Guaranteed following of an isobath

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Follow an isobath

Ile des morts experiment


Plouarzel
Saint-Renan
Kerarbec
Guilers
Bohars
E50
Gouesnou
La Forest-Landerneau
Ploumoguer
Lambézellec
Le Relecq-Kerhuon
L'Élorn
Saint-Marc
N165
Dirinon
Saint-Urbain
Brest
Saint-Pierre
E60
Plouzané
Plougastel-Daoulas
Loperhet
N165
Locmaria-Plouzané
Irvillac
Daoulas
Plougonvelin
Hôpital-Camfrout
Roscanvel
Logonna-Daoulas
E60
Landévennec
Lanvéoc
Kergo
Camaret-sur-Mer
Rosnoën
Crozon
Argol
Rue de Dinan
Telgruc-sur-Mer
Saint-Nic
Ar Valare
Plomodiern


Île des Morts
Celtic Sea
ROSCANVEL
Camping Municipal
Route de la Fraternité
Route de Quélern
D355
Persuel
Route de Keruzou
Le Disloup
Route du Milieu
Route du Ménez
Rostellec
Taladec'h
Rue Auguste Ten


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15
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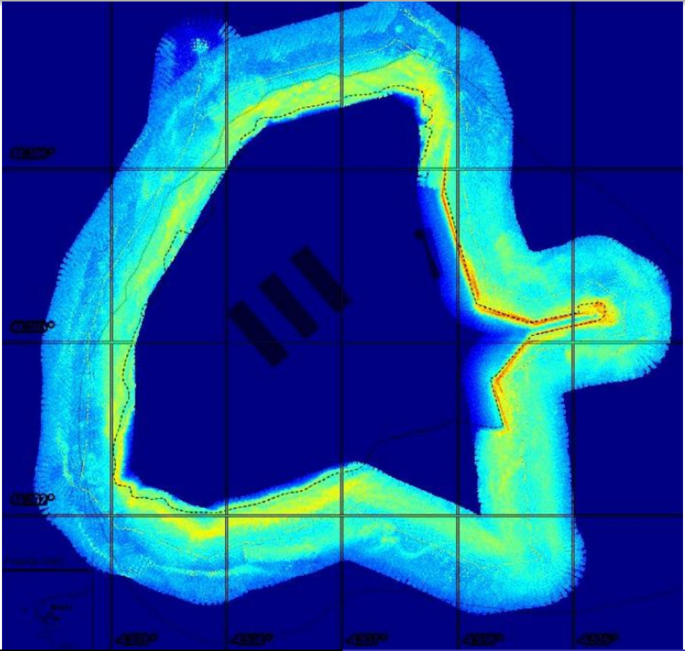
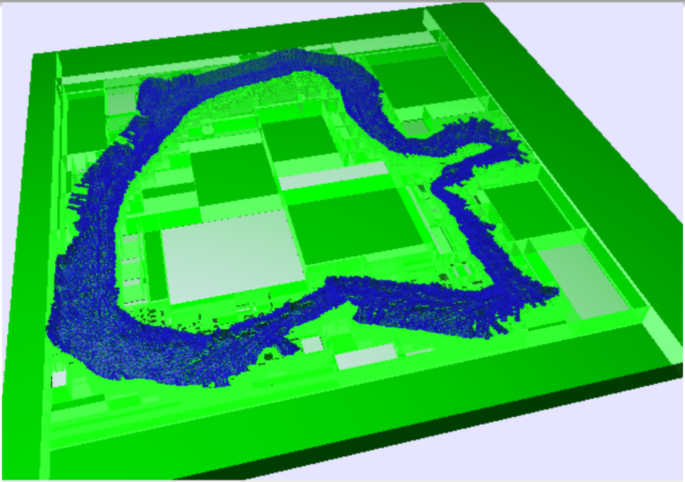
23
50m



24 juillet 2013



Test-case

Consider an underwater robot:

$$\left\{ \begin{array}{lcl} \dot{x} & = & \cos\psi \\ \dot{y} & = & \sin\psi \\ \dot{z} & = & u_1 \\ \dot{\psi} & = & u_2 \end{array} \right.$$

