Separator Algebra for State Estimation

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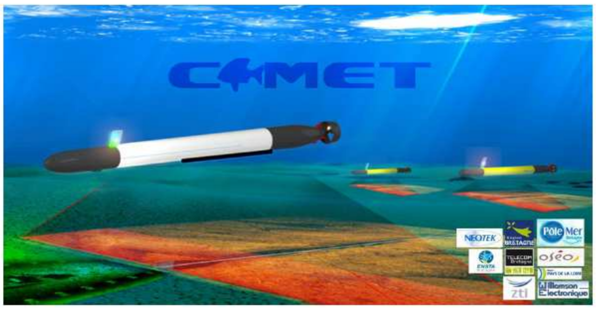
1 Context



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2 Problem

(i) $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), t \in \mathbb{R}$

(ii) $\mathbf{g}(\mathbf{x}(t_k)) \in \mathbb{Y}(k)$, $k \in \mathbb{N}$

Find an inner-outer approximation of the set $\mathbb{X}(t)$ of feasible states

We define by flow map φ_{t_1,t_2}:

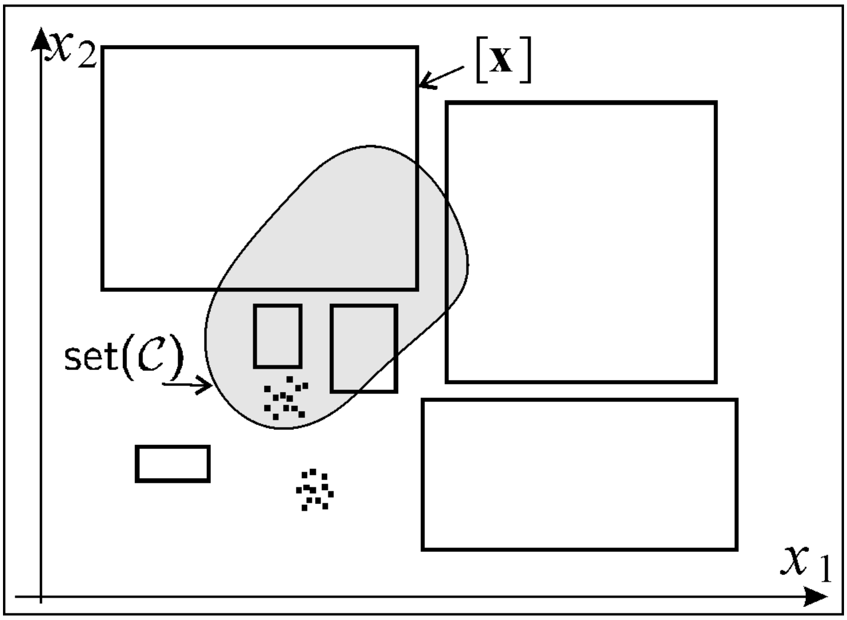
$$\Big(\mathbf{x}(t_1) = \mathbf{x}_1 \text{ and } \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \Rightarrow \mathbf{x}_2 = \boldsymbol{\varphi}_{t_1,t_2}(\mathbf{x}_1)\Big).$$

The set of all *causal feasible states* at time t is

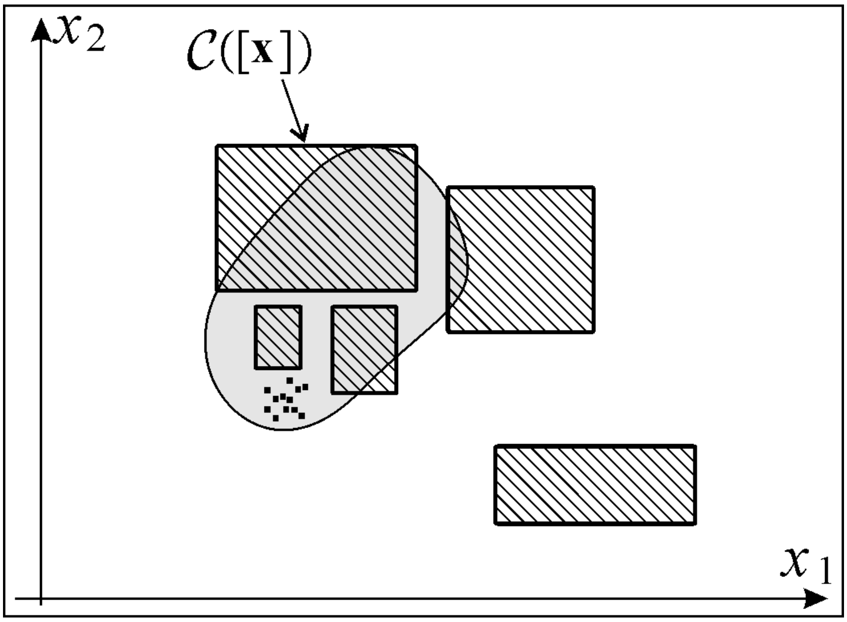
$$\mathbb{X}\left(t\right)=\bigcap_{t_k\leq t}\boldsymbol{\varphi}_{t_k,t}\circ\mathbf{g}^{-1}\left(\mathbb{Y}\left(k\right)\right).$$

