Computing a guaranteed approximation of the viability kernel

SMART 2015

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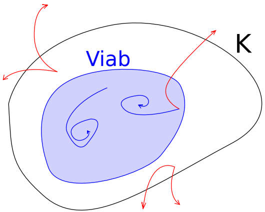
What is viability?

System $\mathcal{S}$ defined by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}),$$
$$\mathbf{f} : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$$

A state $\mathbf{x}$ is viable if at least one evolution of $\mathcal{S}$ from $\mathbf{x}$ can stay indefinitely in a set of constraint $\mathbb{K}$.
The viability kernel of $\mathbb{K}$ under $\mathcal{S}$ noted $Viab_{\mathcal{S}}(\mathbb{K})$ is the set that contains every viable state.



Why viability?

Example: management of renewable resources, economics, robotics,...

Is it possible to avoid the wall?

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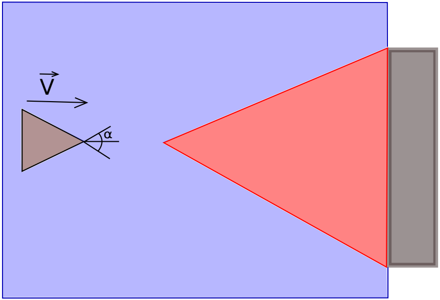
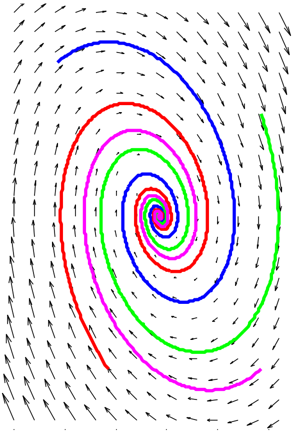


Plan

1. Attraction domains
2. Polygon expansion technique
3. Capture basin
4. Examples
   - Car on the hill
   - Double integrator
5. Conclusion

Attraction domain of a system

Attraction domains of $\mathcal{S}$ are interesting for viability, if they are located in $\mathbb{K}$.

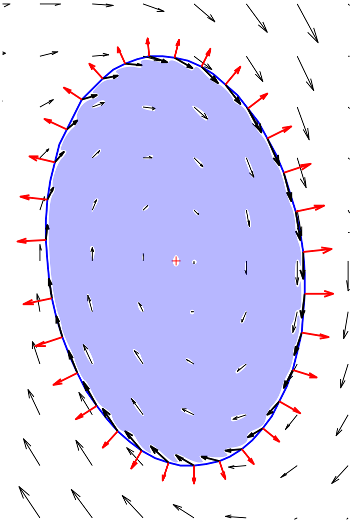


Theorem on viability

Theorem

We consider a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbb{U}$ *the set of possible control and* $\mathbb{K}$ *a closed subset of* $\mathbb{R}^n$.
Let $L \in \mathcal{C}^1(\mathbb{K}, \mathbb{R})$, *and* $\underline{\mathbb{B}_L(r)} = \{\mathbf{x} \in \mathbb{R}^n | L(\mathbf{x}) \leq r\}$, *with* $r \in \mathbb{R}^+$.
If $\mathbb{B}_L(r) \subseteq \mathbb{K}$ *and* $\forall \mathbf{x} \in \overline{\mathbb{B}_L(r)}, \exists \mathbf{u} \in \mathbb{U}$ *such as* $\langle \mathbf{f}(\mathbf{x}, \mathbf{u}), \nabla L(x) \rangle \leq 0$, *then* $\mathbb{B}_L(r) \subseteq Viab_S(\mathbb{K})$.

Illustration of the theorem



Lyapunov function

Definition

A function $L : \mathbb{R}^n \to \mathbb{R}$ is said to be of Lyapunov for the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if:

1. $V(\mathbf{0}) = 0$.
2. $\forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, V(\mathbf{x}) > 0$.
3. $\forall \mathbf{x} \in \mathbb{R}^n, \langle \mathbf{f}(x), \nabla V(\mathbf{x}) \rangle \leq 0$.

How to find a lyapunov function

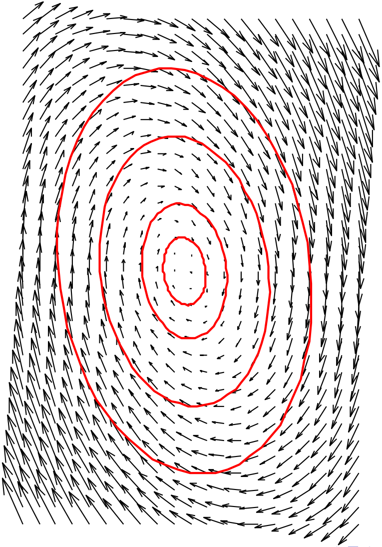
We choose a particular control $\mathbf{u} \in \mathbb{U}$. $\mathcal{S}_{\mathbf{u}}$: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ is an autonomous system.

$\mathbf{x}^*$ is an equilibrium point of $\mathcal{S}_{\mathbf{u}} \iff \mathbf{f}(\mathbf{x}^*, \mathbf{u}) = \mathbf{0}$.

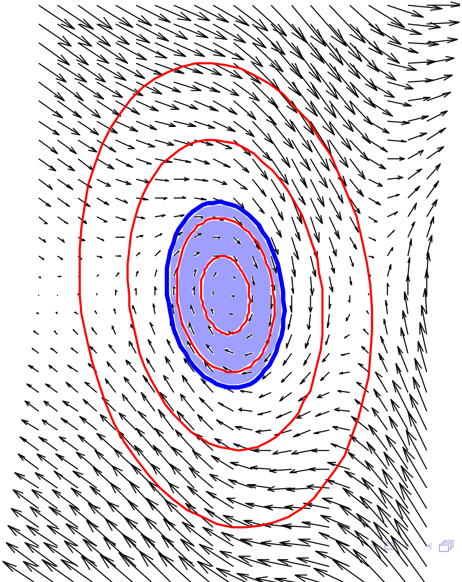
- Linearize $\mathcal{S}_{\mathbf{u}}$ around $\mathbf{x}^*$, we get $\mathcal{S}_{\mathbf{u}}^{\mathbf{x}^*}$ defined by $\dot{\tilde{x}} = A\tilde{\mathbf{x}}, \tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$.
- Solve $A^T W + WA = -I$, where W is the unknown amount.
- Check whether W is positive definite.
- If W is positive definite, then $\frac{1}{2}\tilde{\mathbf{x}}^T W \tilde{\mathbf{x}}$ is a Lyapunov function for the linear system, and $\mathbf{x}^*$ is stable.

If we do not find a Lyapunov function for $\mathcal{S}_{\mathbf{u}}^{\mathbf{x}^*}$, we compute the linear system $\mathcal{S}_{ctrl}$ for which $\mathbf{x}^*$ is a stable equilibrium point.

