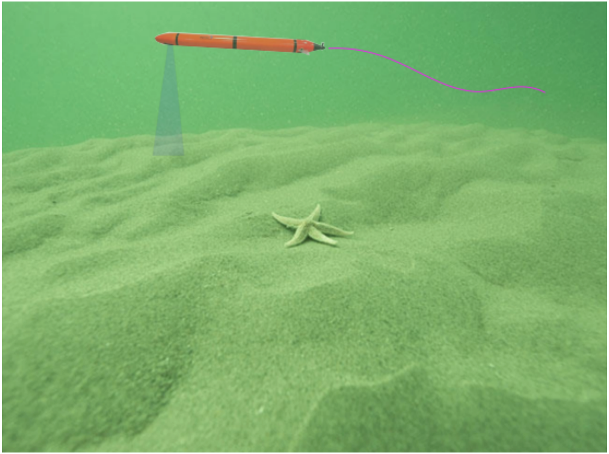
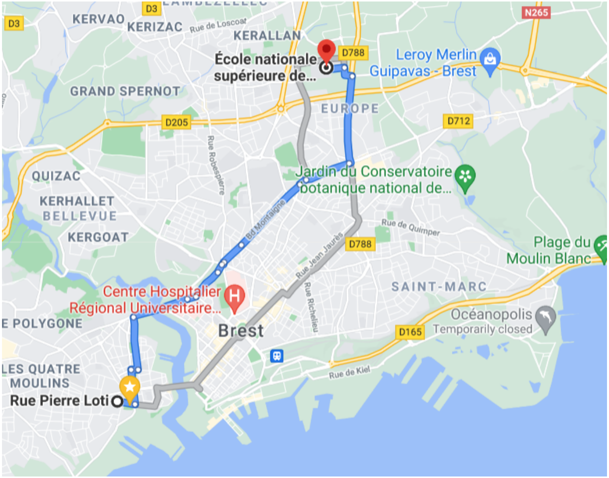
1. Underwater navigation



Explore and return in a minimalist environment

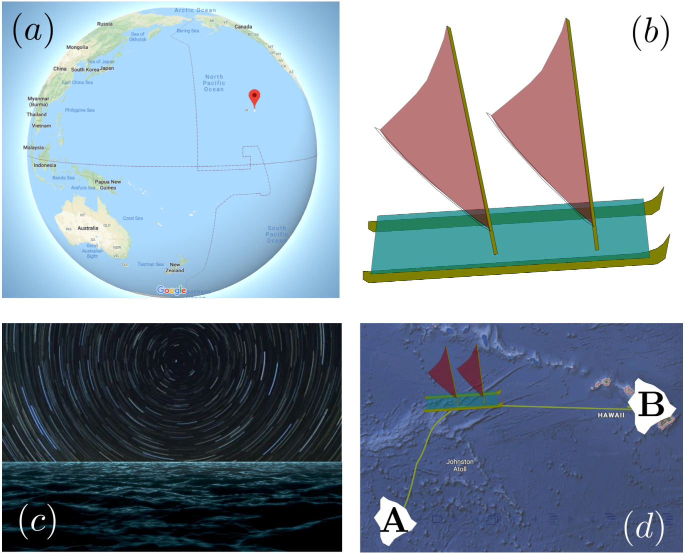


Modern navigation: high cost (computation, infrastructure)

Route-based navigation

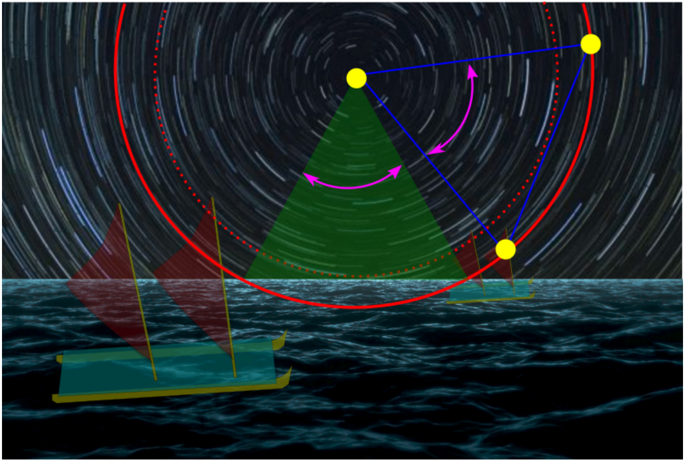
Submeeting 2018

Find the route without GPS, compass, clocks, computer with *wa'a kaulua*

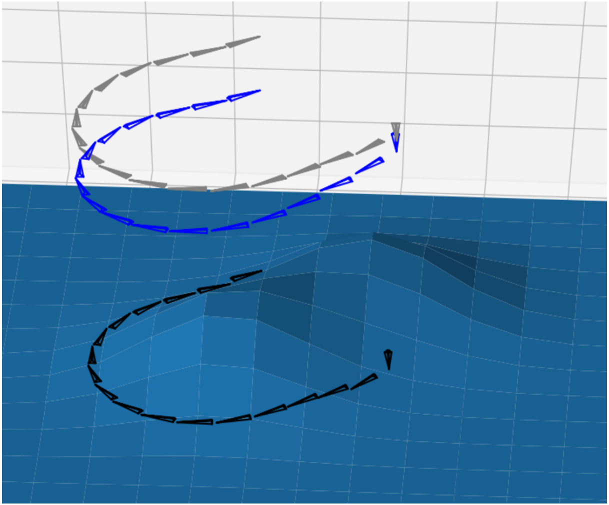

(a)
Arctic Ocean
Russia
Canada
Sea of Okhotsk
Bering Sea
Mongolia
Sea of Japan
China
South Korea
Japan
East China Sea
North Pacific Ocean
Myanmar (Burma)
Thailand
Vietnam
Philippine Sea
Malaysia
Indonesia
Banda Sea
Papua New Guinea
Arafura Sea
Coral Sea
Australia
Great Australian Bight
Tasman Sea
New Zealand
South Pacific Ocean
Google
(b)
(c)
HAWAII
B
Johnston Atoll
A
(d)

