Separators: a new interval tool to bracket solution sets; application to path planning

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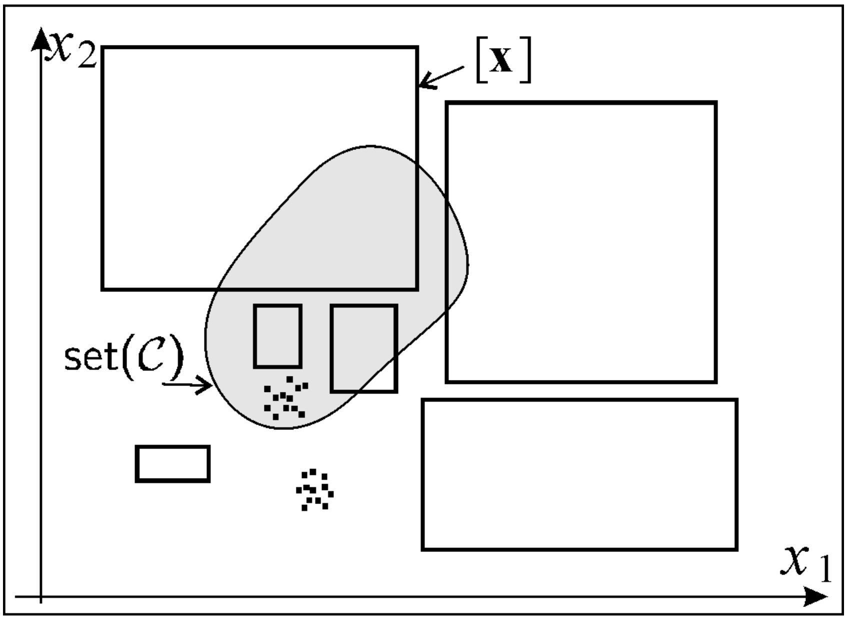
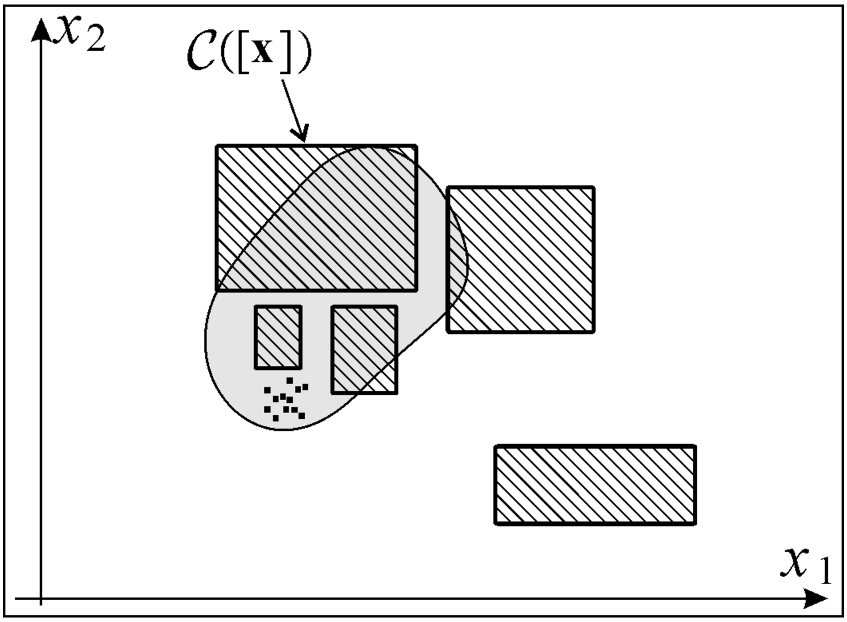
1 Contractors

$$\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \qquad \text{(contractance)}$$
$$[\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]) \qquad \text{(monotonicity)}$$


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
x_1

Inclusion

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall\, [\mathbf{x}] \in \mathbb{IR}^n,\, \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]).$$

A set $\mathbb{S}$ is *consistent* with $\mathcal{C}$ (we write $\mathbb{S} \sim \mathcal{C}$) if

$$\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}.$$

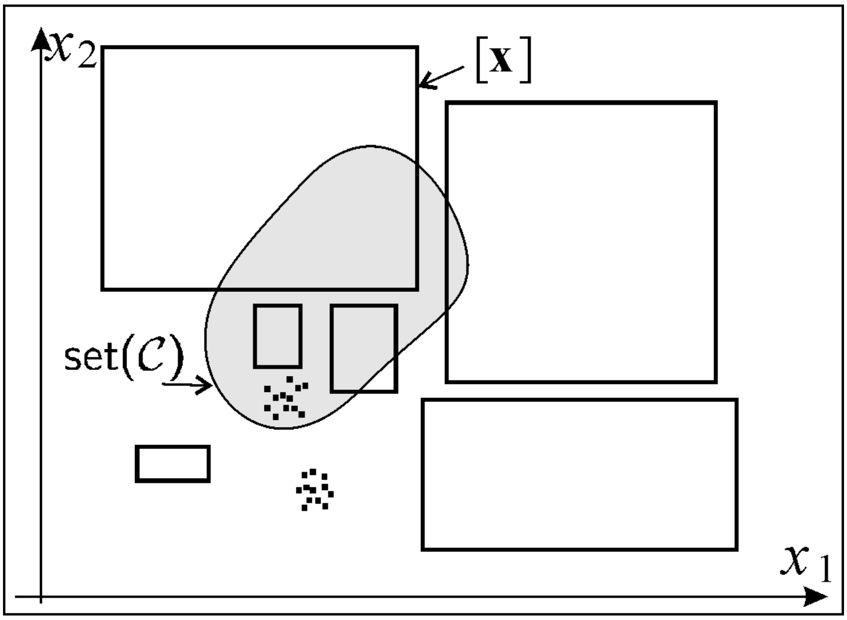
$\mathcal{C}$ is *minimal* if

$$\left.\begin{array}{c} \mathbb{S} \sim \mathcal{C} \\ \mathbb{S} \sim \mathcal{C}_1 \end{array}\right\} \Rightarrow \mathcal{C} \subset \mathcal{C}_1.$$