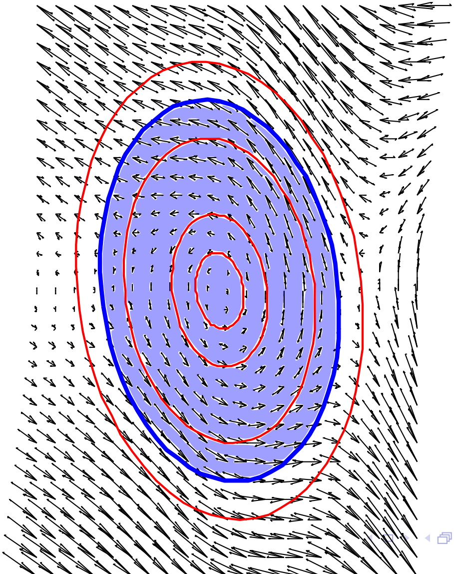
Lyapunov function and linearized system $\mathcal{S}_{\mathbf{u}}^{\mathbf{x}^*}$



Lyapunov function and autonomous system $\mathcal{S}_{\mathbf{u}}$



Lyapunov function and system $\mathcal{S}$



Viable set characterization algorithm

- Choose a control $\mathbf{u} \in \mathbb{U}$.
- Find an equilibrium point $\mathbf{x}^* \in \mathbb{K}$.
- Linearize $\mathcal{S}_{\mathbf{u}}$ around $\mathbf{x}^*$.
- Try to compute a Lyapunov function of $\mathcal{S}_{\mathbf{u}}^{\mathbf{x}^*}$.
- If no function found, compute $\mathcal{S}_{ctrl}$.
- Try to compute a Lyapunov function of $\mathcal{S}_{ctrl}$.
- Find $r \in \mathbb{R}^+$ such as conditions of the theorem are met.

Plan

Theorem

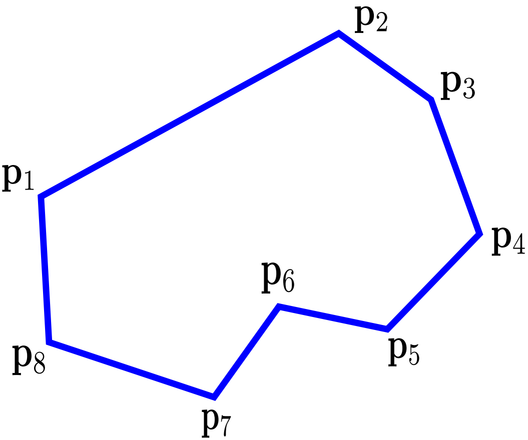
Theorem

Let $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ *be a polygon included in* $\mathbb{K}$*. We suppose* $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ *sorted in clockwise order.*
If

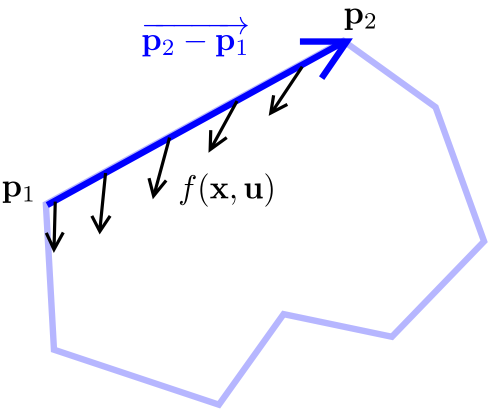
$$\forall i, \forall x \in segment\ [\mathbf{p}_i, \mathbf{p}_{i+1}], \exists \mathbf{u} \in \mathbb{U},$$
$$det(\mathbf{p}_{i+1} - \mathbf{p}_i, \mathbf{f}(\mathbf{x}, \mathbf{u})) \leq 0.$$

Then $P \subseteq Viab_S(\mathbb{K})$

Theorem illustration



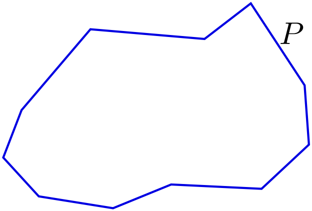
Theorem illustration



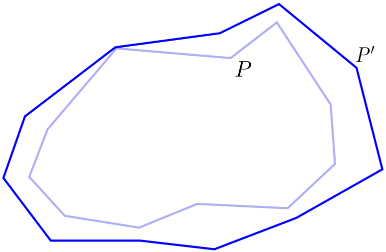
Polygon expansion algorithm

1. Find a polygon $P \subseteq Viab_{\mathcal{S}}(\mathbb{K})$.
2. Compute a larger polygon P'.
3. If $P' \subseteq Viab_{\mathcal{S}}(\mathbb{K})$, $P = P'$, go to 1.
4. Else compute another polygon P', go to 3.

Polygon expansion algorithm



Polygon expansion algorithm



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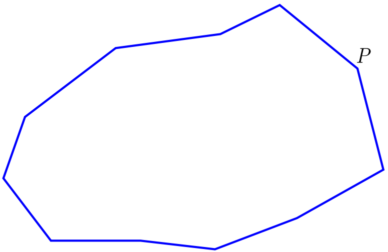

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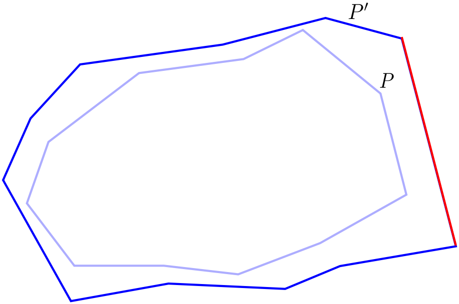
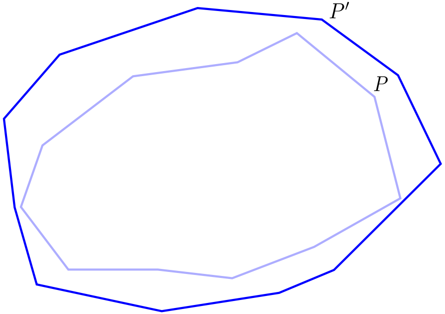
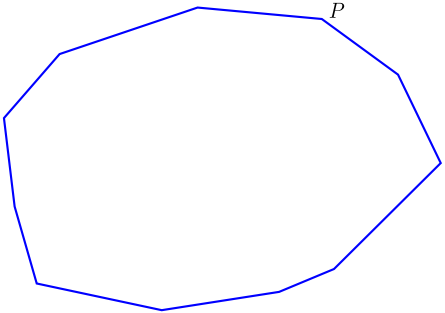


Polygon expansion algorithm



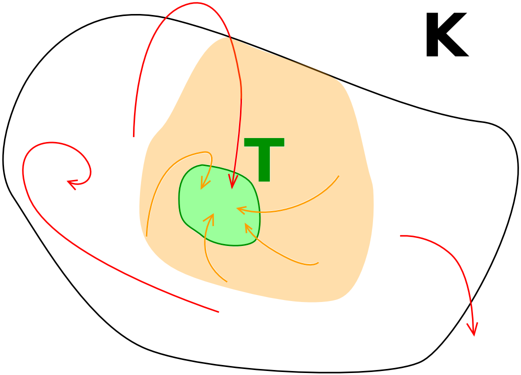
Polygon expansion algorithm



Plan

Capture basin problem

The capture basin of a set $\mathbb{T} \subset \mathbb{K}$ viable in $\mathbb{K}$ noted $Capt_S(\mathbb{K}, \mathbb{T})$ is composed of every states $\mathbf{x}$ such as $\mathcal{S}$ can reach $\mathbb{T}$ from $\mathbf{x}$ in a finite time without leaving $\mathbb{K}$.