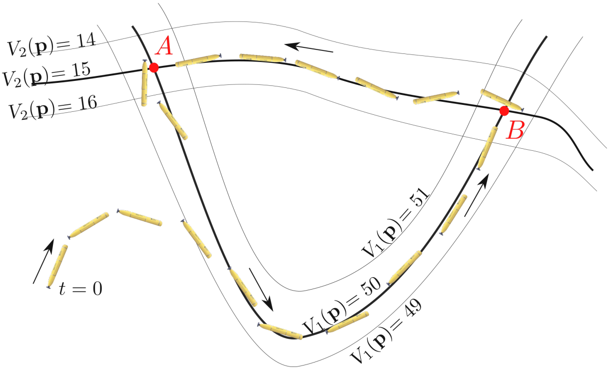
Follow a route

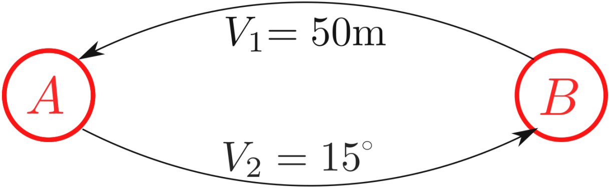
Given a function $h:\mathbb{R}^2 \mapsto \mathbb{R}$, a route in defined by $h(\mathbf{p}) = 0$.
h could be the temperature, the radiation, the pressure, the altitude, the time shift between two periodic events.



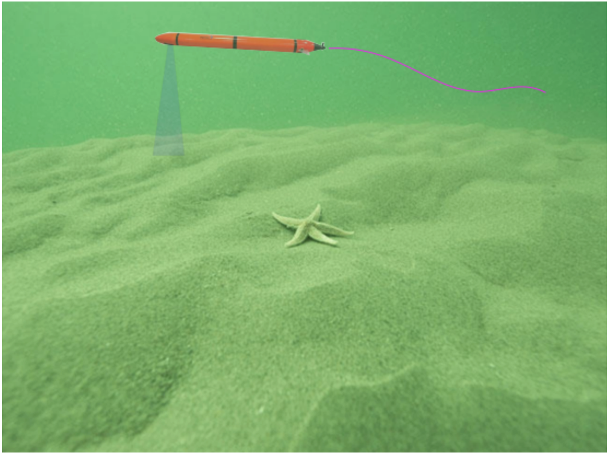
When one star sets the other rises



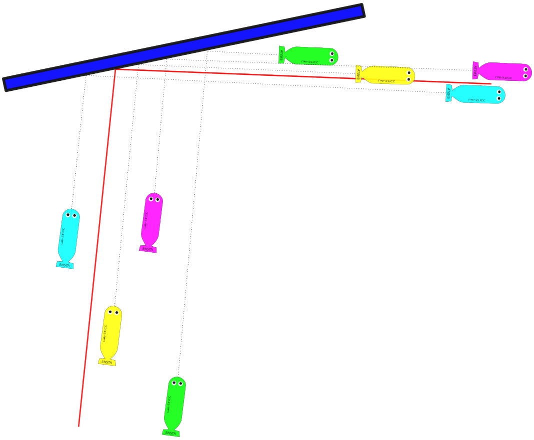

$V_2(\mathbf{p}) = 14$
$V_2(\mathbf{p}) = 15$
$V_2(\mathbf{p}) = 16$
A
B
$V_1(\mathbf{p}) = 51$
$V_1(\mathbf{p}) = 50$
$V_1(\mathbf{p}) = 49$
$t = 0$