The robot is able to measure its altitude, the angle of the gradient of h and its depth

$$\left\{\begin{array}{lcl} y_1 & = & z - h(x,y) \\ y_2 & = & angle(\nabla h(x,y)) - \psi \\ y_3 & = & -z \end{array}\right.$$

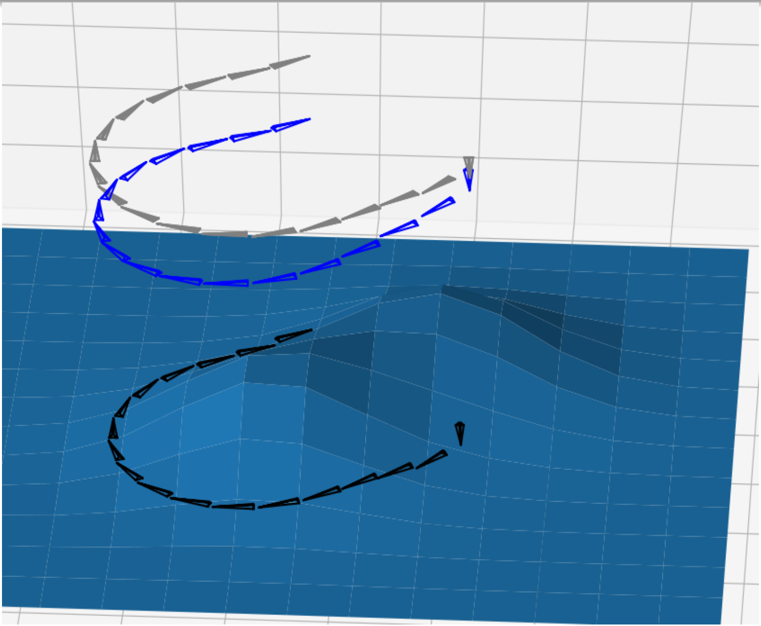
If $\mathbf{y} = \left(-7, \frac{\pi}{2}, 2\right)$, the robot knows that it is following an isobath corresponding to $-7-2=-9$m, at a depth of 2m.

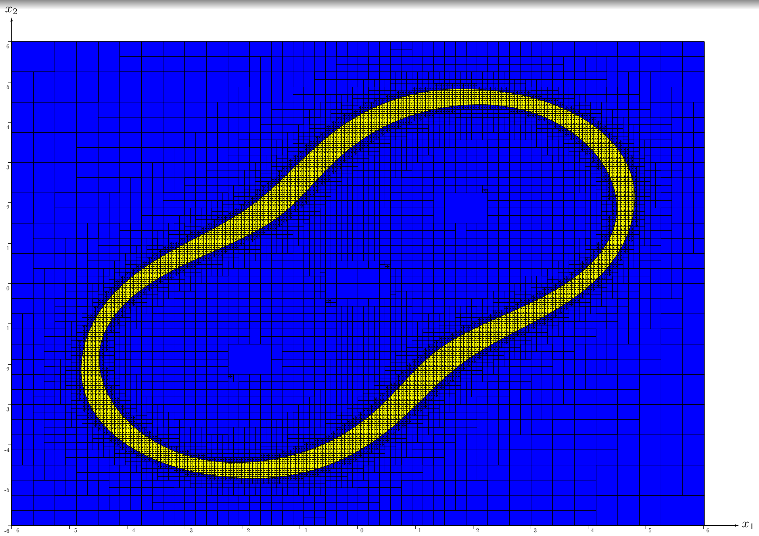
The seafloor is described by

$$h(x,y) = 2 \cdot e^{-\frac{(x+2)^2+(y+2)^2}{10}} + 2 \cdot e^{-\frac{(x-2)^2+(y-2)^2}{10}} - 10.$$

We take the controller may thus be given by

$$\mathbf{u} = \begin{pmatrix} y_3 - \overline{y}_3 \\ \text{-tanh}(h_0 + y_3 + y_1) + \text{sawtooth}(y_2 + \frac{\pi}{2}) \end{pmatrix}.$$




x_2
6
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0
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-2
-3
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-6
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-3
-2
-1
0
1
2
3
4
5
6
x_1