3 Contractors

$$\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \qquad \text{(contractance)}$$
$$[\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]) \qquad \text{(monotonicity)}$$



x_2
$[\mathbf{x}]$
set$(\mathcal{C})$
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
x_1

Inclusion

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall\, [\mathbf{x}] \in \mathbb{IR}^n,\, \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]).$$

A set $\mathbb{S}$ is *consistent* with $\mathcal{C}$ (we write $\mathbb{S} \sim \mathcal{C}$) if

$$\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}.$$

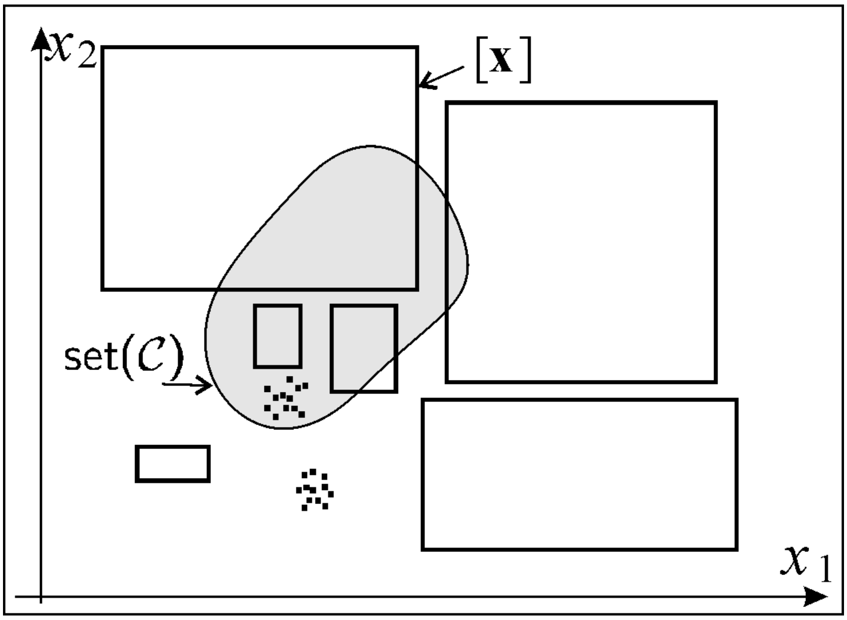
$\mathcal{C}$ is *minimal* if

$$\left.\begin{array}{c} \mathbb{S} \sim \mathcal{C} \\ \mathbb{S} \sim \mathcal{C}_1 \end{array}\right\} \Rightarrow \mathcal{C} \subset \mathcal{C}_1.$$

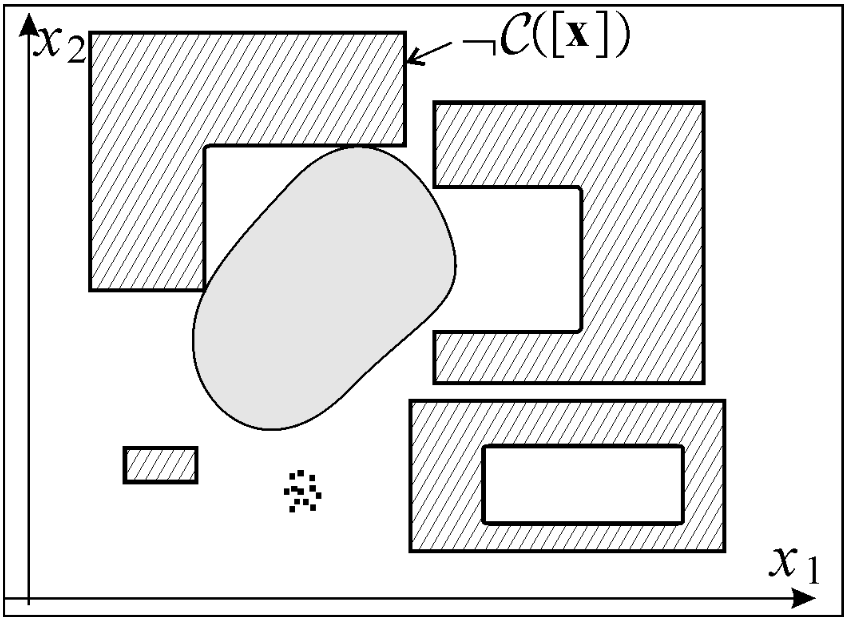
The *negation* $\neg\mathcal{C}$ of $\mathcal{C}$ is defined by