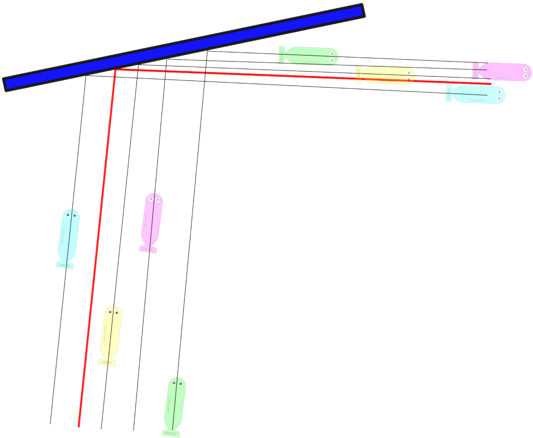

$V_1 = 50\text{m}$
A
B
$V_2 = 15^\circ$

2. Stable bouncing (phd of Quentin Brateau)

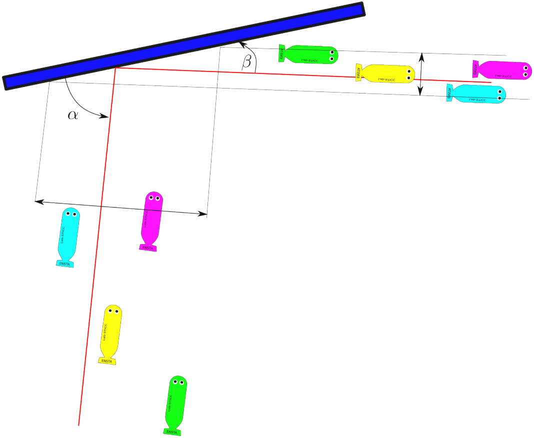


No route exists

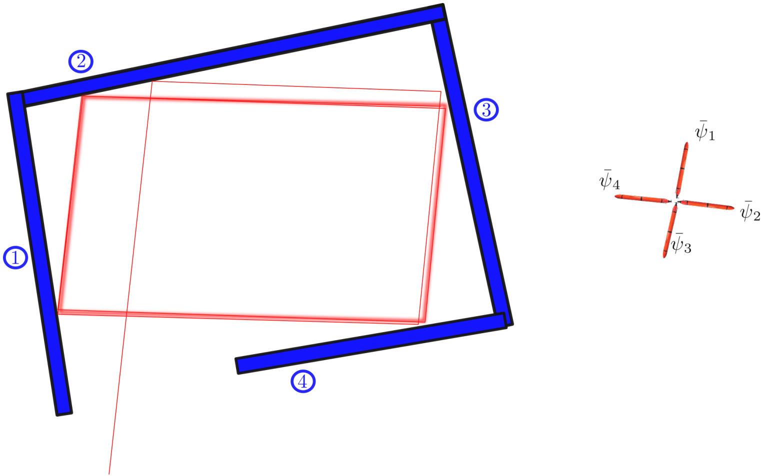


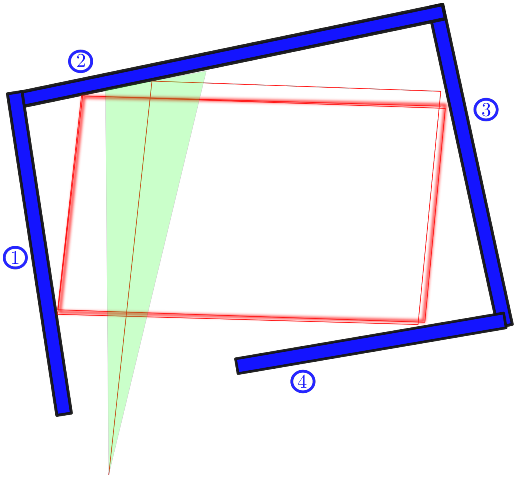


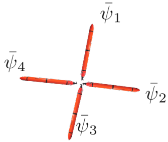
Contraction of the distance

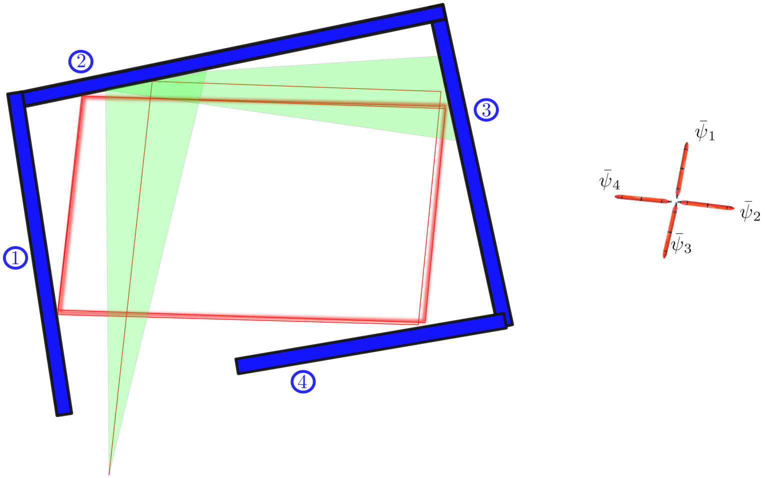


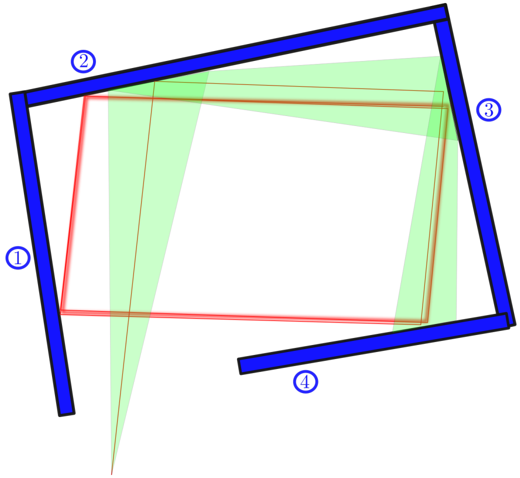
Contraction if $\frac{\sin\beta}{\sin\alpha} < 1$