Theorem on the viability of a capture basin

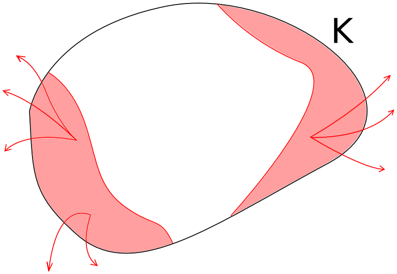
Theorem

Let $\mathcal{S}$ a dynamical system, $\mathbb{K}$ a closed subset of the state space of $\mathcal{S}$ and $\mathbb{T} \subset \mathbb{K}$.
If $\mathbb{T}$ is viable in $\mathbb{K}$,
then $Capt_{\mathcal{S}}(\mathbb{K}, \mathbb{T})$ is viable in $\mathbb{K}$.

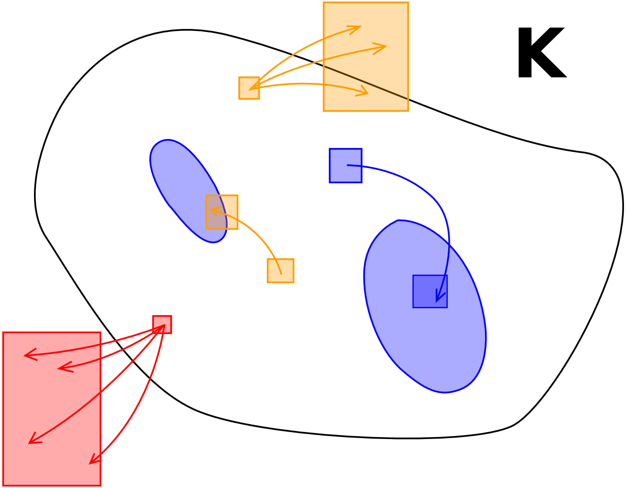
The set $\mathbb{V}_{in} = \mathbb{T} \cup Capt_{\mathcal{S}}(\mathbb{K}, \mathbb{T})$ is an inner approximation of $Viab_{\mathcal{S}}(\mathbb{K})$.

Over approximation of the viability kernel

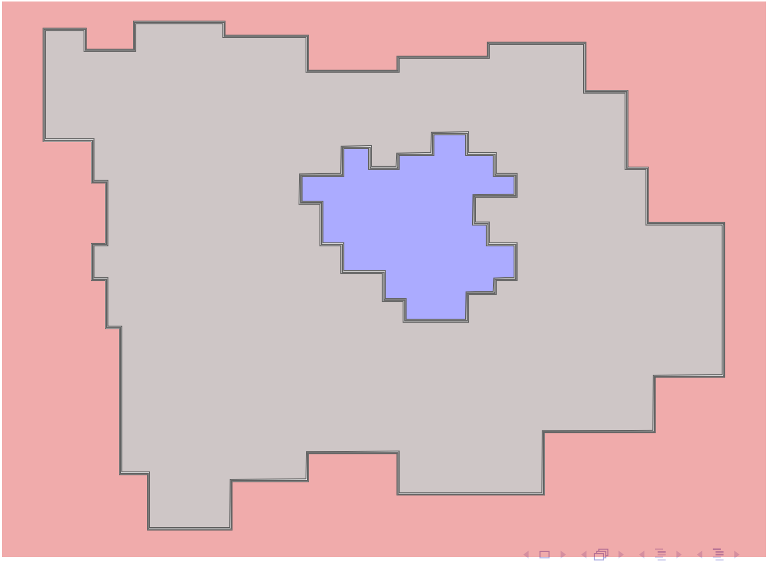
- We try to find an over approximation of $Viab_{\mathcal{S}}(\mathbb{K})$ to get an enclosure of $Viab_{\mathcal{S}}(\mathbb{K})$.
- If $\forall \mathbf{u} \in \mathbb{U}$, $\mathcal{S}$ cannot stay in $\mathbb{K}$ from a state $\mathbf{x} \in \mathbb{K}$, then $\mathbf{x} \notin Viab_{\mathcal{S}}(\mathbb{K})$.



Guaranteed integration of a box [Chaputot, 2015]



Under approximation algorithm



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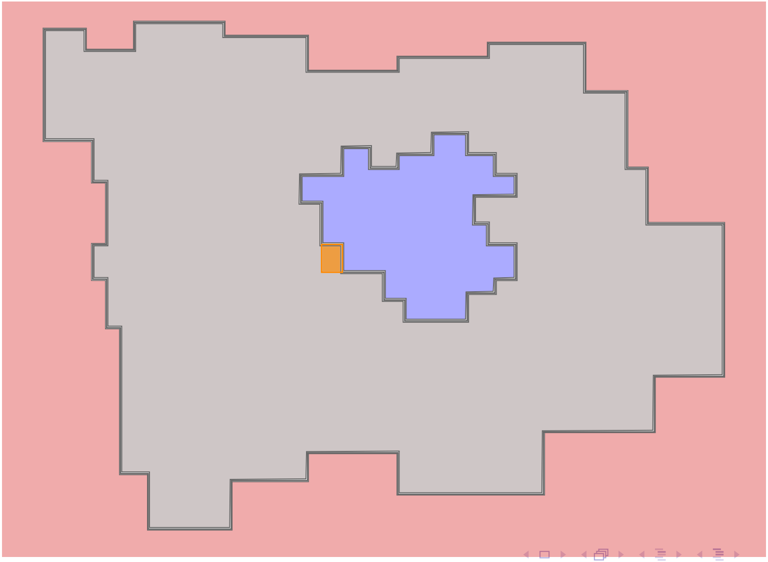