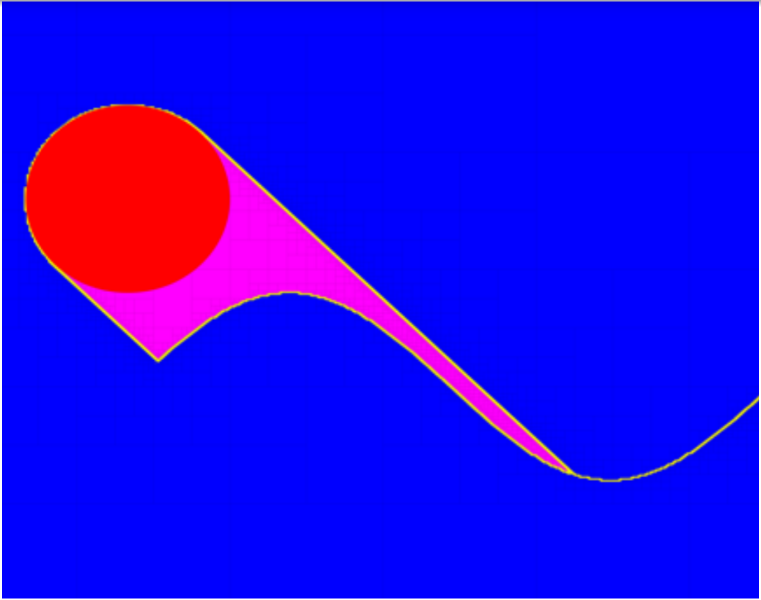
Guaranteed integration

We consider a state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

Let φ be the flow map.
The *forward reach set* of $\mathbb{X} \subset \mathbb{R}^n$ is:

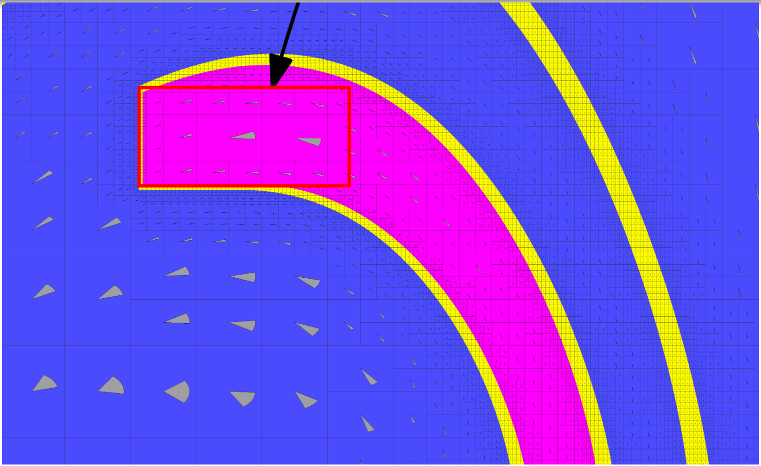
$$\overrightarrow{\mathbb{T}}_{\varphi}(\mathbb{X}) = \{\mathbf{x} \mid \exists \mathbf{x}_0 \in \mathbb{X}, \exists t \geq 0, \mathbf{x} = \varphi(t, \mathbf{x}_0)\}.$$

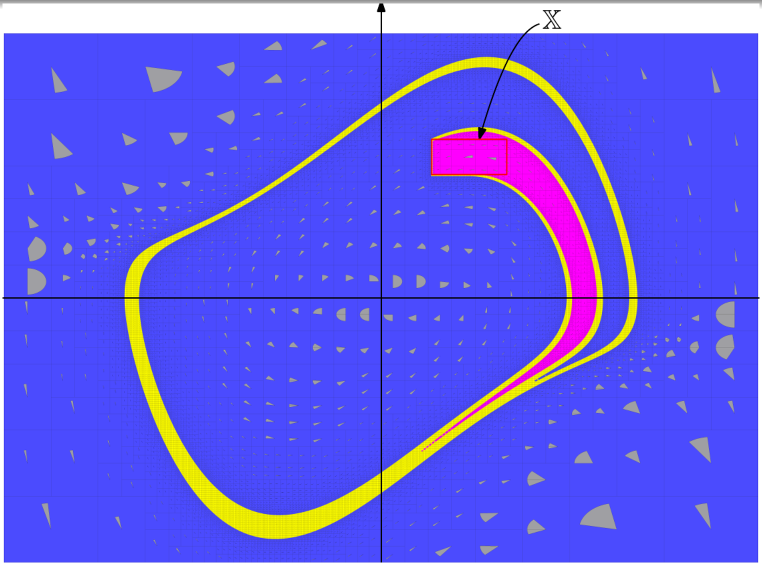
$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= \operatorname{sign}(\sin(x_1) - x_2) \end{aligned}$$