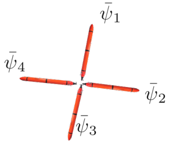

2
3
1
4
$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

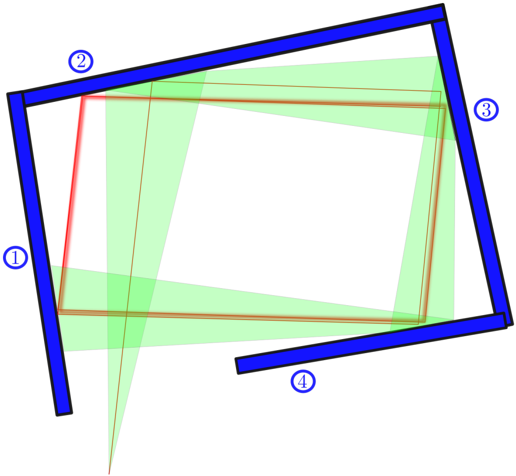

②
③
①
④

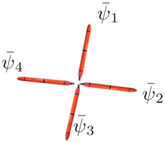

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

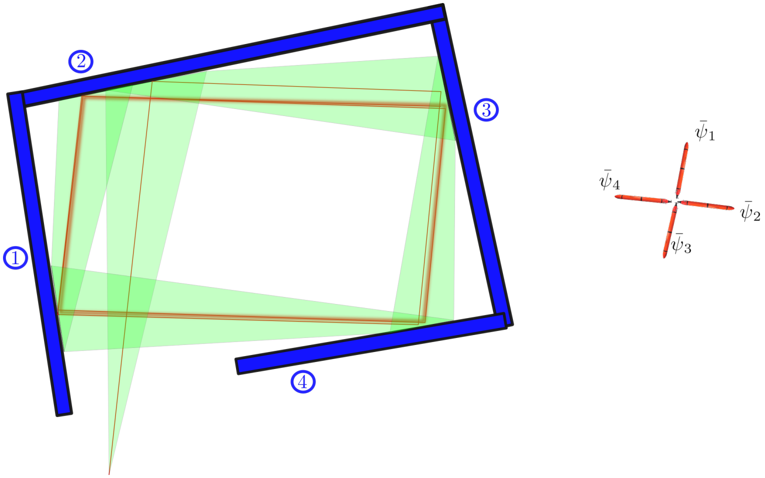

2
3
1
4
$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

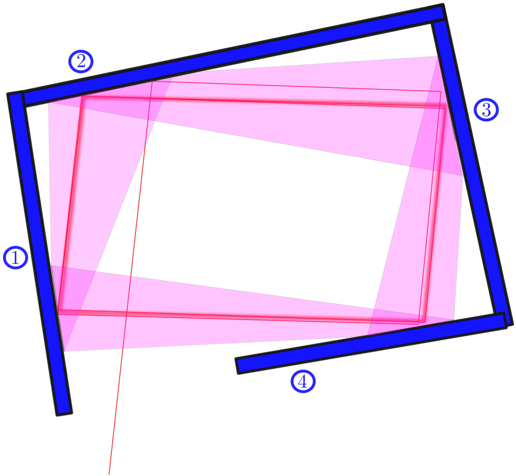

2
3
1
4

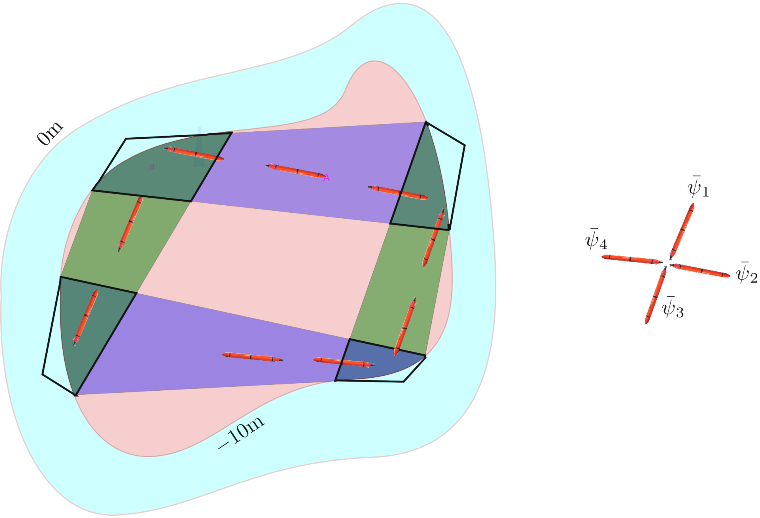

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

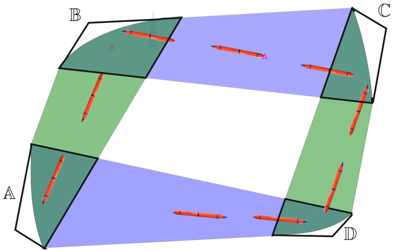

2
3
1
4

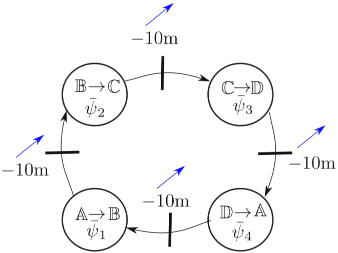

$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$

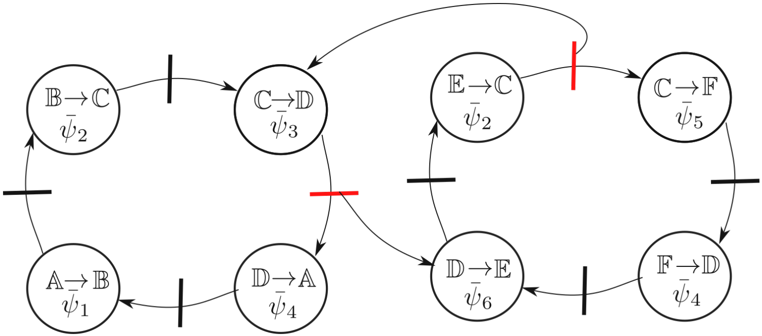

1
2
3
4
$\bar{\psi}_1$
$\bar{\psi}_2$
$\bar{\psi}_3$
$\bar{\psi}_4$


1
2
3
4

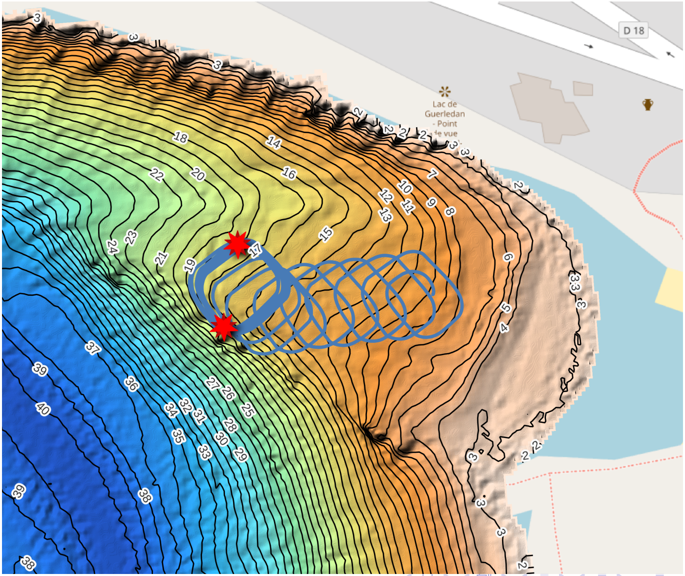

0m
−10m
A
B
$\bar{\psi}_1$
$\bar{\psi}_2$
$\bar{\psi}_3$
$\bar{\psi}_4$


$\mathbb{B}$
$\mathbb{C}$
$\mathbb{A}$
$\mathbb{D}$


-10m
$\mathbb{B}\!\to\!\mathbb{C}$ $\bar{\psi}_2$
$\mathbb{C}\!\to\!\mathbb{D}$ $\bar{\psi}_3$
-10m
-10m
-10m
$\mathbb{A}\!\to\!\mathbb{B}$ $\bar{\psi}_1$
$\mathbb{D}\!\to\!\mathbb{A}$ $\bar{\psi}_4$


$\mathbb{B}\to\mathbb{C}$