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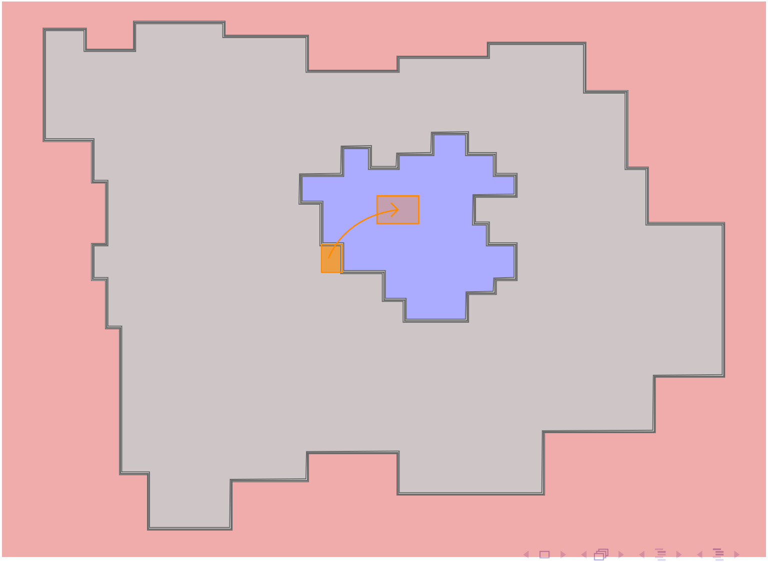

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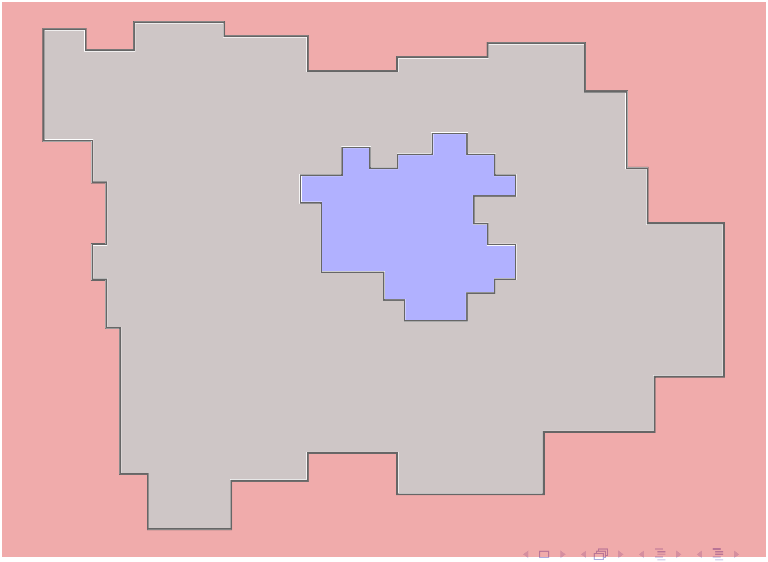

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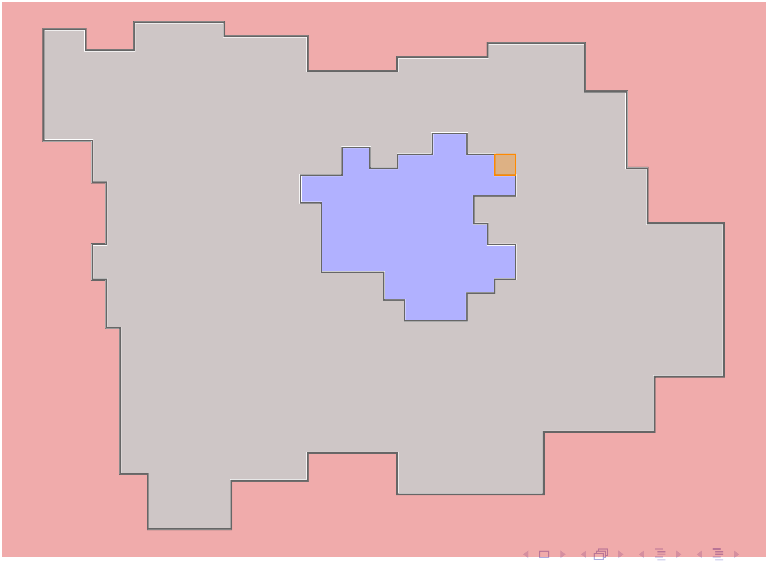

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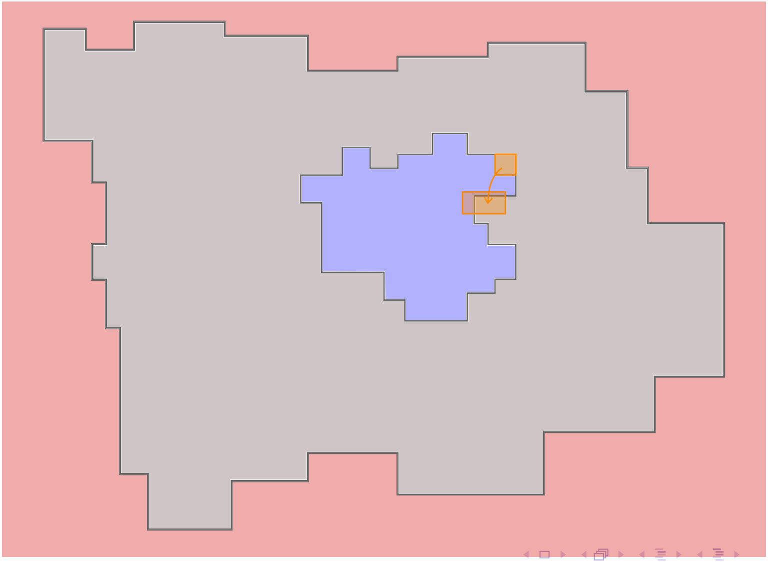

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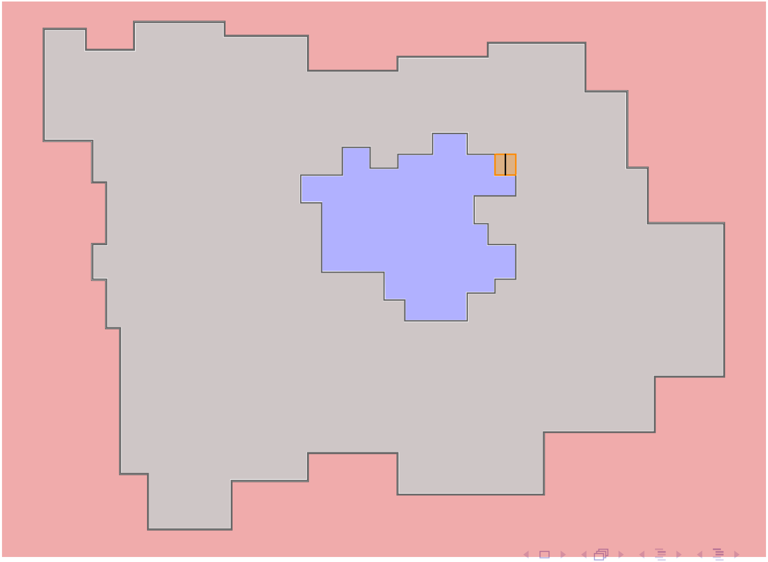