$$\neg\mathcal{C}\left([\mathbf{x}]\right) = \left\{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{x} \notin \mathcal{C}\left([\mathbf{x}]\right)\right\}.$$

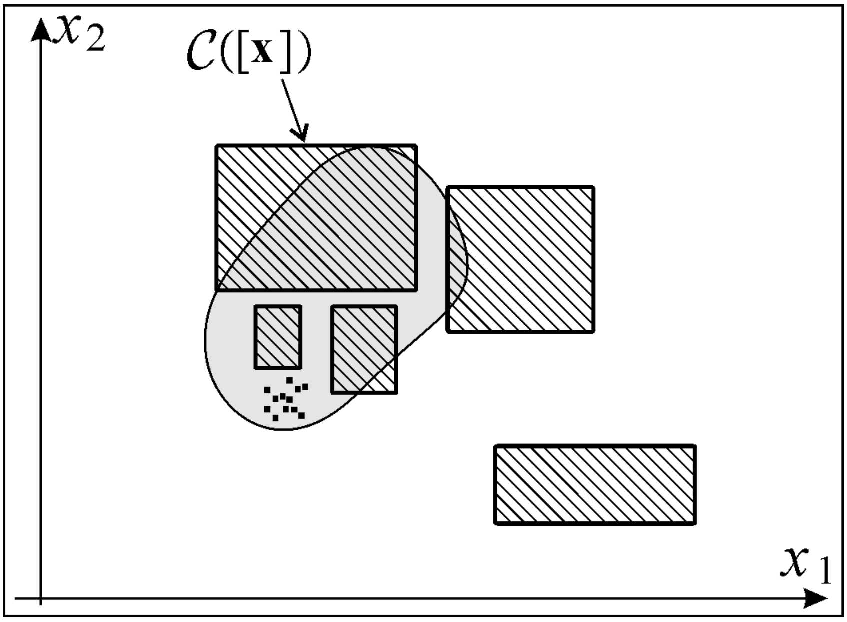
It is not a box in general.



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\neg\mathcal{C}([\mathbf{x}])$
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
x_1

4 Separators

A *separator* $\mathcal{S}$ is pair of contractors $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ such that

$$\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}] \qquad \text{(complementarity).}$$

A set $\mathbb{S}$ is *consistent* with $\mathcal{S}$ (we write $\mathbb{S} \sim \mathcal{S}$), if

$$\mathbb{S} \sim \mathcal{S}^{\text{out}} \text{ and } \overline{\mathbb{S}} \sim \mathcal{S}^{\text{in}}.$$

The *remainder* of $\mathcal{S}$ is

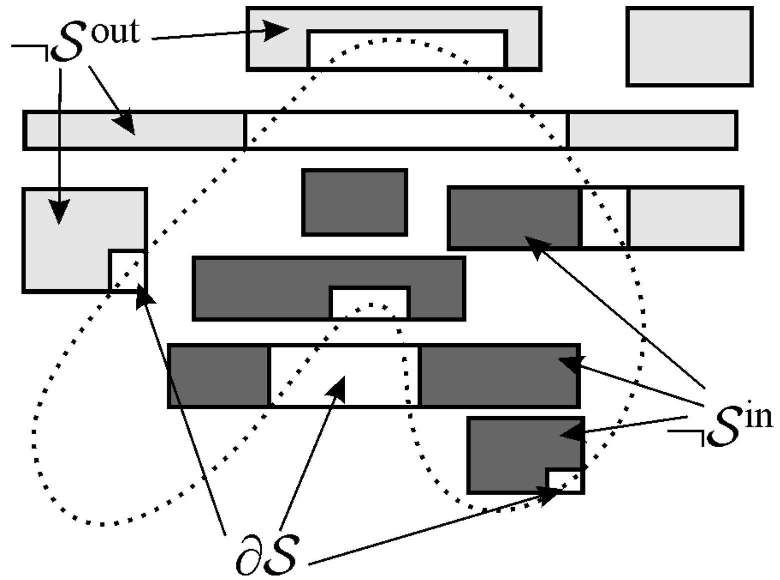
$$\partial\mathcal{S}([\mathbf{x}]) = \mathcal{S}^{\mathsf{in}}([\mathbf{x}]) \cap \mathcal{S}^{\mathsf{out}}([\mathbf{x}]).$$

$\partial\mathcal{S}$ is a contractor, not a separator.