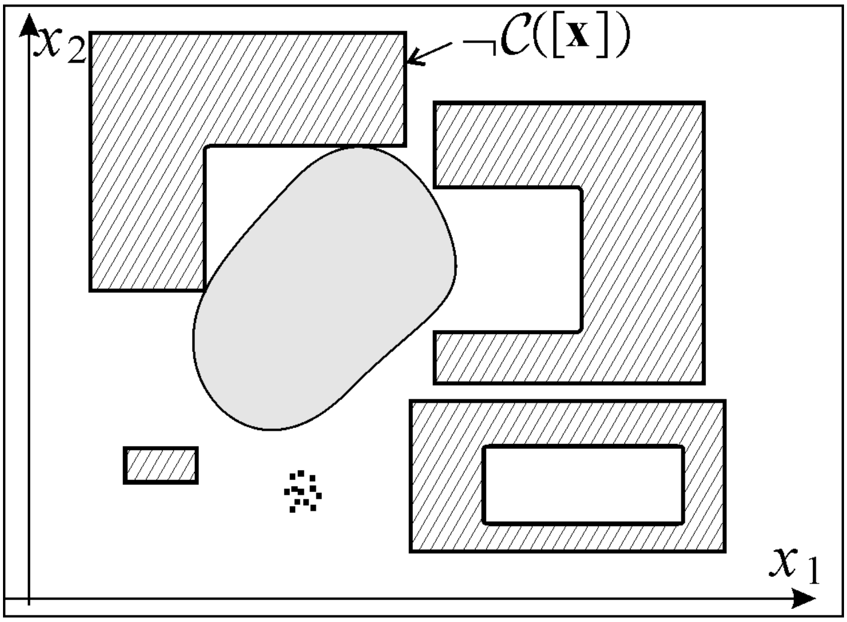
The *negation* $\neg\mathcal{C}$ of $\mathcal{C}$ is defined by

$$\neg\mathcal{C}\left([\mathbf{x}]\right) = \left\{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{x} \notin \mathcal{C}\left([\mathbf{x}]\right)\right\}.$$

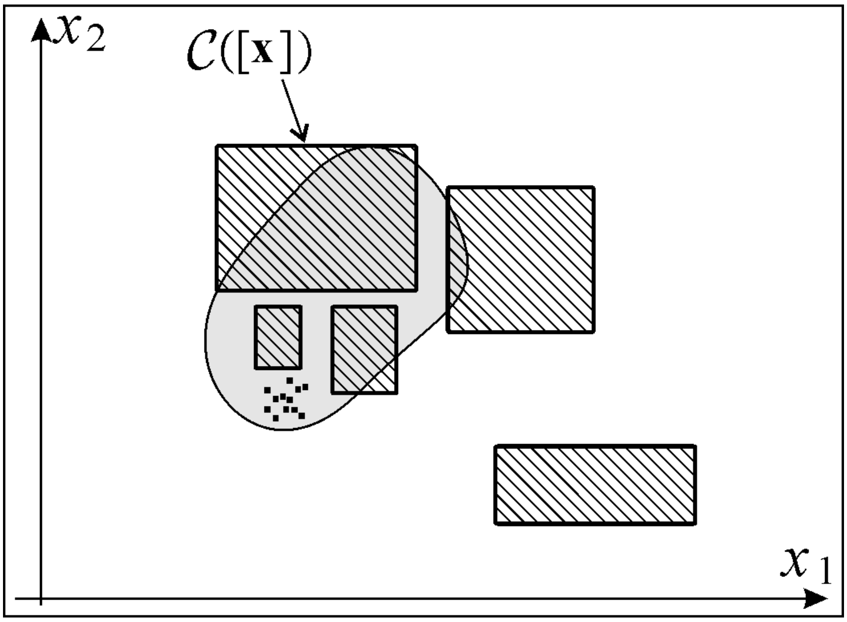
It is not a box in general.



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\neg\mathcal{C}([\mathbf{x}])$
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

2 Separators

A *separator* $\mathcal{S}$ is pair of contractors $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ such that

$$\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}] \qquad \text{(complementarity).}$$

A set $\mathbb{S}$ is *consistent* with $\mathcal{S}$ (we write $\mathbb{S} \sim \mathcal{S}$), if

$$\mathbb{S} \sim \mathcal{S}^{\mathsf{out}} \text{ and } \overline{\mathbb{S}} \sim \mathcal{S}^{\mathsf{in}}.$$

The *remainder* of $\mathcal{S}$ is

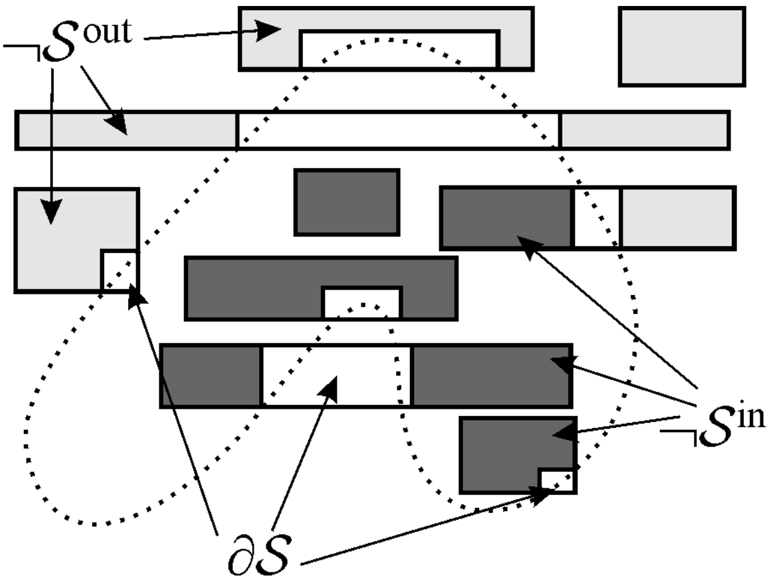
$$\partial\mathcal{S}([\mathbf{x}]) = \mathcal{S}^{\mathsf{in}}([\mathbf{x}]) \cap \mathcal{S}^{\mathsf{out}}([\mathbf{x}]).$$

$\partial\mathcal{S}$ is a contractor, not a separator.