$\bar{\psi}_2$
$\mathbb{C}\to\mathbb{D}$
$\bar{\psi}_3$
$\mathbb{E}\to\mathbb{C}$
$\bar{\psi}_2$
$\mathbb{C}\to\mathbb{F}$
$\bar{\psi}_5$
$\mathbb{A}\to\mathbb{B}$
$\bar{\psi}_1$
$\mathbb{D}\to\mathbb{A}$
$\bar{\psi}_4$
$\mathbb{D}\to\mathbb{E}$
$\bar{\psi}_6$
$\mathbb{F}\to\mathbb{D}$
$\bar{\psi}_4$

Experiment (phd of Quentin Brateau)

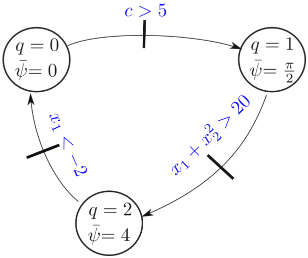

D 18
Lac de
Guerledan
- Point
de vue

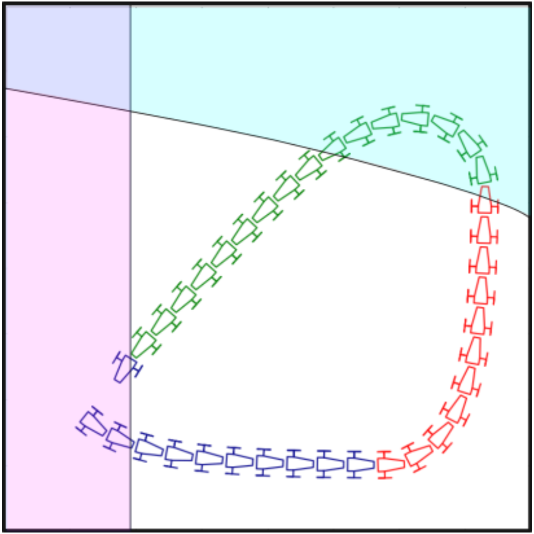
3. Proving the stability

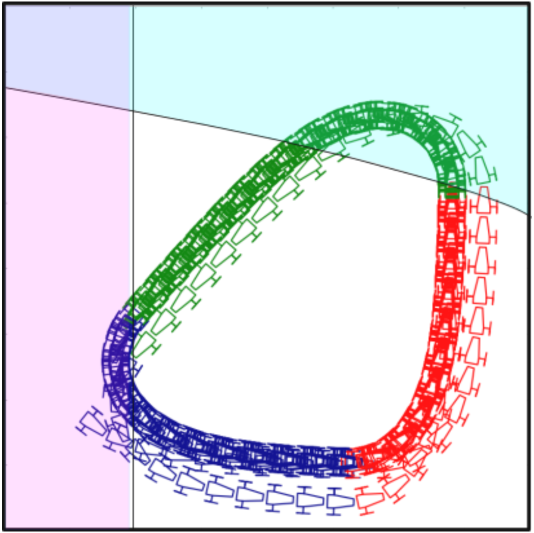
Consider the robot

$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

with the heading control $u = \sin(\bar{\psi} - x_3)$.







Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= ?, \\ [-1,3]\cdot[2,5] &= ?, \\ \mathsf{abs}\,([-7,1]) &= ? \end{aligned}$$

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8],\\
[-1,3]\cdot[2,5] &= [-5,15],\\
\mathsf{abs}([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

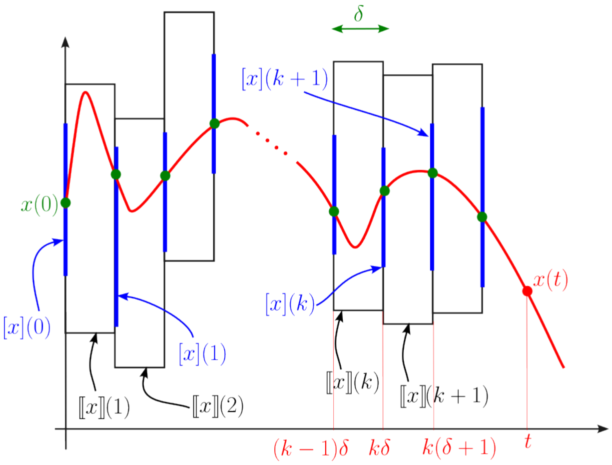
is

$$\begin{aligned}[f]([x_1],[x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos[x_2] \\ &\quad + \sin[x_1] \cdot \sin[x_2] + 2.\end{aligned}$$

Theorem (Moore, 1970)