We have

$$\neg\mathcal{S}^{\mathsf{in}}([\mathbf{x}]) \cup \neg\mathcal{S}^{\mathsf{out}}([\mathbf{x}]) \cup \partial\mathcal{S}([\mathbf{x}]) = [\mathbf{x}]\,.$$

Moreover, they do not overlap.



$\neg\mathcal{S}^{\mathsf{in}}([\mathbf{x}])$, $\neg\mathcal{S}^{\mathsf{out}}([\mathbf{x}])$ and $\partial\mathcal{S}([\mathbf{x}])$

Inclusion

$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \mathcal{S}_1^{\text{in}} \subset \mathcal{S}_2^{\text{in}} \text{ and } \mathcal{S}_1^{\text{out}} \subset \mathcal{S}_2^{\text{out}}.$$

Here $\subset$ means *more accurate.*

$\mathcal{S}$ is *minimal* if

$$\mathcal{S}_1 \subset \mathcal{S} \Rightarrow \mathcal{S}_1 = \mathcal{S}.$$

i.e., if $\mathcal{S}^{\text{in}}$ and $\mathcal{S}^{\text{out}}$ are both minimal.

5 Paver

We want to compute $\mathbb{X}^-, \mathbb{X}^+$ such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Algorithm Paver(in: $[\mathbf{x}], \mathcal{S}$; out: $\mathbb{X}^-$, $\mathbb{X}^+$)

1 $\mathcal{L} := \{[\mathbf{x}]\}$;
2 Pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 $\left\{[\mathbf{x}^{\text{in}}], [\mathbf{x}^{\text{out}}]\right\} = \mathcal{S}([\mathbf{x}])$;
4 Store $[\mathbf{x}] \backslash [\mathbf{x}^{\text{in}}]$ into $\mathbb{X}^-$ and also into $\mathbb{X}^+$;
5 $[\mathbf{x}] = [\mathbf{x}^{\text{in}}] \cap [\mathbf{x}^{\text{out}}]$;
6 If $w([\mathbf{x}]) < \varepsilon$, then store $[\mathbf{x}]$ in $\mathbb{X}^+$,
7 Else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$ the two childs
8 If $\mathcal{L} \neq \emptyset$, go to 2.

6 Algebra

Contractor algebra does not allow decreasing operations, i.e.,

$$\forall i, \mathcal{C}_i \subset \mathcal{C}_i^{'} \Rightarrow \mathcal{E}\left(\mathcal{C}_1, \mathcal{C}_2, \dots\right) \subset \mathcal{E}\left(\mathcal{C}_1^{'}, \mathcal{C}_2^{'}, \dots\right).$$

The complementary $\overline{\mathcal{C}}$ of a contractor $\mathcal{C}$, the restriction $\mathcal{C}_1 \backslash \mathcal{C}_2$, etc. cannot be defined.

Separators extend the operations allowed for contractors to non monotonic expressions.

The *complement* of $\mathcal{S} = \left\{\mathcal{S}^{\mathsf{in}}, \mathcal{S}^{\mathsf{out}}\right\}$ is

$$\overline{\mathcal{S}} = \left\{\mathcal{S}^{\mathsf{out}}, \mathcal{S}^{\mathsf{in}}\right\}.$$

If $\mathcal{S}_i = \left\{\mathcal{S}_i^{\text{in}}, \mathcal{S}_i^{\text{out}}\right\}, i \geq 1$, are separators, we define

$$\begin{array}{rcll}
\mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\right\} & \text{(intersection)} \\
\mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\right\} & \text{(union)} \\
\overset{\{q\}}{\bigcap} \mathcal{S}_i & = & \left\{ \overset{\{m-q-1\}}{\bigcap} \mathcal{S}_i^{\text{in}}, \overset{\{q\}}{\bigcap} \mathcal{S}_i^{\text{out}} \right\} & \text{(relaxed intersection)} \\
\mathcal{S}_1 \backslash \mathcal{S}_2 & = & \mathcal{S}_1 \cap \overline{\mathcal{S}_2}. & \text{(difference)}
\end{array}$$