We have

$$\neg\mathcal{S}^{\mathsf{in}}([\mathbf{x}]) \cup \neg\mathcal{S}^{\mathsf{out}}([\mathbf{x}]) \cup \partial\mathcal{S}([\mathbf{x}]) = [\mathbf{x}]\,.$$

Moreover, they do not intersect.



$\neg\mathcal{S}^{\mathsf{in}}([\mathbf{x}])$, $\neg\mathcal{S}^{\mathsf{out}}([\mathbf{x}])$ and $\partial\mathcal{S}([\mathbf{x}])$

Inclusion

$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \mathcal{S}_1^{\mathsf{in}} \subset \mathcal{S}_2^{\mathsf{in}} \text{ and } \mathcal{S}_1^{\mathsf{out}} \subset \mathcal{S}_2^{\mathsf{out}}.$$

Here $\subset$ means *more accurate.*

$\mathcal{S}$ is *minimal* if

$$\mathcal{S}_1 \subset \mathcal{S} \Rightarrow \mathcal{S}_1 = \mathcal{S}.$$

i.e., if $\mathcal{S}^{\mathsf{in}}$ and $\mathcal{S}^{\mathsf{out}}$ are both minimal.

3 Paver

We want to compute $\mathbb{X}^-, \mathbb{X}^+$ such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Algorithm Paver(in: $[\mathbf{x}], \mathcal{S}$; out: $\mathbb{X}^-$, $\mathbb{X}^+$)

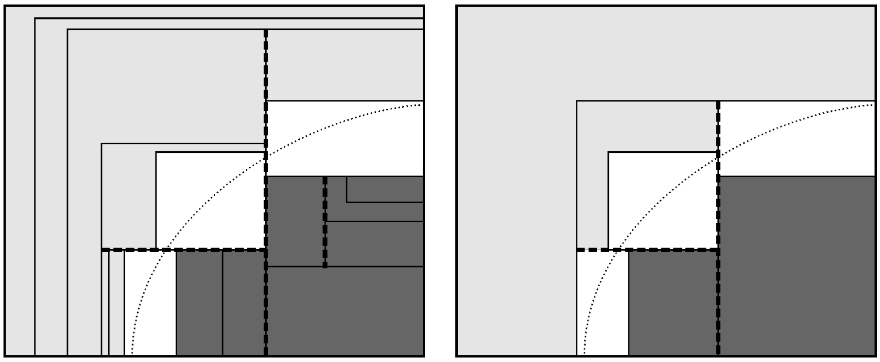
1 $\mathcal{L} := \{[\mathbf{x}]\}$;
2 Pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 $\left\{[\mathbf{x}^{\text{in}}], [\mathbf{x}^{\text{out}}]\right\} = \mathcal{S}([\mathbf{x}])$;
4 Store $[\mathbf{x}] \backslash [\mathbf{x}^{\text{in}}]$ into $\mathbb{X}^-$ and also into $\mathbb{X}^+$;
5 $[\mathbf{x}] = [\mathbf{x}^{\text{in}}] \cap [\mathbf{x}^{\text{out}}]$;
6 If $w([\mathbf{x}]) < \varepsilon$, then store $[\mathbf{x}]$ in $\mathbb{X}^+$,
7 Else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$ the two childs
8 If $\mathcal{L} \neq \emptyset$, go to 2.

For the implementation, the paving is represented by a binary tree.

The ith node of the tree contains two boxes: $[\mathbf{x}^{\text{in}}](i)$ and $[\mathbf{x}^{\text{out}}](i)$.

The binary tree is said to be *minimal* if for any node i_1 with brother i_2 and father j, we have

$$\left\{\begin{array}{lll} \text{(i)} & [\mathbf{x}^{\text{in}}](i_1) \neq \emptyset,\ [\mathbf{x}^{\text{out}}](i_1) \neq \emptyset & \\ \text{(ii)} & [\mathbf{x}^{\text{in}}](j) \cap [\mathbf{x}^{\text{out}}](j) = & \Big([\mathbf{x}^{\text{in}}](i_1) \cap [\mathbf{x}^{\text{out}}](i_1)\Big) \\ & & \sqcup \Big([\mathbf{x}^{\text{in}}](i_2) \cap [\mathbf{x}^{\text{out}}](i_2)\Big) \end{array}\right.$$



4 Algebra

Contractor algebra does not allow decreasing operations, i.e., we should have expression $\mathcal{E}$ such that

$$\forall i, \mathcal{C}_i \subset \mathcal{C}_i^{'} \Rightarrow \mathcal{E}\left(\mathcal{C}_1, \mathcal{C}_2, \dots\right) \subset \mathcal{E}\left(\mathcal{C}_1^{'}, \mathcal{C}_2^{'}, \dots\right).$$

The complementary $\overline{\mathcal{C}}$ of a contractor $\mathcal{C}$ or the restriction $\mathcal{C}_1 \backslash \mathcal{C}_2$ of two contractors cannot be defined.

Separators extend the operations allowed for contractors to non monotonic expressions.