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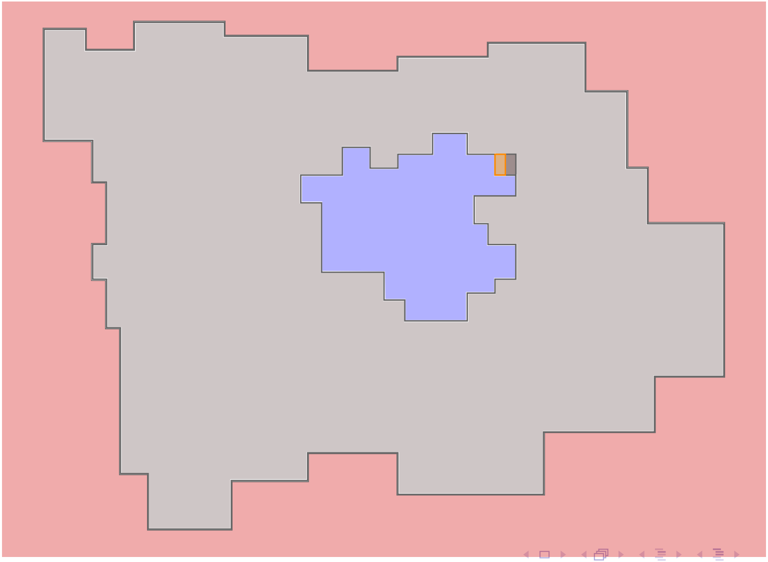
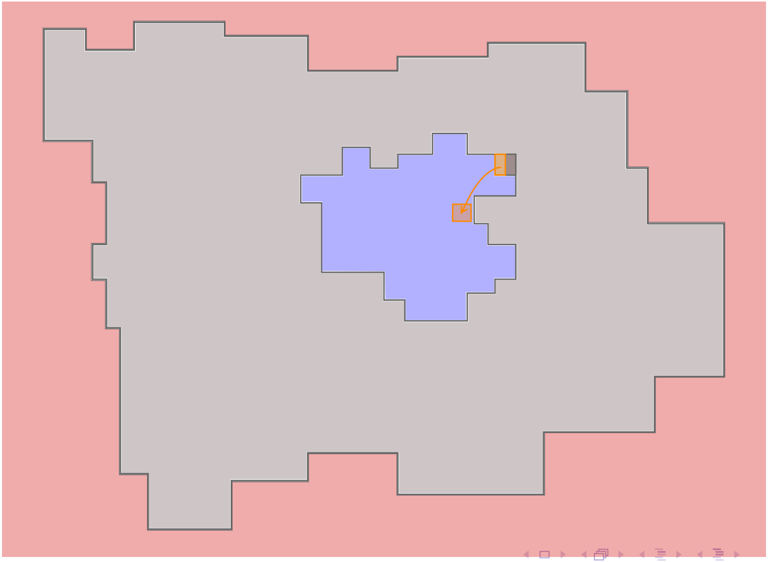
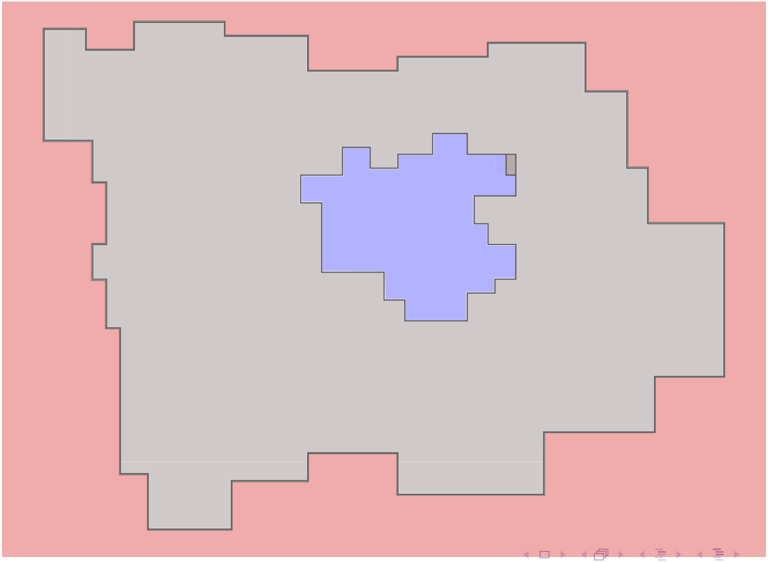


Under approximation algorithm



Under approximation algorithm



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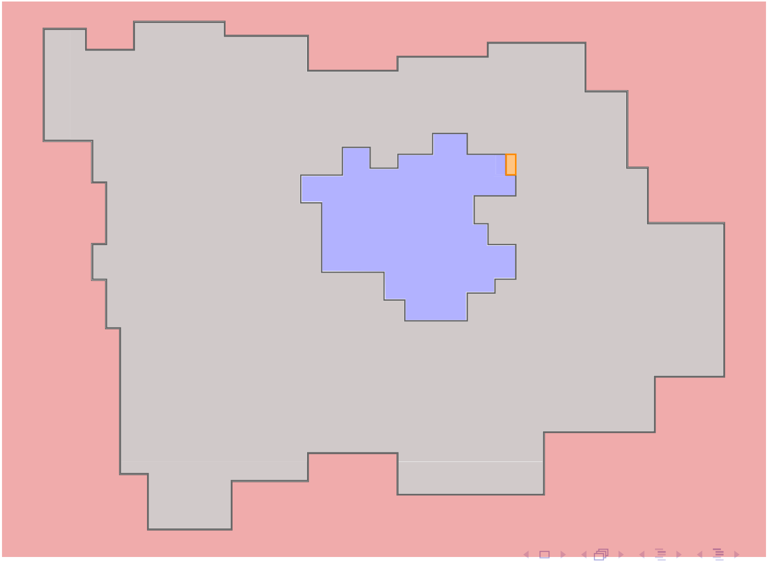

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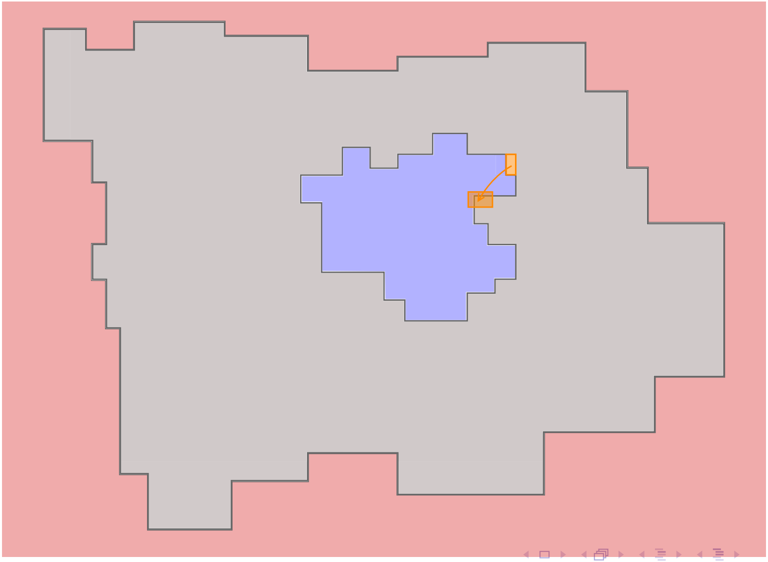