Theorem. If $\mathbb{S}_i$ are subsets of $\mathbb{R}^n$, we have

$$\begin{array}{rccc}
\text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathcal{S}_1 \cap \mathcal{S}_2 \\
\text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathcal{S}_1 \cup \mathcal{S}_2 \\
\text{(iii)} & \overline{\mathbb{S}}_i & \sim & \overline{\mathcal{S}_i} \\
\text{(iv)} & \mathbb{S}_i & \sim & \mathcal{S}_i^k,\ k \geq 0 \\
\text{(v)} & \bigcap\limits^{\{q\}} \mathbb{S}_i & \sim & \bigcap\limits^{\{q\}} \mathcal{S}_i \\
\text{(vi)} & \mathbb{S}_1 \backslash \mathbb{S}_2 & \sim & \mathcal{S}_1 \backslash \mathcal{S}_2.
\end{array}$$

7 Transformation of separators

A transformation is an invertible function $\mathbf{f}$ such as an inclusion function is known for both $\mathbf{f}$ and $\mathbf{f}^{-1}$.

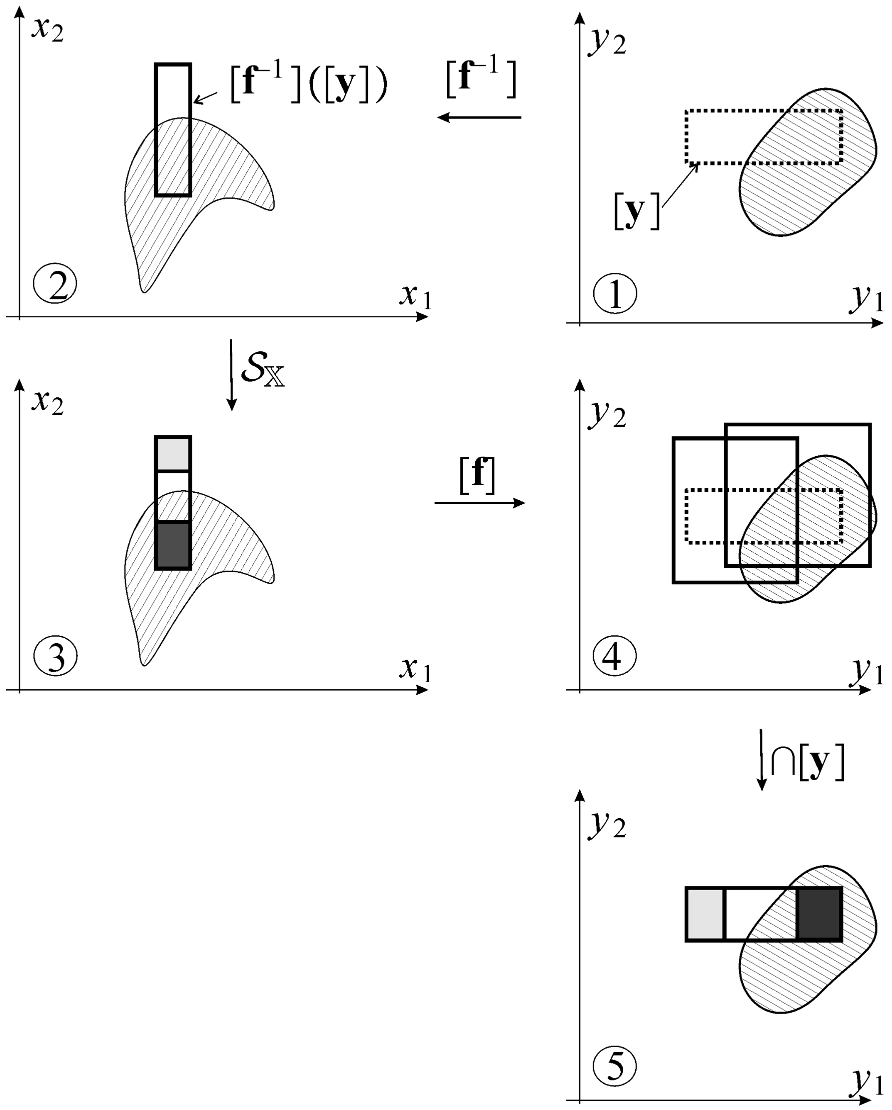
The set of transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a group with respect to the composition $\circ$. Symmetries, translations, homotheties, rotations, ... are linear transformations.