The *complement* of $\mathcal{S} = \left\{\mathcal{S}^{\mathsf{in}}, \mathcal{S}^{\mathsf{out}}\right\}$ is

$$\overline{\mathcal{S}} = \left\{\mathcal{S}^{\mathsf{out}}, \mathcal{S}^{\mathsf{in}}\right\}.$$

The *exponentiation* of $\mathcal{S} = \left\{ \mathcal{S}^{\mathsf{in}}, \mathcal{S}^{\mathsf{out}} \right\}$ is defined as

$$\begin{array}{ll} \mathcal{S}^0 & = \{\top, \top\} \\ \mathcal{S}^{k+1} & = \left\{ \neg \mathcal{S}^{k\ \mathsf{out}} \sqcup (\mathcal{S}^{\mathsf{in}} \circ \partial \mathcal{S}^k), \neg \mathcal{S}^{k\ \mathsf{in}} \sqcup (\mathcal{S}^{\mathsf{out}} \circ \partial \mathcal{S}^k) \right\}. \end{array}$$

Example.

$$\begin{aligned}\mathcal{S}^1 \quad &= \quad \{ \quad \neg(\mathcal{S}^0)^{\mathsf{out}} \sqcup \mathcal{S}^{\mathsf{in}} \circ \left(\left(\mathcal{S}^0\right)^{\mathsf{in}} \cap \left(\mathcal{S}^0\right)^{\mathsf{out}}\right), \\ &\qquad\quad \neg(\mathcal{S}^0)^{\mathsf{in}} \sqcup \mathcal{S}^{\mathsf{out}} \circ \left(\left(\mathcal{S}^0\right)^{\mathsf{in}} \cap \left(\mathcal{S}^0\right)^{\mathsf{out}}\right) \quad \} \\ &= \left\{\neg\top \sqcup \mathcal{S}^{\mathsf{in}} \circ (\top \cap \top), \neg\top \sqcup \mathcal{S}^{\mathsf{out}} \circ (\top \cap \top)\right\} \\ &= \left\{\mathcal{S}^{\mathsf{in}}, \mathcal{S}^{\mathsf{out}}\right\} = \mathcal{S}.\end{aligned}$$

If $\mathcal{S}_i = \left\{\mathcal{S}_i^{\mathsf{in}}, \mathcal{S}_i^{\mathsf{out}}\right\}, i \geq 1$, are separators, we define

$$\begin{array}{rcll}
\mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\mathsf{in}} \cup \mathcal{S}_2^{\mathsf{in}}, \mathcal{S}_1^{\mathsf{out}} \cap \mathcal{S}_2^{\mathsf{out}}\right\} & \text{(intersection)} \\
\mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\mathsf{in}} \cap \mathcal{S}_2^{\mathsf{in}}, \mathcal{S}_1^{\mathsf{out}} \cup \mathcal{S}_2^{\mathsf{out}}\right\} & \text{(union)} \\
\bigcap^{\{q\}} \mathcal{S}_i & = & \left\{\bigcap^{\{m-q-1\}} \mathcal{S}_i^{\mathsf{in}}, \bigcap^{\{q\}} \mathcal{S}_i^{\mathsf{out}}\right\} & \text{(relaxed intersection)} \\
\mathcal{S}_1 \backslash \mathcal{S}_2 & = & \mathcal{S}_1 \cap \overline{\mathcal{S}_2}. & \text{(difference)}
\end{array}$$

Theorem 1. If $\mathbb{S}_i$ are subsets of $\mathbb{R}^n$, we have

$$
\begin{array}{rccc}
\text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathcal{S}_1 \cap \mathcal{S}_2 \\
\text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathcal{S}_1 \cup \mathcal{S}_2 \\
\text{(iii)} & \overline{\mathbb{S}}_i & \sim & \overline{\mathcal{S}_i} \\
\text{(iv)} & \mathbb{S}_i & \sim & \mathcal{S}_i^k,\ k \geq 0 \\
\text{(v)} & \bigcap^{\{q\}} \mathbb{S}_i & \sim & \bigcap^{\{q\}} \mathcal{S}_i \\
\text{(vi)} & \mathbb{S}_1 \backslash \mathbb{S}_2 & \sim & \mathcal{S}_1 \backslash \mathcal{S}_2.
\end{array}
$$

5 Inversion of separators

The inverse of $\mathbb{Y} \subset \mathbb{R}^n$ by $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) = \{\mathbf{x} \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\}.$$

$\mathbf{f}$ can be a translation, rotation, homothety, projection, ...

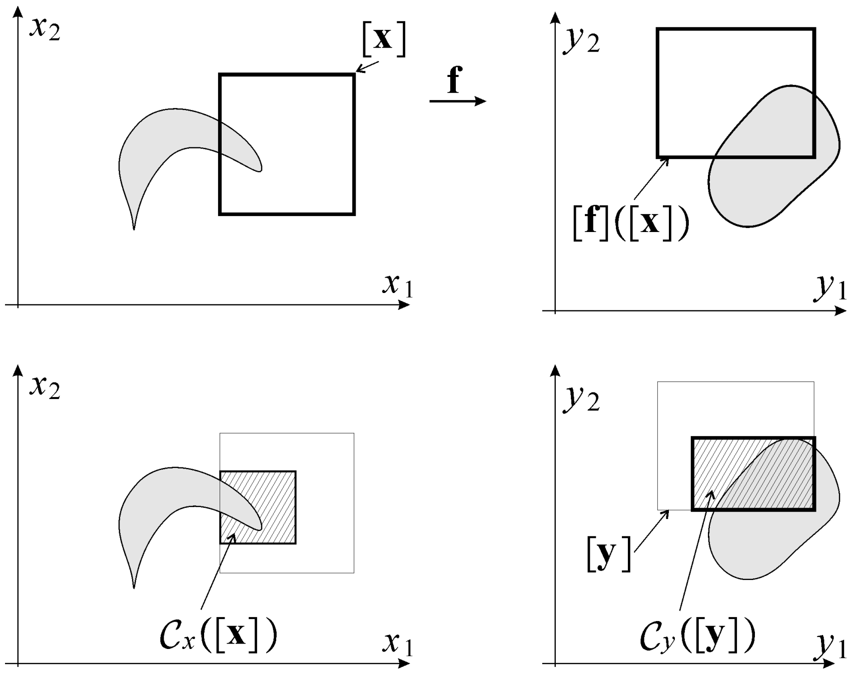
Inverse of a contractor

We have

$$\mathcal{C}_x \sim \mathbf{f}^{-1}(\mathcal{C}_y)$$

if

$$\mathcal{C}_x \sim \mathbf{f}^{-1}(\mathsf{set}(\mathcal{C}_y)).$$



x_2
$[\mathbf{x}]$
$\mathbf{f}$
y_2
$[\mathbf{f}]([\mathbf{x}])$
x_1
y_1
x_2
y_2
$[\mathbf{y}]$
$\mathcal{C}_x([\mathbf{x}])$
x_1
$\mathcal{C}_y([\mathbf{y}])$
y_1