The Van der Pol system

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$

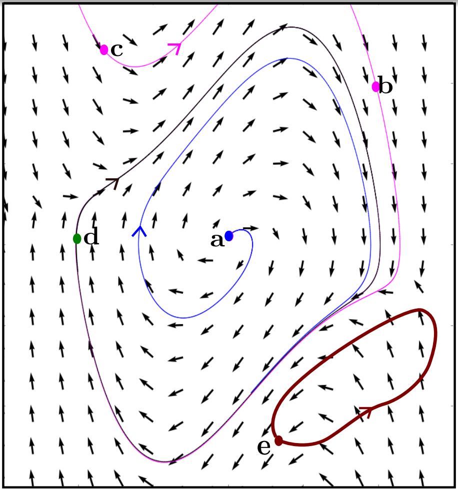



X

Largest positive invariant sets

Example: Consider

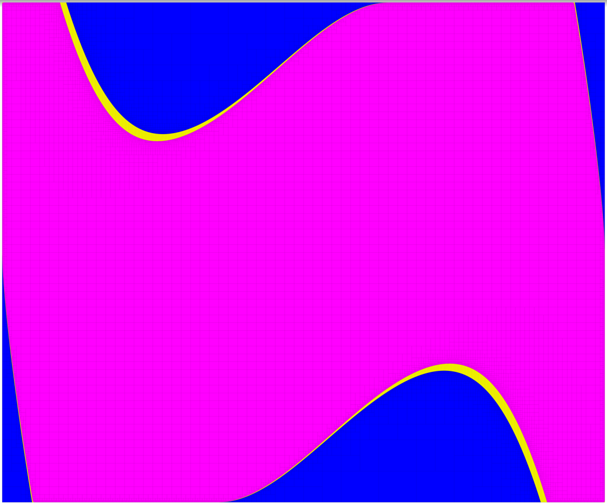
$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$



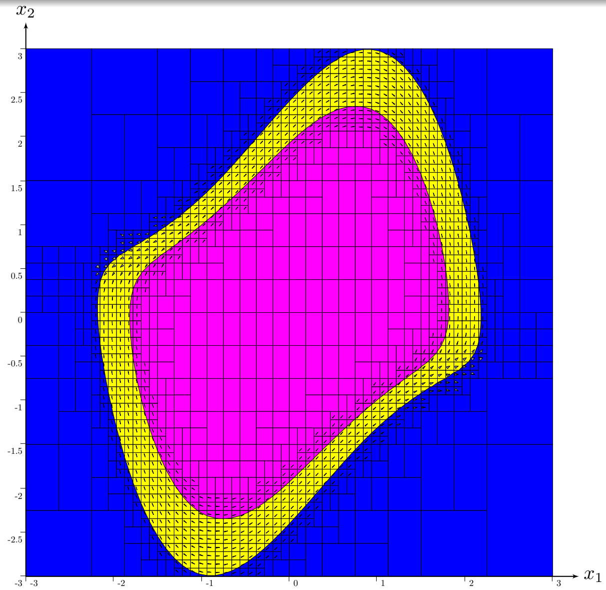
Positive invariant sets : $\mathbb{I}_{\varphi}^{+}(\mathbb{X})$ with $\mathbb{X} = [-4,4] \times [-4,4]$.