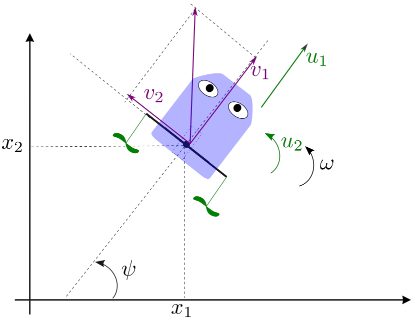
$$\{f(x_1,x_2)\,|\,x_1 \in [x_1], x_2 \in [x_2])\} \;\subset\; [f]([x_1],[x_2])$$

Tubes



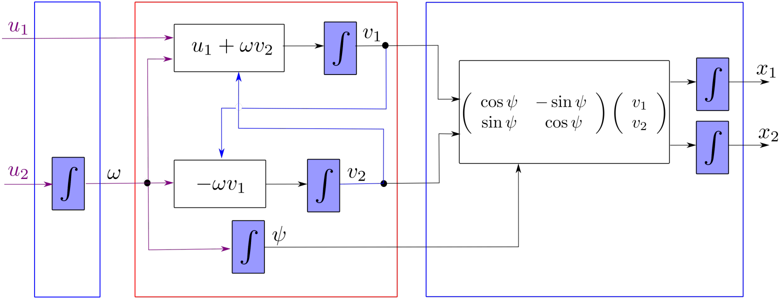
$$x(t) \in [x](t)$$

The hovercraft


u_1
v_1
v_2
u_2
ω
x_2
ψ
x_1

The state equations are given by

$$\begin{cases} \dot{x}_1 &= v_1 \cos\psi - v_2 \sin\psi \\ \dot{x}_2 &= v_1 \sin\psi + v_2 \cos\psi \\ \dot{v}_1 &= u_1 + \omega v_2 \\ \dot{v}_2 &= -\omega v_1 \\ \dot{\psi} &= \omega \\ \dot{\omega} &= u_2 \end{cases}$$

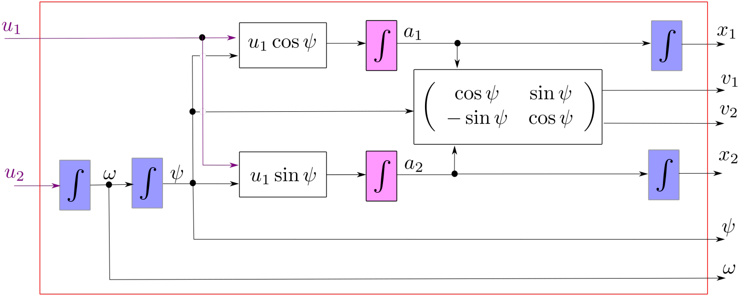

u_1
u_2
$\int$
ω
$u_1 + \omega v_2$
$-\omega v_1$
v_1
v_2
ψ
$\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
x_1
x_2

An integral formulation of the hovercraft is

$$\left\{\begin{array}{ccc} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = & \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} \int(\cos\psi\cdot v_1 - \sin\psi\cdot v_2) \\ \int(\sin\psi\cdot v_1 + \cos\psi\cdot v_2) \end{pmatrix} \\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & = & \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \left(\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} + \begin{pmatrix} \int(u_1\cos\psi) \\ \int(u_1\sin\psi) \end{pmatrix} \right) \\ \psi & = & \psi(0) + \int\omega \\ \omega & = & \omega(0) + \int u_2 \end{array}\right.$$

where

$$\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} \cos\psi(0) & -\sin\psi(0) \\ \sin\psi(0) & \cos\psi(0) \end{pmatrix} \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}.$$


u_1
u_2
$\int$
ω
ψ
$u_1 \cos\psi$
$u_1 \sin\psi$
a_1
a_2
$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$
x_1
v_1
v_2
x_2
ψ
ω

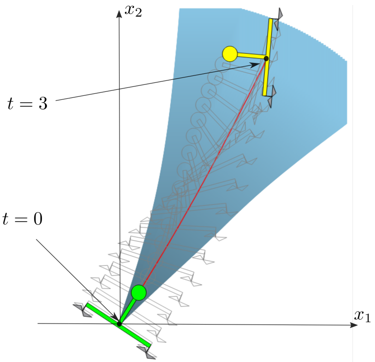
Take

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \begin{pmatrix} [u_1](t) \\ [u_2](t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} [-0.01, 0.01] \\ [-0.01, 0.01] \end{pmatrix}$$

$$\mathbf{x}(0) \in [\mathbf{x}](0) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.2, 0.2] \\ [-0.001, 0.001] \end{pmatrix}$$

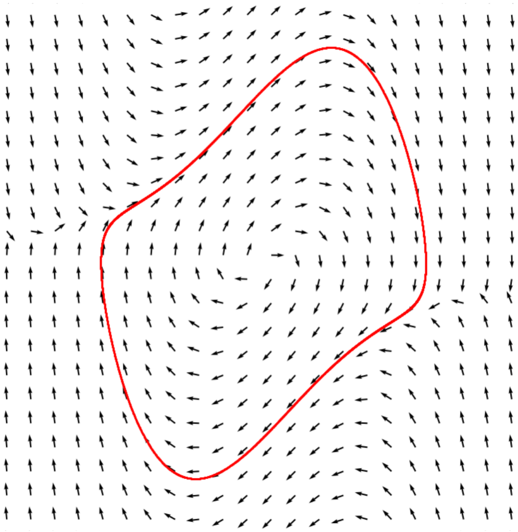
The interval trajectory in the (x_1, x_2)-space is obtained by