The flow $\boldsymbol{\varphi}_{t_1,t_2}$ is a transformation. Indeed:

$$\boldsymbol{\varphi}_{t_2,t_1}^{-1} = \boldsymbol{\varphi}_{t_1,t_2}.$$

Theorem. Consider a set $\mathbb{X}$ and a transformation $\mathbf{f}$. If $\mathcal{S}_{\mathbb{X}}$ is a separator for $\mathbb{X}$ then a separator $\mathcal{S}_{\mathbb{Y}}$ for $\mathbb{Y} = \mathbf{f}(\mathbb{X})$ is

$$[\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}} \circ \left[\mathbf{f}^{-1}\right] \cap \mathsf{Id}.$$


x_2
$[\mathbf{f}^{-1}]([\mathbf{y}])$
$[\mathbf{f}^{-1}]$
y_2
$[\mathbf{y}]$
2
x_1
1
y_1
$\mathcal{S}_{\mathbb{X}}$
x_2
$[\mathbf{f}]$
y_2
3
x_1
4
y_1
$\cap[\mathbf{y}]$
y_2
5
y_1

Example. Consider the constraint

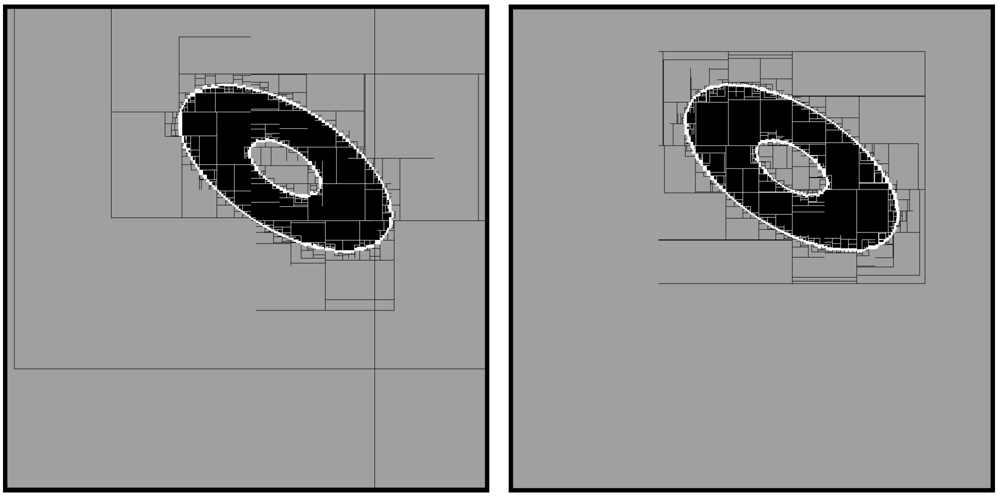
$$\left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 - 1 \\ y_2 - 2 \end{pmatrix} \right\| \in [1, 3] \,.$$

Define

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

The constraint can be rewritten as

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \,, \text{ and } \|\mathbf{x}\| \in [1, 3] \,.$$



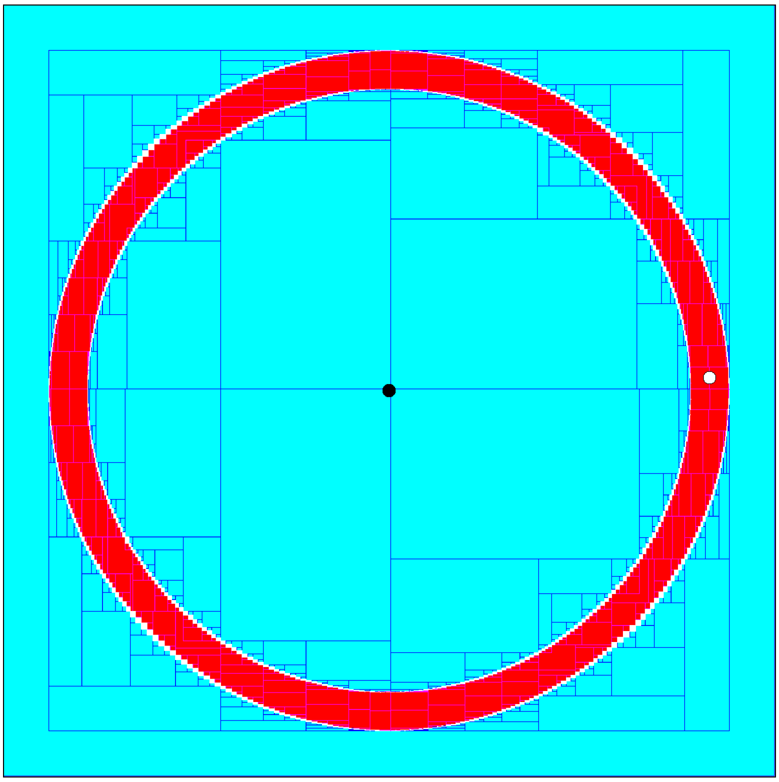
8 State estimation

If $\mathcal{S}_{\mathbb{Y}(k)}$ are separators for $\mathbb{Y}(k)$, then a separator for the set $\mathbb{X}(t)$ is