The *largest positive invariant set* in $\mathbb{X} \subset \mathbb{R}^n$ is:

$$\mathbb{I}_{\varphi}^{+}(\mathbb{X}) = \{\mathbf{x}_0 \mid \forall t \geq 0, \varphi(t, \mathbf{x}_0) \in \mathbb{X}\}.$$



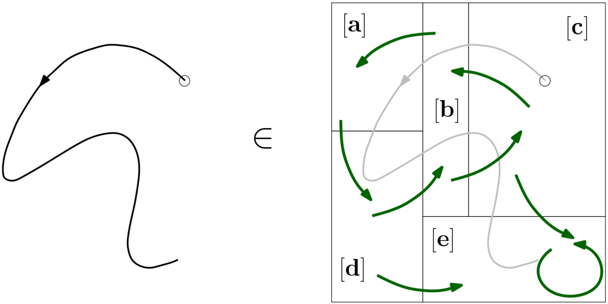
Positive invariant sets : $\mathbb{I}^{+}_{\varphi}(\mathbb{X})$ with $\mathbb{X} = [-4,4] \times [-4,4]$.


x_2
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2
-1
0
1
2
3
x_1

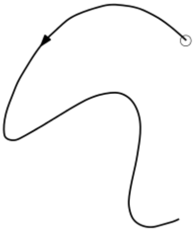
Positive and negative invariant sets : $\mathbb{I}^{+}_{\varphi^{-1}}(\mathbb{X}) \cap \mathbb{I}^{+}_{\varphi}(\mathbb{X})$

Mazes

A maze [2][1] is a set of trajectories.



Mazes can be made more accurate by adding polygones.



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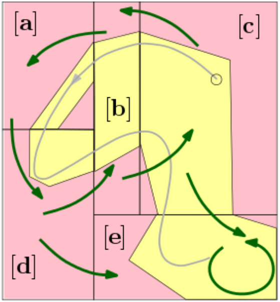
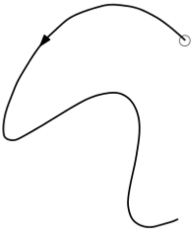


Or using doors instead of a graph



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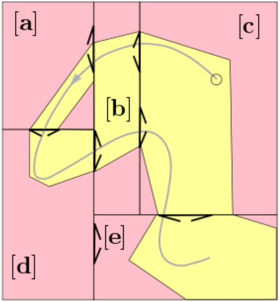
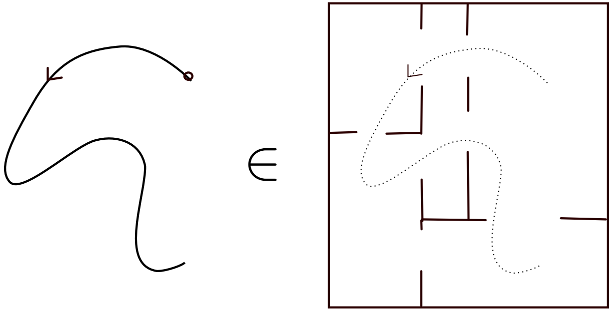


Here, we use bi-directional doors



The trajectory $\mathbf{x}(\cdot)$ belongs to the maze $[\mathbf{x}](\cdot)$

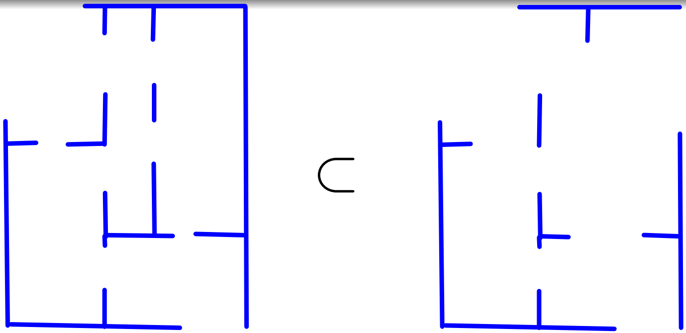
Here, a **maze** $\mathscr{L}$ is composed of

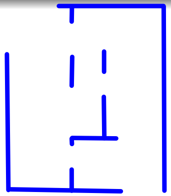
- A paving $\mathscr{P}$
- Doors between adjacent boxes

The set of mazes forms a lattice with respect to $\subset$.