$$
\begin{array}{rcl}
\text{In:} & & [\mathbf{x}](0),[\mathbf{v}](0),[\psi](0),[\omega](0),[\mathbf{u}](t) \\
[\mathbf{a}](0) & = & \begin{pmatrix} \cos([\psi](0)) & -\sin([\psi](0)) \\ \sin([\psi](0)) & \cos([\psi](0)) \end{pmatrix} \cdot [\mathbf{v}](0) \\
[\mathbf{a}](t) & = & [\mathbf{a}](0) + \begin{pmatrix} \int_0^t [u_1](\tau) \cdot \cos([\psi](\tau)) \cdot d\tau \\ \int_0^t [u_1](\tau) \cdot \sin([\psi](\tau)) \cdot d\tau \end{pmatrix} \\
[\omega](t) & = & [\omega](0) + \int_0^t [u_2](\tau) d\tau \\
[\psi](t) & = & [\psi](0) + \int_0^t [\omega](\tau) d\tau \\
[\mathbf{v}](t) & = & \begin{pmatrix} \cos([\psi](t)) & \sin([\psi](t)) \\ -\sin([\psi](t)) & \cos([\psi](t)) \end{pmatrix} \cdot [\mathbf{a}](t) \\
[\mathbf{x}](t) & = & [\mathbf{x}](0) + \begin{pmatrix} \cos([\psi](t)) & -\sin([\psi](t)) \\ \sin([\psi](t)) & \cos([\psi](t)) \end{pmatrix} \cdot [\mathbf{v}](t)
\end{array}
$$


x_2
$t = 3$
$t = 0$
x_1

Stability with Poincaré map

System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
How to prove that the system has a cycle ?
How to prove that the system is stable ?

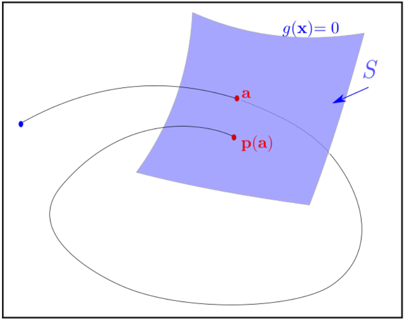


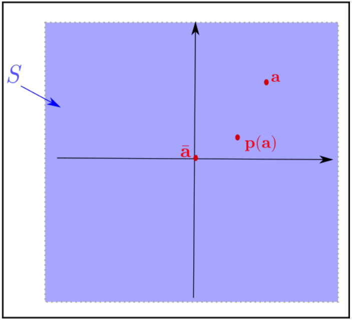
System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
Poincaré section $\mathscr{G}$: $g(\mathbf{x}) = 0$

We define

$$\mathbf{p}: \begin{array}{ccc} \mathscr{G} & \rightarrow & \mathscr{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where $\mathbf{p}(\mathbf{a})$ is the point of $\mathscr{G}$ such that the trajectory initialized at $\mathbf{a}$ intersects $\mathscr{G}$ for the first time.


$g(\mathbf{x}) = 0$
S
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$

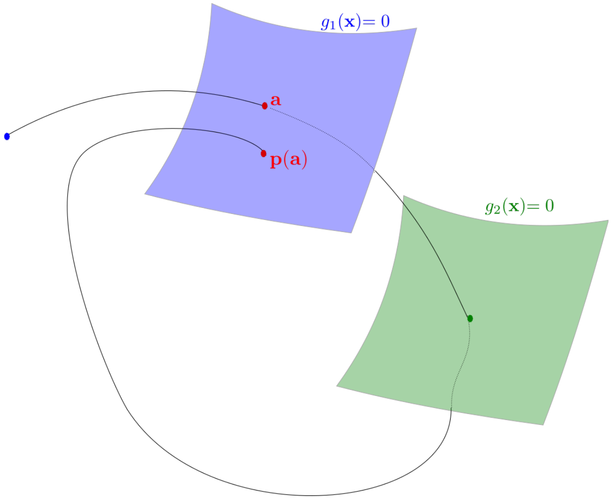

S
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$\bar{\mathbf{a}}$

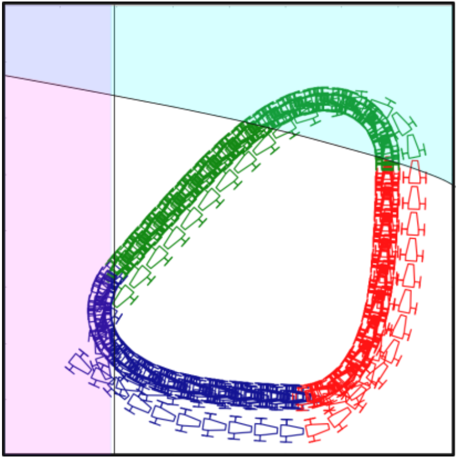
The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$

With hybrid systems

Systems: $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}), i \in \{1, \ldots, m\}$
Section i: $g_i(\mathbf{x}) = 0$


$g_1(\mathbf{x})=0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$g_2(\mathbf{x})=0$