We define

$$\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}) = \left\{ \mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}^{\mathsf{in}}), \mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}^{\mathsf{out}}) \right\}.$$

Theorem 2. The separator $\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}})$ is a separator associated with the set $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$, i.e.,

$$\mathbf{f}^{-1}(\mathbb{Y}) \sim \mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}) .$$

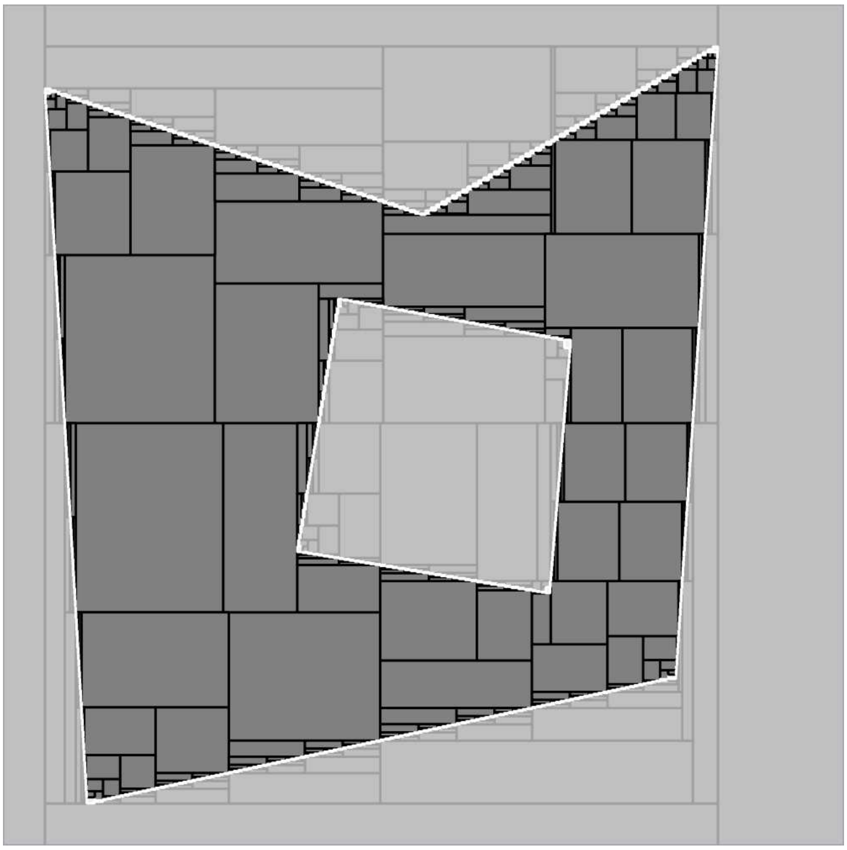
Example. If

$$\mathbf{f}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix},$$

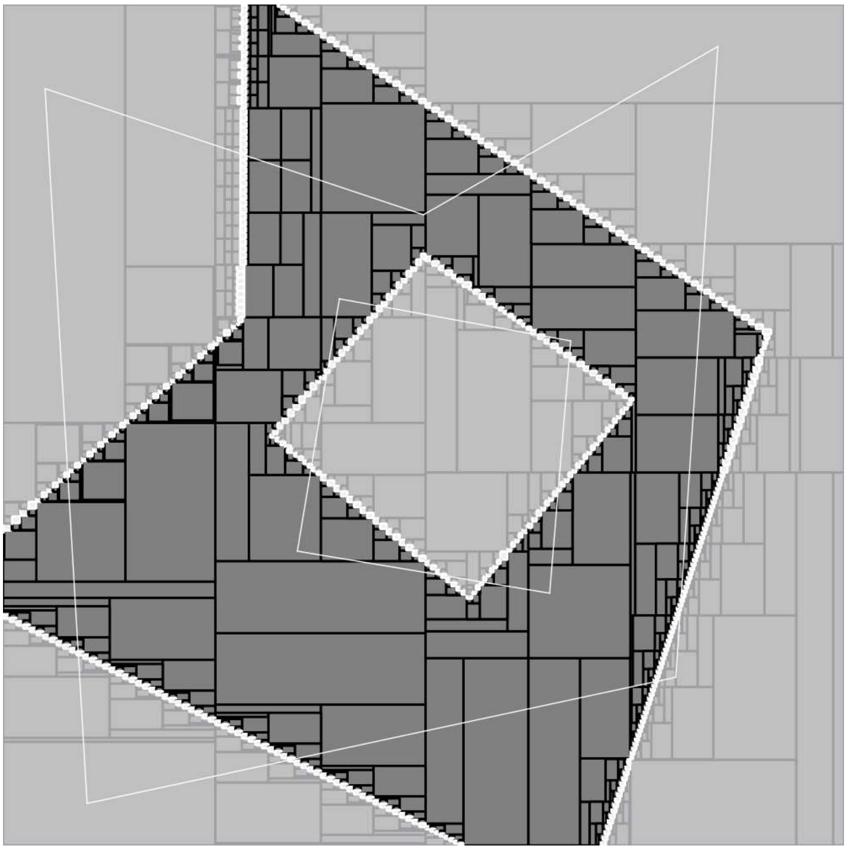
a separator $\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}})$ obtained by the following algorithm.