$\mathscr{L}_a \subset \mathscr{L}_b$ means :

- the boxes of $\mathscr{L}_a$ are subboxes of the boxes of $\mathscr{L}_b$.
- The doors of $\mathscr{L}_a$ are thinner than those of $\mathscr{L}_b$.





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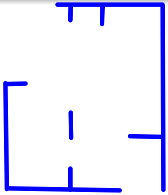

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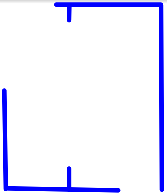
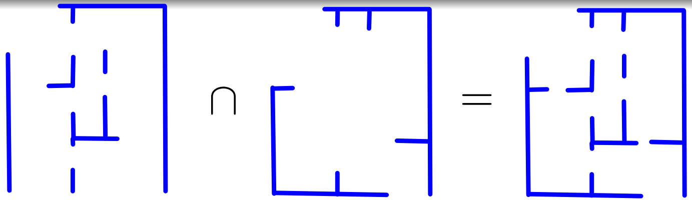
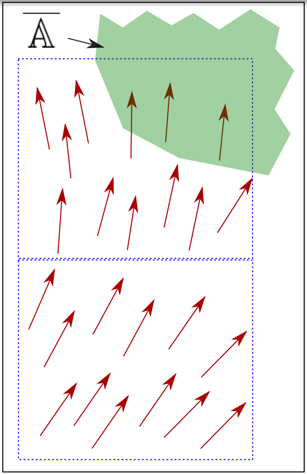
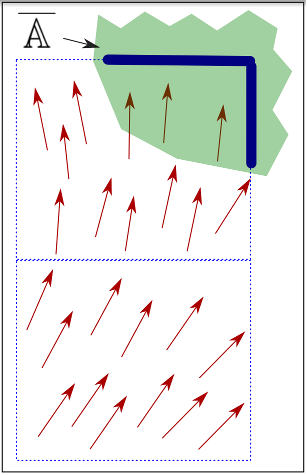
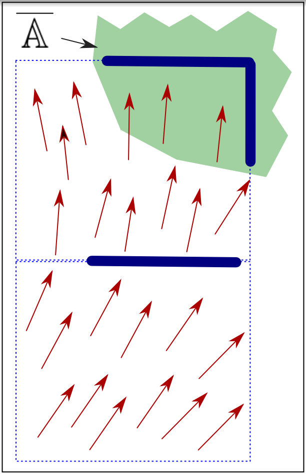