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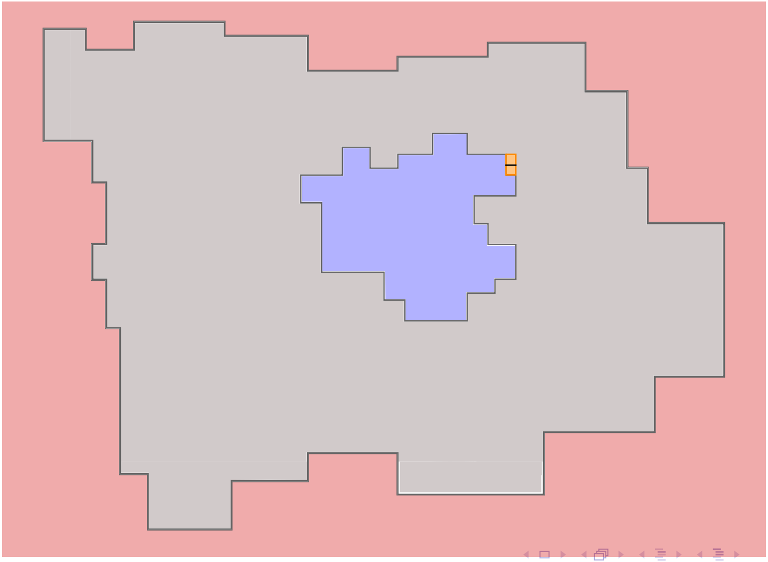

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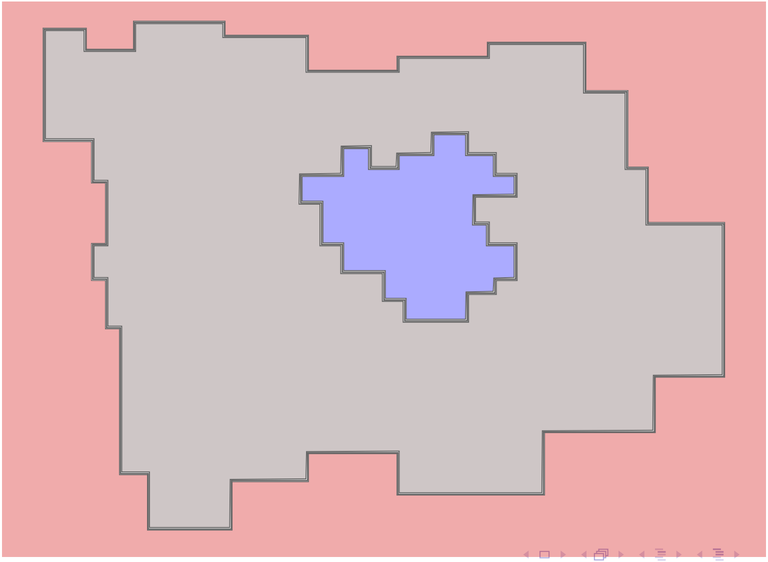

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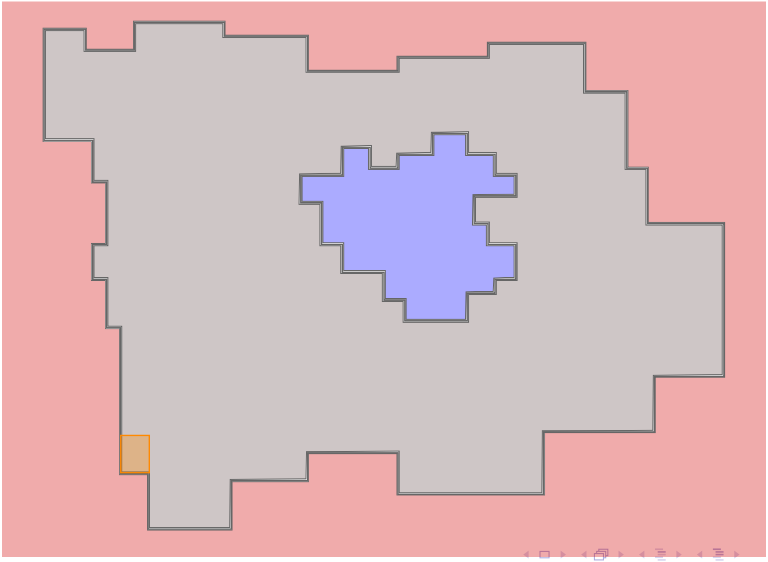

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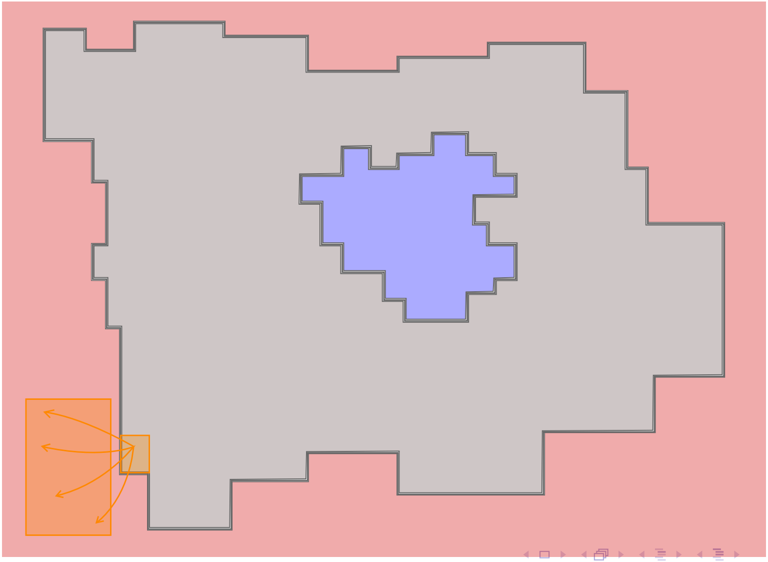