Separator $\mathcal{S}_{\mathbb{X}}$(in: $[\mathbf{x}]\,,\mathcal{S}_{\mathbb{Y}}$; out: $\{[\mathbf{x}^{\text{in}}],[\mathbf{x}^{\text{out}}]\}$)

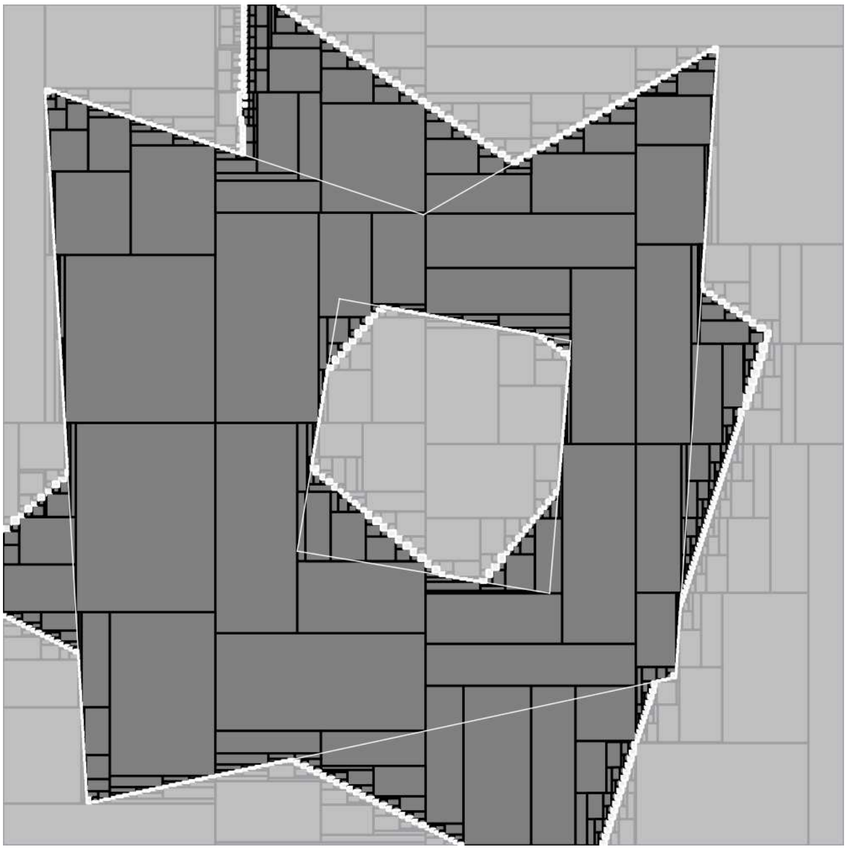
$$[y_1]=[x_1]+2[x_2];$$
$$[y_2]=[x_1]-[x_2];$$
$$[\mathbf{y}^{\text{in}}]=\mathcal{S}_{\mathbb{Y}}^{\text{in}}\left([\mathbf{y}]\right);\;[\mathbf{y}^{\text{out}}]=\mathcal{S}_{\mathbb{Y}}^{\text{out}}\left([\mathbf{y}]\right);$$
$$[\mathbf{x}^{\text{in}}]=[\mathbf{x}];[\mathbf{x}^{\text{out}}]=[\mathbf{x}];$$
$$[x_1^{\text{in}}]=[x_1^{\text{in}}]\cap\left([y_2^{\text{in}}]+[x_2^{\text{in}}]\right);$$
$$[x_2^{\text{in}}]=[x_2^{\text{in}}]\cap\left([x_1^{\text{in}}]-[y_2^{\text{in}}]\right);$$
$$[x_1^{\text{out}}]=[x_1^{\text{out}}]\cap\left([y_2^{\text{out}}]+[x_2^{\text{out}}]\right);$$
$$[x_2^{\text{out}}]=[x_2^{\text{out}}]\cap\left([x_1^{\text{out}}]-[y_2^{\text{out}}]\right);$$
$$[x_1^{\text{in}}]=[x_1^{\text{in}}]\cap[y_1^{\text{in}}]-2[x_2^{\text{in}}];$$
$$[x_2^{\text{in}}]=[x_2^{\text{in}}]\cap\frac{1}{2}\left([y_1^{\text{in}}]-[x_1^{\text{in}}]\right);$$
$$[x_1^{\text{out}}]=[x_1^{\text{out}}]\cap[y_1^{\text{out}}]-2[x_2^{\text{out}}];$$
$$[x_2^{\text{out}}]=[x_2^{\text{out}}]\cap\frac{1}{2}\left([y_1^{\text{out}}]-[x_1^{\text{out}}]\right).$$