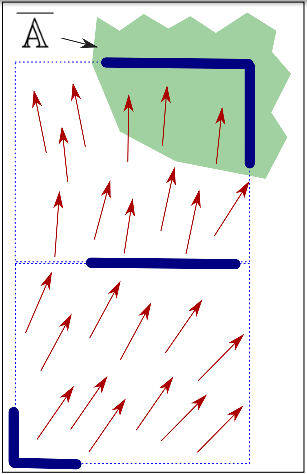


Contract trajectories that certainly go to $\overline{\mathbb{A}}$

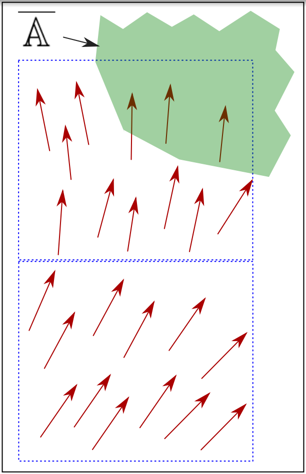

$\overline{\mathbb{A}}$

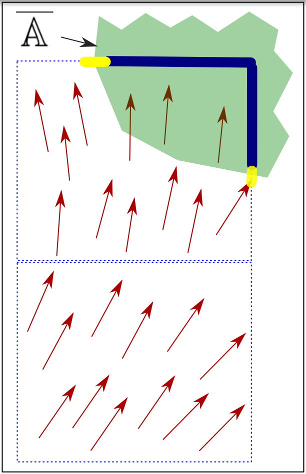

$\bar{\mathbb{A}}$


$\overline{\mathbb{A}}$


$\mathbb{A}$

Contract trajectories that possibly go to $\overline{\mathbb{A}}$


A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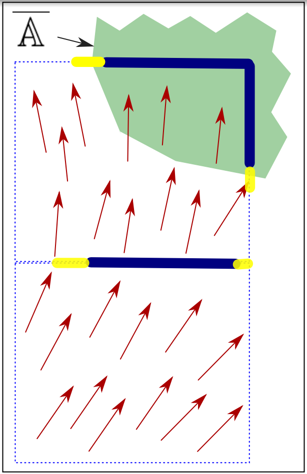
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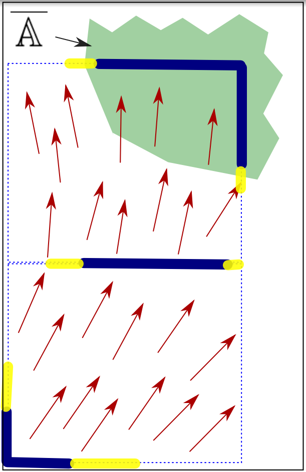
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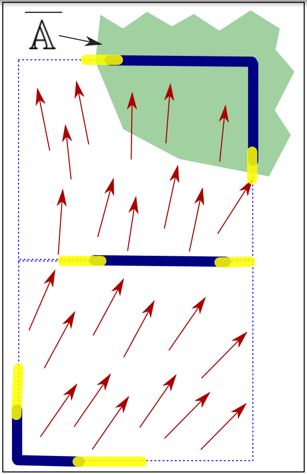
Getting the largest positive invariant set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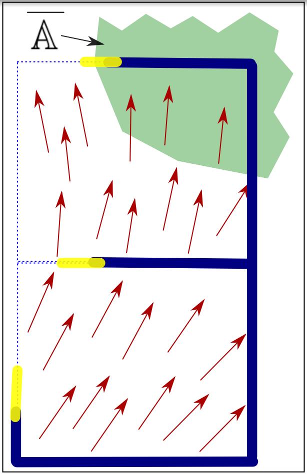
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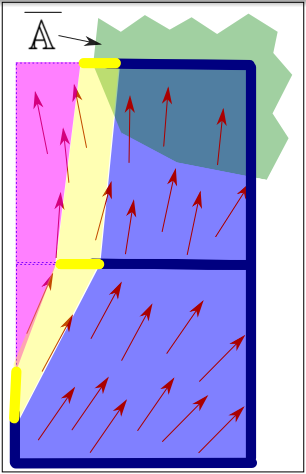
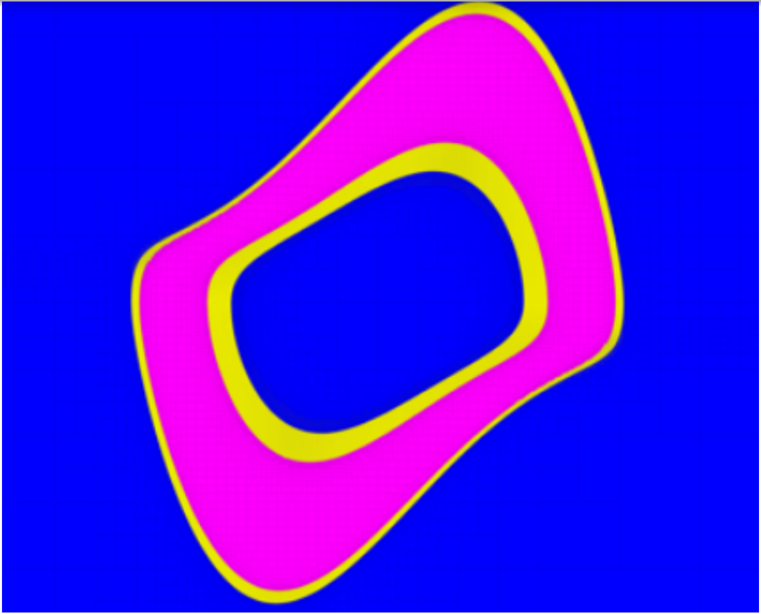

$\overline{\mathbb{A}}$

Viability kernel



Dimension 2|1|1

Consider the system:

$$\begin{cases} \dot{x} & = & v\cos\psi \\ \dot{y} & = & v\sin\psi \\ \dot{\psi} & = & u_1 \\ \dot{v} & = & u_2 \end{cases}$$

which is of dimension 4 or more presily $2|1|1$ (which is easier).