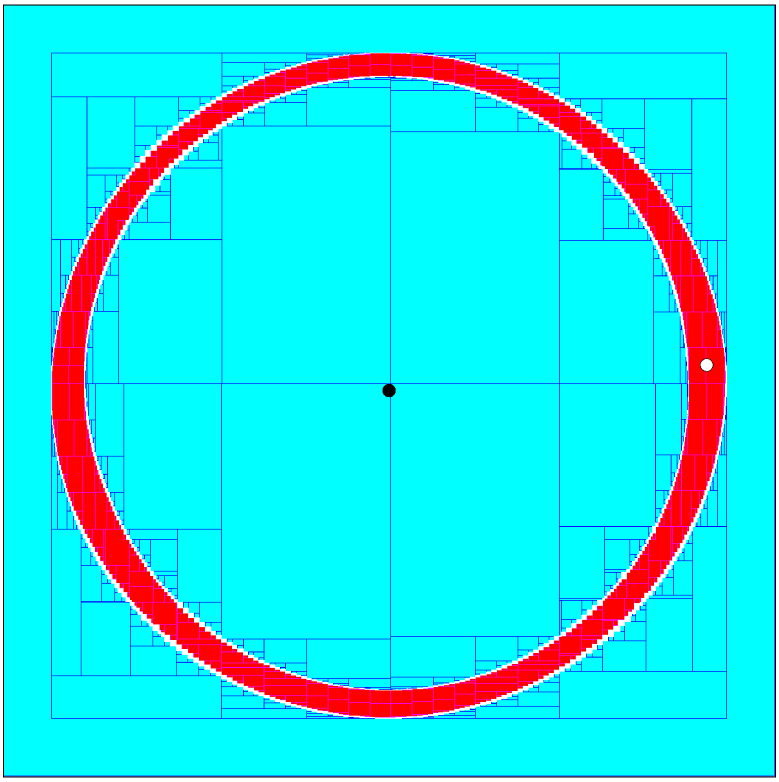
$$\mathcal{S}_{\mathbb{X}(t)} = \bigcap_{t_k \leq t} \boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right).$$

Example

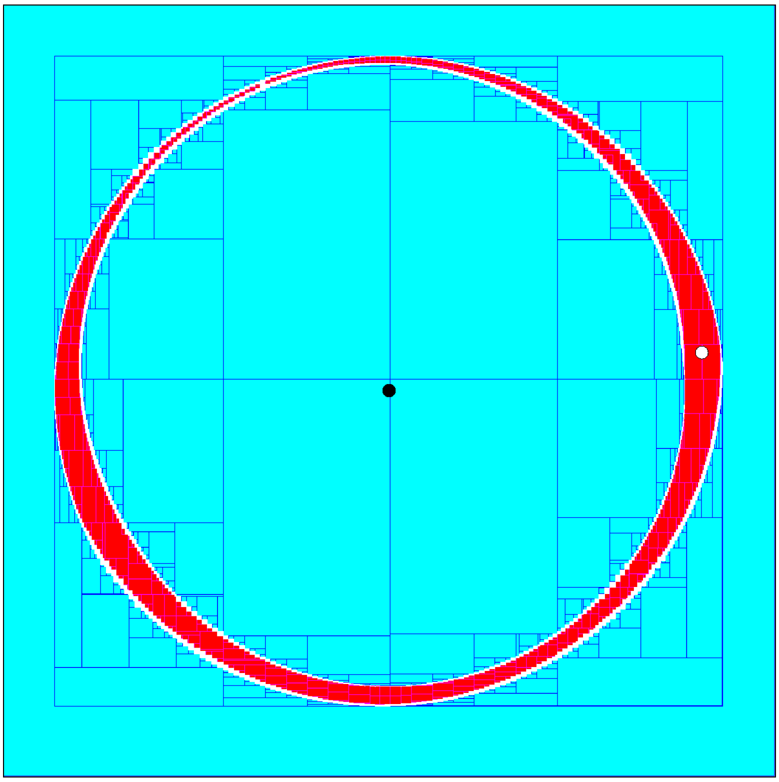
$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{pmatrix} v(t)\cos\theta(t) \\ v(t)\sin\theta(t) \end{pmatrix} \\ \|\mathbf{x}(t_k)\| \in y(t_k) + [-0.3, 0.3],\ t_k = 0.1 \cdot k,\ \ k \in \mathbb{N} \end{cases}$$