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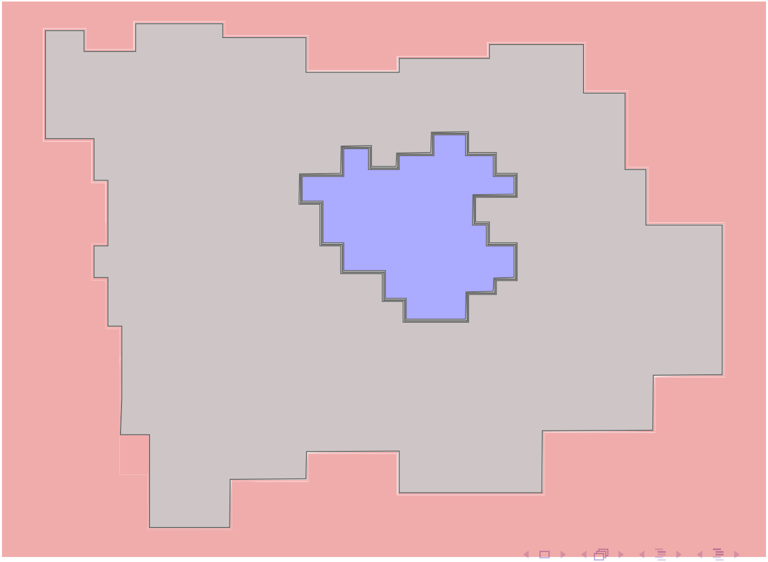

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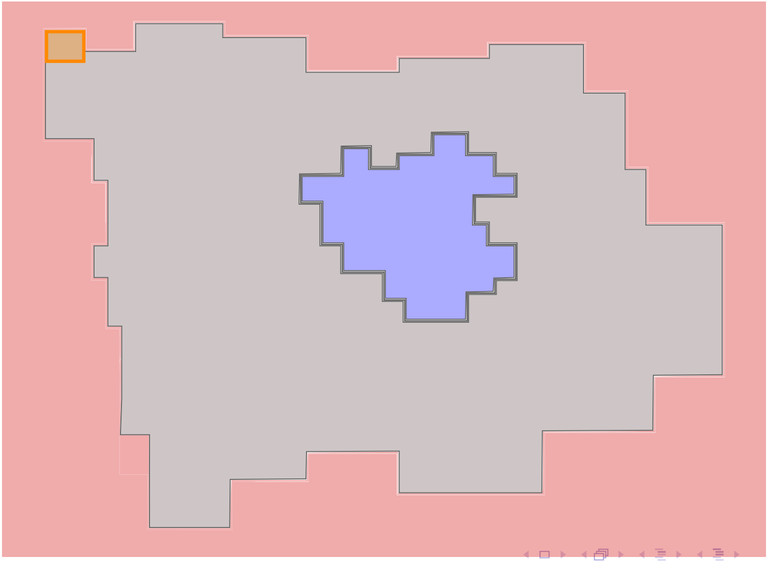
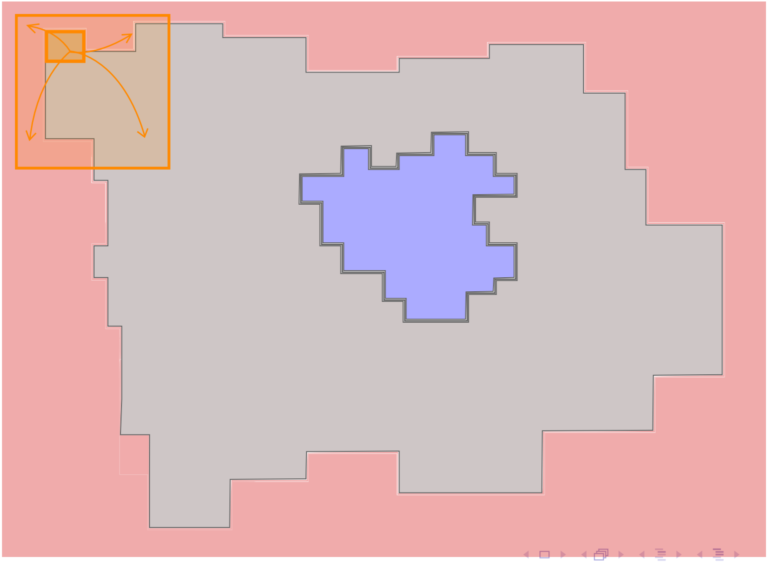


Over approximation algorithm



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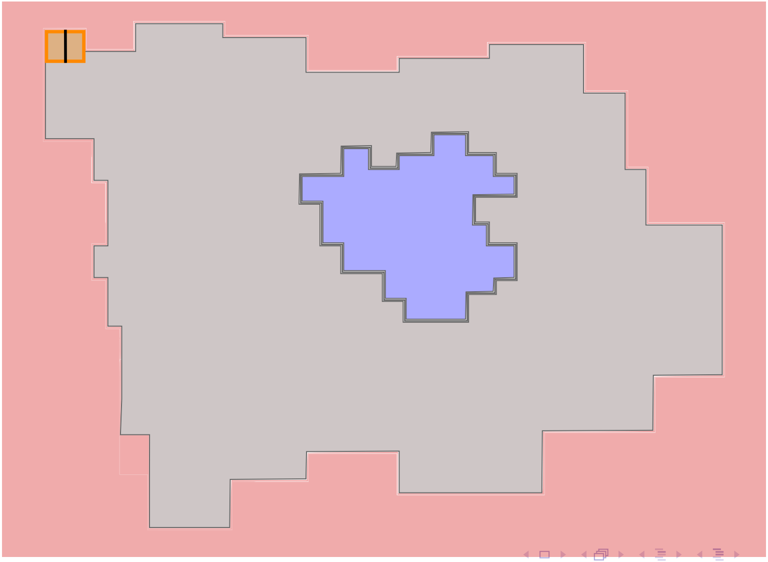


Plan

Plan

Car on the hill problem

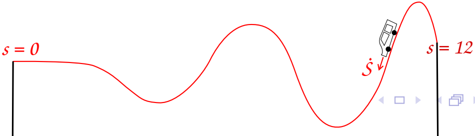
- The landscape is represented by the parametric function

$$g : s \to \frac{\frac{-1.1}{1.2}cos(1.2s) + \frac{1.2}{1.1}cos(1.1s)}{2}$$

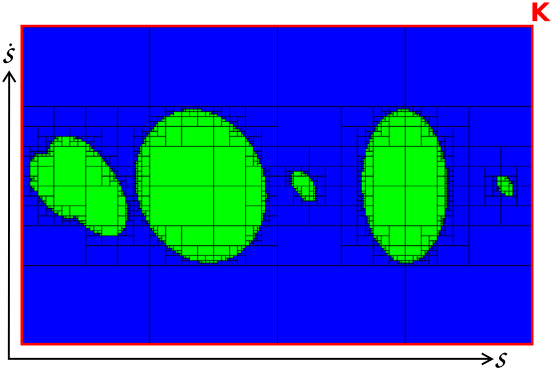
- State vector: $\mathbf{x} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- Evolution function:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -9.81 sin(\frac{dg}{dx_1}(x_1)) - 0.7x_2 + u \end{cases}$$