A map $\mathbb{M}$



$\mathrm{Rot}(\mathbb{M})$



$\mathsf{Rot}(\mathbb{M}) \cup \mathbb{M}$

6 Atomic separators

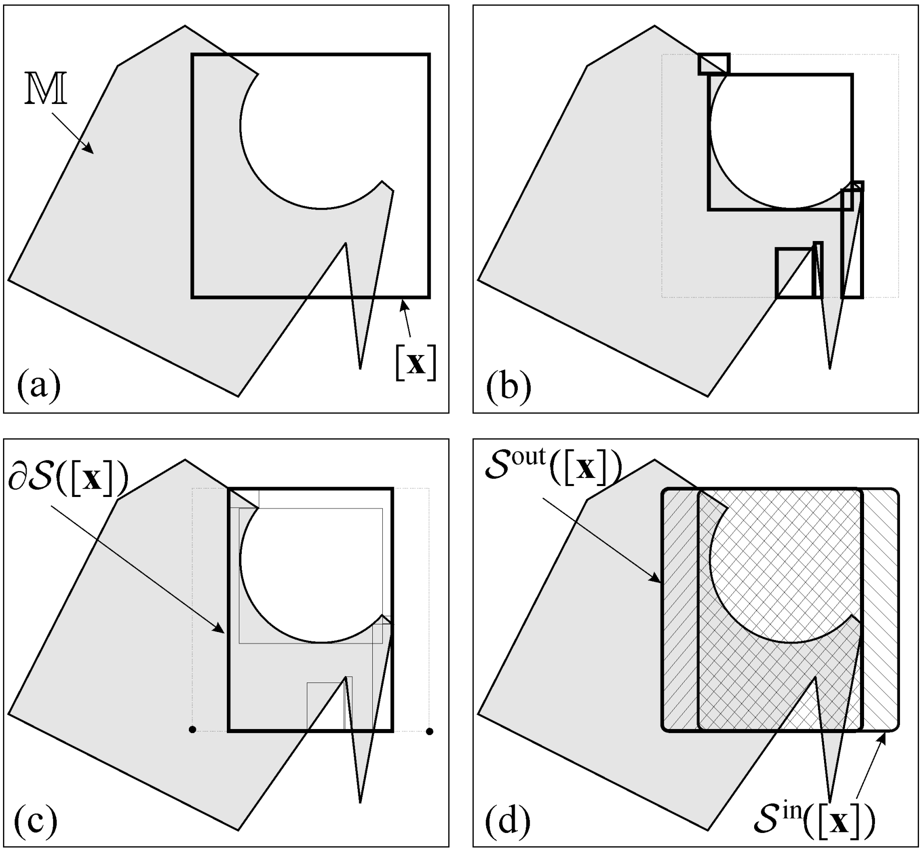
6.1 Equation-based separators

If

$$\mathbb{X} = \{f(\mathbf{x}) \leq 0\},$$

the pair $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$, where $\mathcal{S}^{\text{out}}$: $f(\mathbf{x}) \leq 0$ and $\mathcal{S}^{\text{in}}$: $f(\mathbf{x}) \geq 0$, is a separator for $\mathbb{X}$.

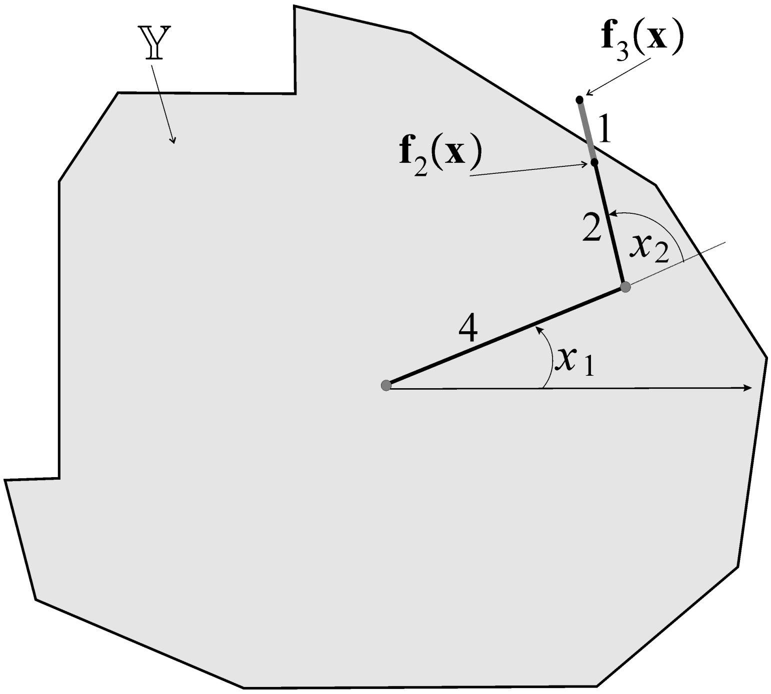
6.2 Database-based separators



Optimal separator using boundaries

7 Path planning

Wire loop game : a metal loop on a handle and a curved wire. The player holds the loop in one hand and attempts to guide it along the curved wire without touching.



Wire loop game. Is it possible to perform to complete circular path ?

The feasible configuration space is

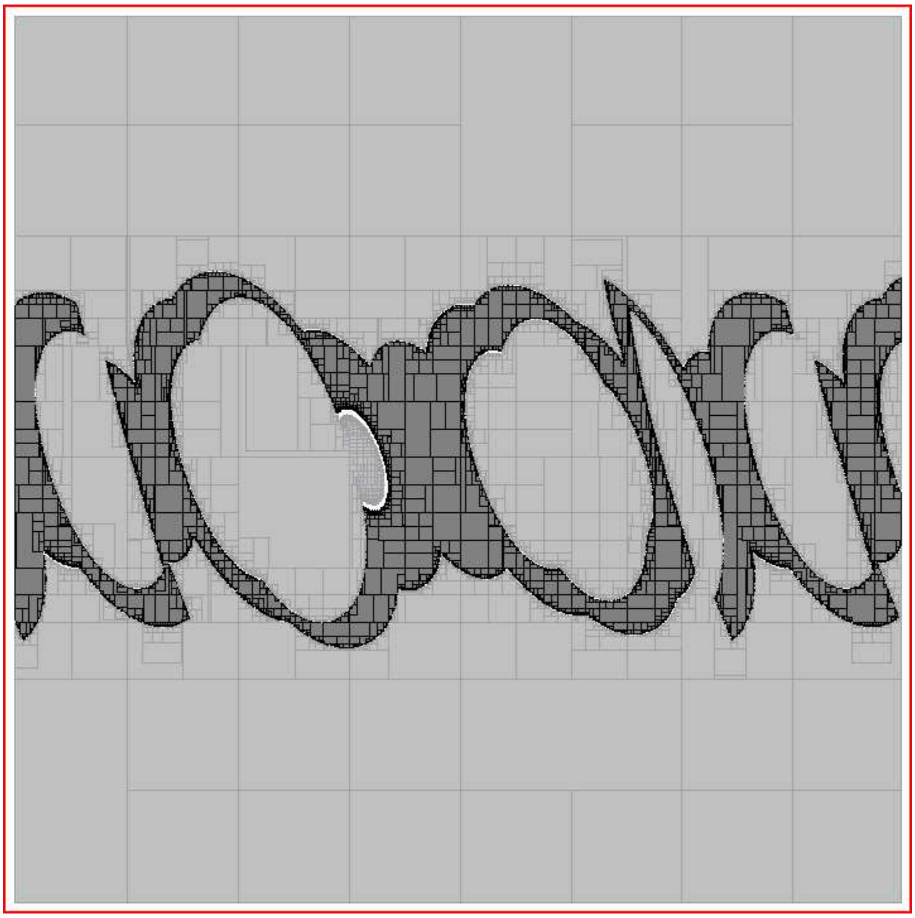
$$\mathbb{M} = \{(x_1, x_2) \in [-\pi, \pi] \mid \mathbf{f}_2(\mathbf{x}) \in \mathbb{Y} \text{ and } \mathbf{f}_3(\mathbf{x}) \notin \mathbb{Y}\}$$