From flatness tools, taking (x, y) as outputs, we may express all state variables and derivative as a function of $(x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dots)$. Indeed:

$$\begin{cases} \psi & = & atan2(\dot{y}, \dot{x}) \\ v & = & \sqrt{\dot{x}^2 + \dot{y}^2} \\ \dot{x} & = & \dot{x} \\ \dot{y} & = & \dot{y} \\ \dot{\psi} & = & \frac{1}{\left(\frac{\dot{y}}{\dot{x}}\right)^2 + 1} \left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \right) \\ \dot{v} & = & \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\ \dots & & \dots \end{cases}$$

If we follow the dynamics

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= (1-x^2)\cdot y - x \end{cases}$$

then all variables $\psi, v, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dot{\psi}, \dot{v}, \dots$ are functions of x, y:

$$\begin{array}{rcl}
\psi & = & atan2\left(\dot{y},\dot{x}\right) = atan2\left(\left(1-x^2\right)\cdot y - x, y\right) \\
& = & \psi_d\left(x,y\right) \\
v & = & \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{y^2+\left(\left(1-x^2\right)\cdot y - x\right)^2} \\
& = & v_d\left(x,y\right) \\
\ddot{x} & = & \dot{y} = \left(1-x^2\right)\cdot y - x \\
& = & \ddot{x}_d\left(x,y\right) \\
\ddot{y} & = & -2x\dot{x}y + \left(1-x^2\right)\dot{y} - \dot{x} = -2xy^2 + \left(1-x^2\right)\left(\left(1-x^2\right)\cdot y - x\right) - y \\
& = & \ddot{y}_d\left(x,y\right) \\
\dot{\psi} & = & \frac{1}{\left(\frac{\dot{y}}{\dot{x}}\right)^2+1}\left(\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{x}^2}\right) = \dot{\psi}\left(x,y\right) = \cdots \\
& = & \dot{\psi}_d\left(x,y\right) \\
\dot{v} & = & \dot{v}\left(x,y\right) = \cdots \\
& = & \dot{v}_d\left(x,y\right)
\end{array}$$

If we take

$$\begin{aligned} u_1 &= \psi_d(x,y) - \psi + \dot{\psi}_d(x,y) \\ u_2 &= v_d(x,y) - v + \dot{v}_d(x,y) \end{aligned}$$

then both $|\psi_d - \psi|$ and $|v_d - v|$decrease in e^{-t}.

If for instance $|\psi_d(y,x) - \psi| < \varepsilon_0$ and $|v_d(y,x) - v| < \varepsilon_0$ $t = 0$, it will be the case for all $t > 0$. Thus we have

$$\begin{cases} \dot{x} &= v\cos\psi \in v_d\cos(\psi_d) + [\varepsilon] \\ \dot{y} &= v\sin\psi \in v_d\sin(\psi_d) + [\varepsilon] \end{cases}$$

where $[\varepsilon] = [-2v_{max}, 2v_{max}]$ (draw the pie in the polar plane).

Denote by $\mathbb{I}_{xy}^{+}$ a robust positive invariant set for

$$\begin{cases} \dot{x} & \in & y+[\varepsilon] \\ \dot{y} & \in & \left(1-x^2\right)\cdot y-x+[\varepsilon] \end{cases}$$

A positive invariant set for our system is

$$\left\{(x,y,v,\psi)\mid(x,y)\in\mathbb{I}_{xy}^{+},v\in v_d\left(x,y\right)+[\varepsilon],\psi\in\psi_d\left(x,y\right)+[\varepsilon]\right\}$$

T. Le Mézo L. Jaulin and B. Zerr.
Bracketing the solutions of an ordinary differential equation with uncertain initial conditions.
Applied Mathematics and Computation, 2017.

T. Le Mézo, L. Jaulin, and B. Zerr.
An interval approach to compute invariant sets.
IEEE Transaction on Automatic Control, 2017.