  $u \in [-2, 2]$
- The car must stay on the landscape, i.e $s \in [0, 12]$

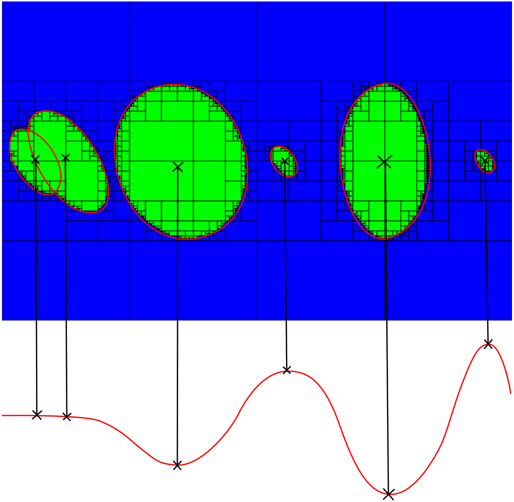


Result of viable set characterization algorithm



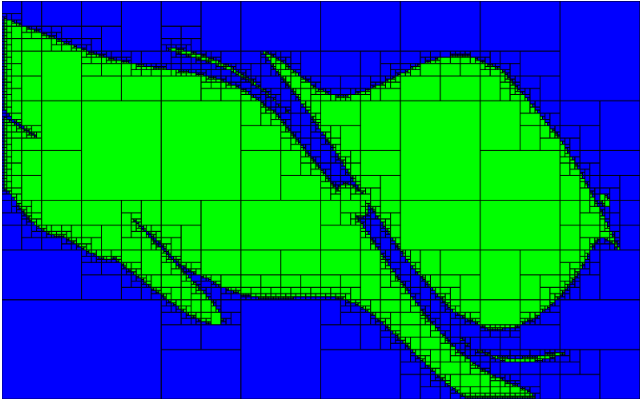
Viable sets computed with $u = 0$. Computation time: 25 sec.

Result explained



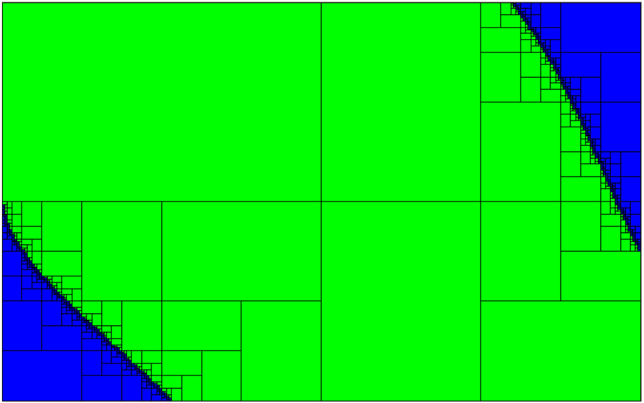
Result of polygon expansion algorithm

Polygons are initialized with viable sets computed previously.



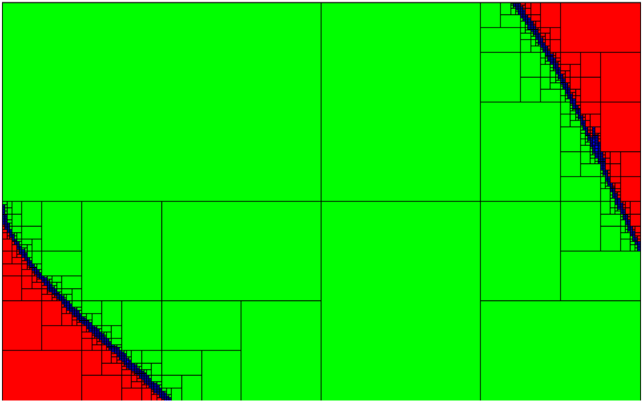
Computation time: 45 sec.

Result of inner approximation algorithm



Computation time ≈ 60 minutes

Result of over approximation algorithm



Computation time ≈ 30 minutes

Plan

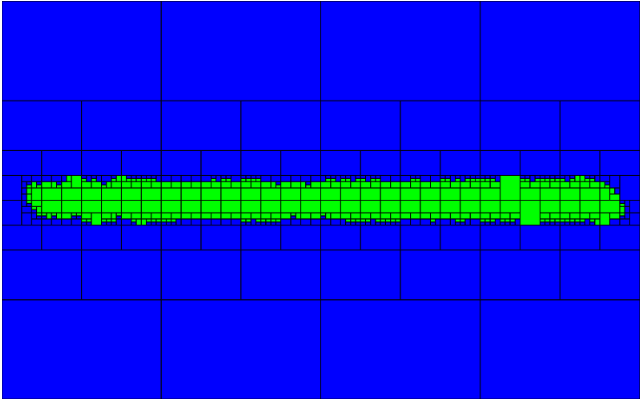
Double integrator equations

Evolution function: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \quad u \in [-1, 1]$

Constraints:

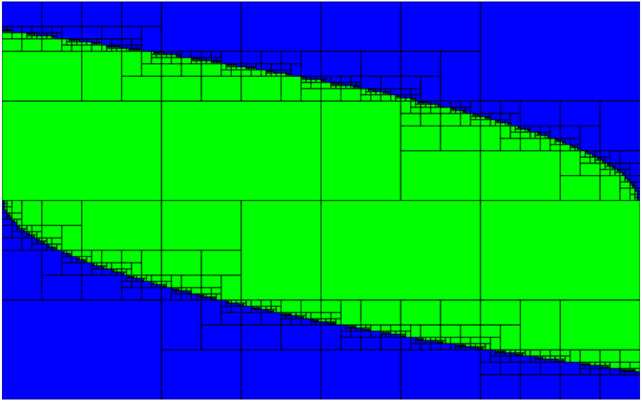
- $x_1 \in [-5, 5]$
- $x_2 \in [-5, 5]$

Results of viable set characterization algorithm



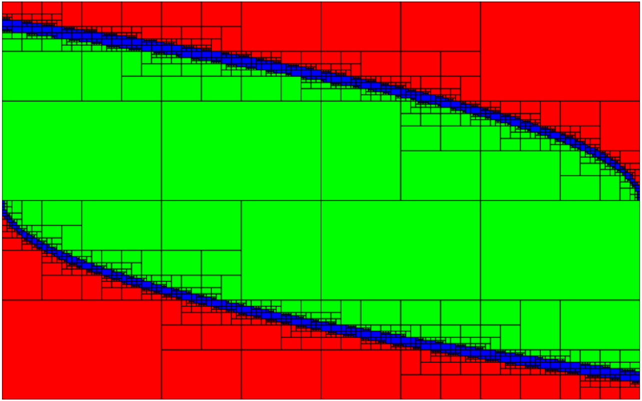
Viable sets computed with $u = 0$. Computation time: 40 sec.

Result of inner approximation algorithm



Computation time: 5 minutes.

Result of over approximation algorithm



Computation time: 7 minutes.

Plan

Conclusion

- We are able to deal with many viability problems in a guaranteed way.
- The system must have at least one equilibrium point.
- We can deal with 2D problems, but inner and over approximation algorithms are not efficient for higher dimensional problems
- We approached viability problem with new methods based on the study of the frontier of closed sets.