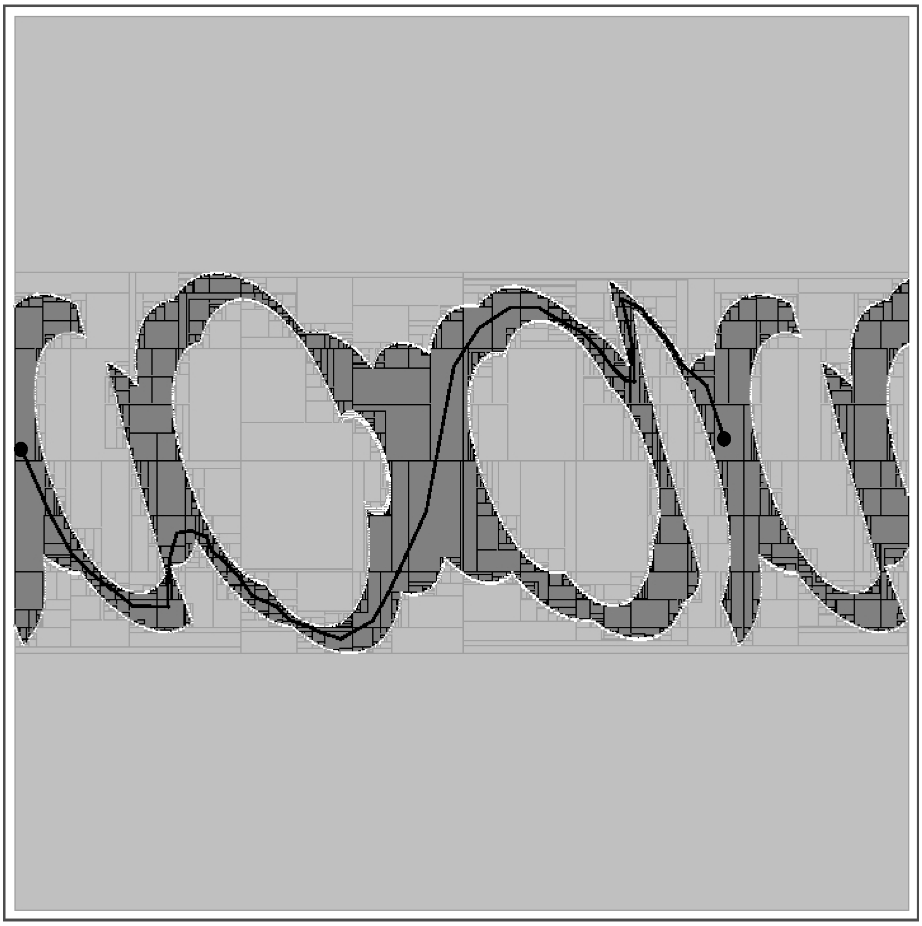
where

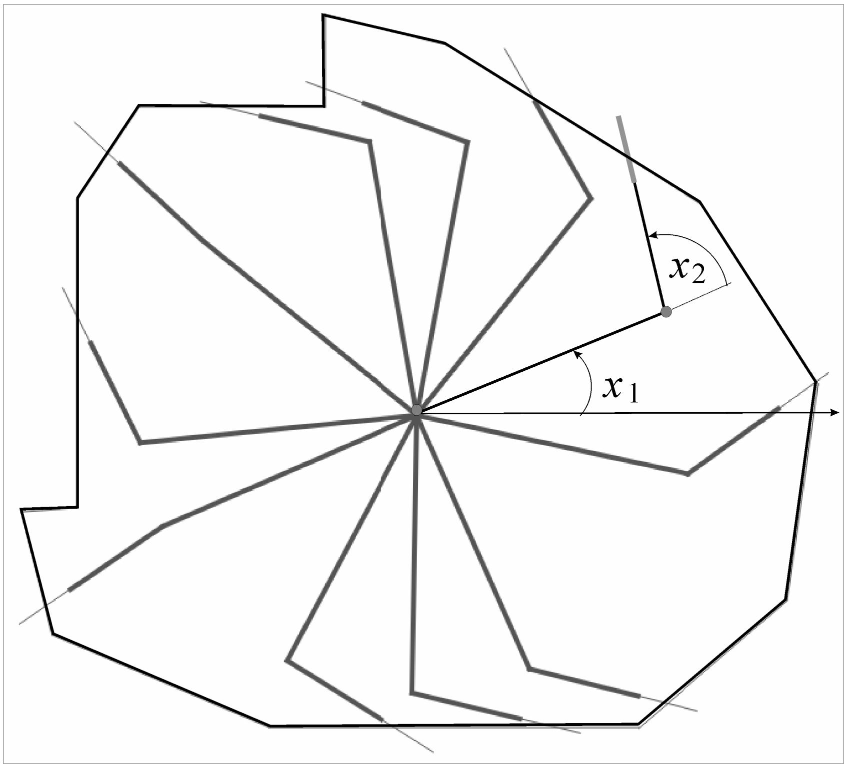
$$\mathbf{f}_\ell(\mathbf{x}) = 4 \begin{pmatrix} \cos x_1 \\ \sin x_1 \end{pmatrix} + \ell \begin{pmatrix} \cos(x_1 + x_2) \\ \sin(x_1 + x_2) \end{pmatrix}.$$

A separator for $\mathbb{M}$ is

$$\mathcal{S}_{\mathbb{M}} = \mathbf{f}_2^{-1}(\mathcal{S}_{\mathbb{Y}}) \cap \mathbf{f}_3^{-1}\left(\overline{\mathcal{S}_{\mathbb{Y}}}\right).$$







x_2
x_1

References

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L. Jaulin and B. Desrochers (2014). Introduction to the Algebra of Separators with Application to Path Planning. Engineering Applications of Artificial Intelligence, **33**, pp. 141-147,