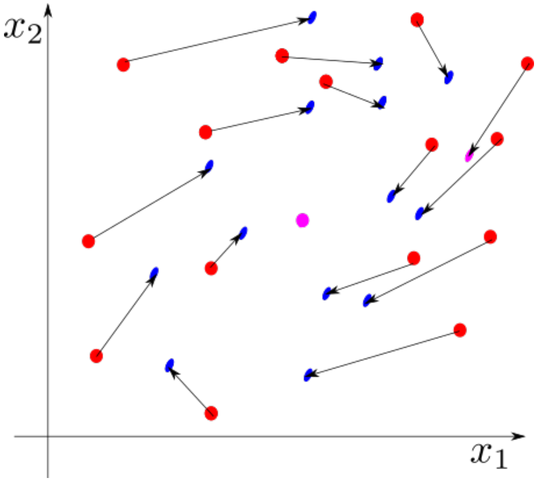


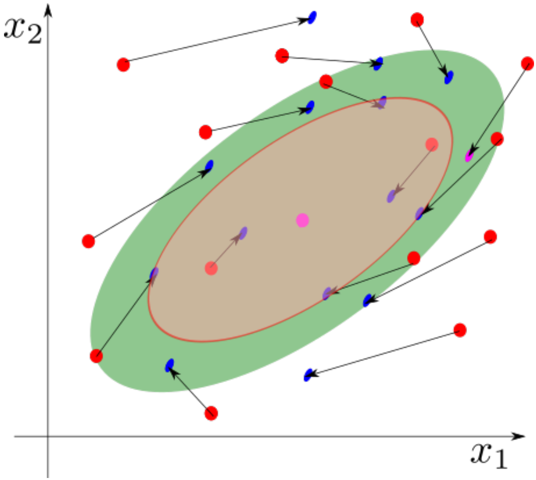
Proving the stability

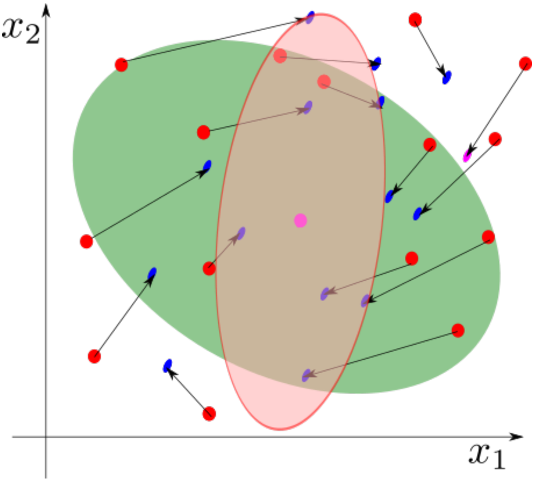
Consider the discrete time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.


x_2
x_1


x_2
x_1


x_2
x_1

We have to find

$$\mathscr{E}_{\mathbf{x}} : \mathbf{x}^{\mathsf{T}} \cdot \mathbf{P} \cdot \mathbf{x} \leq \varepsilon$$

Such that

$$\mathbf{f}(\mathscr{E}_{\mathbf{x}}) \subset \mathscr{E}_{\mathbf{x}}$$

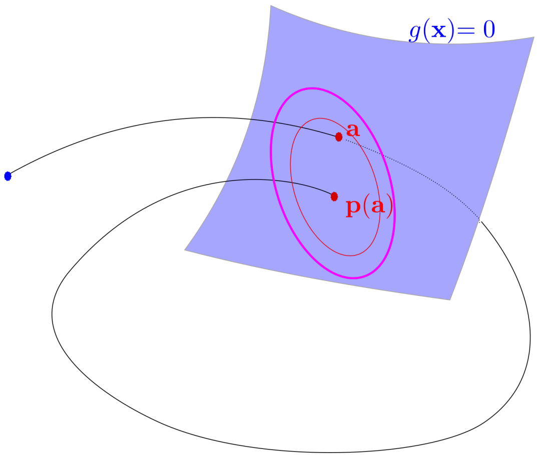
For this, we solve the *axis-aligned* Lyapunov equation

$$\mathbf{A}^{\mathsf{T}} \cdot \mathbf{P} \cdot \mathbf{A} - \mathbf{P} = -\mathbf{A}^{\mathsf{T}}\mathbf{A}$$

Stability of cycles

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$


$g(\mathbf{x}) = 0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$

References

1. Interval and stability [2][11]
2. Route following [4][8]
3. Navigation with stable cycles [3]
4. Tubes [10][1]
5. Integral formulation [5]
6. Ellipse and guaranteed integration [9]
7. Ellipses and guaranteed stability [7]
8. Axis-aligned Lyapunov equation [6]

F. L. Bars.
Analyse par intervalles pour la localisation et la cartographie simultanées ; Application à la robotique sous-marine.
PhD dissertation, Université de Bretagne Occidentale, Brest, France, 2011.

A. Bourgois and L. Jaulin.
Interval centred form for proving stability of non-linear discrete-time systems.
Electronic Proceedings in Theoretical Computer Science, 331:1–17, jan 2021.

Q. Brateau, F. L. Bars, and L. Jaulin.
Navigation without localization using stable cycles.
In *ICRA 2025, Submitted*, 2025.

L. Jaulin.
Naviguer comme les polynésiens.

Interstices, 2019.

L. Jaulin.
Outer approximation of the occupancy set left by a mobile robot.
SWIM 2024, Maastricht, 2024.

M. Louedec.
Guaranteed ellipsoidal numerical method for the stability analysis of the formation control of a group of underwater robots.
PhD dissertation, Université de Bretagne Occidentale, ENSTA-Bretagne, France, November 2024.

M. Louedec, L. Jaulin, and C. Viel.
Computational tractable guaranteed numerical method to study the stability of n-dimensional time-independent nonlinear systems with bounded perturbation.
Automatica, 153:110981, 2023.

T. Nico, L. Jaulin, and B. Zerr.
Guaranteed Polynesian Navigation.
In *SWIM'19, Paris, France*, 2019.

A. Rauh, A. Bourgois, L. Jaulin, and J. Kersten.
Ellipsoidal enclosure techniques for a verified simulation of initial value problems for ordinary differential equations.
In *2021 International Conference on Control, Automation and Diagnosis (ICCAD)*. IEEE, nov 2021.

S. Rohou.
Codac (Catalog Of Domains And Contractors), available at `http://codac.io/`*.*
Robex, Lab-STICC, ENSTA-Bretagne, 2021.

W. Tucker.
A Rigorous ODE Solver and Smale's 14th Problem.

Foundations of Computational Mathematics, 2(1):53–117, 2002.