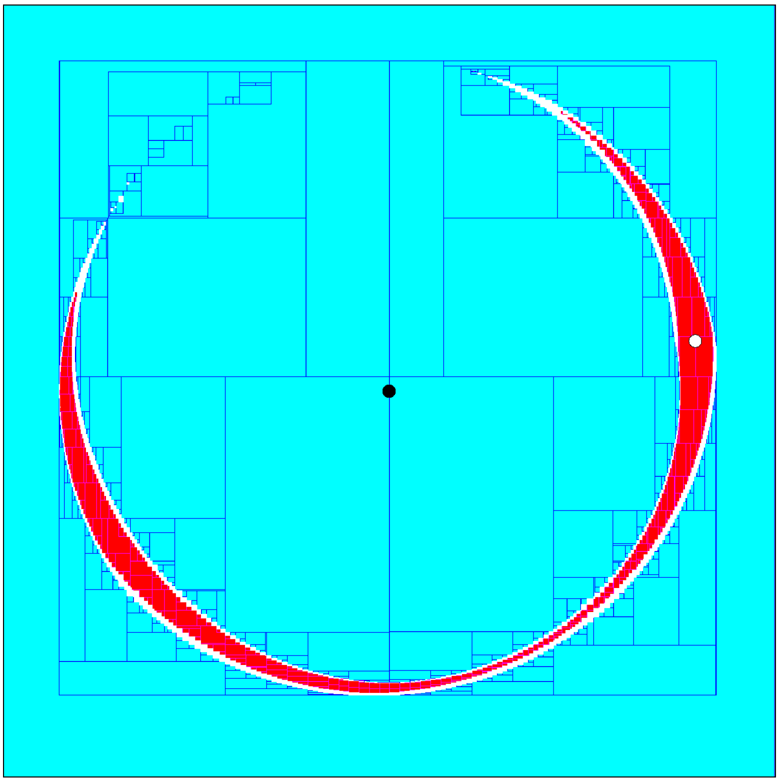
$t = 0$



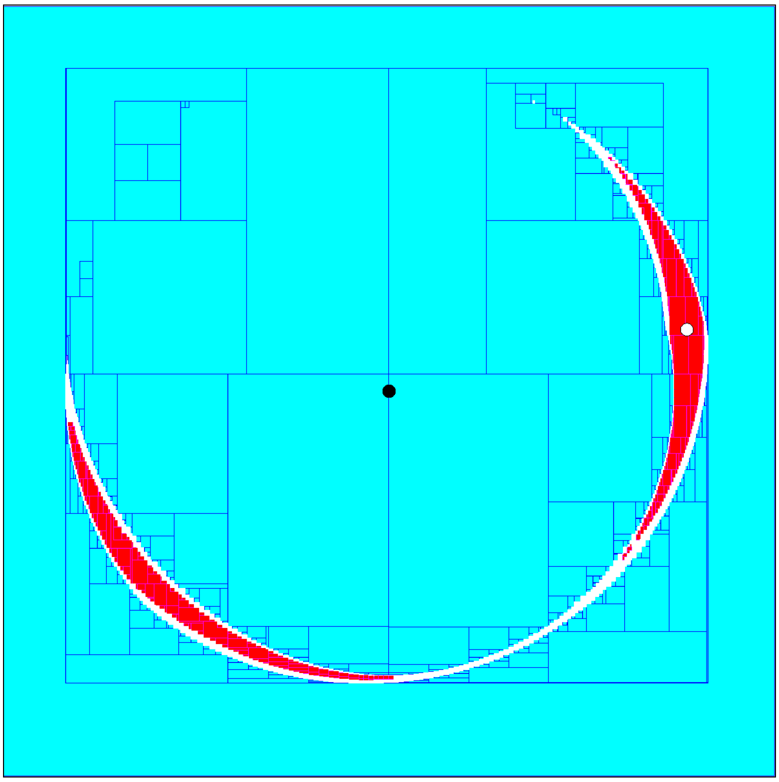
$t = 0.1$



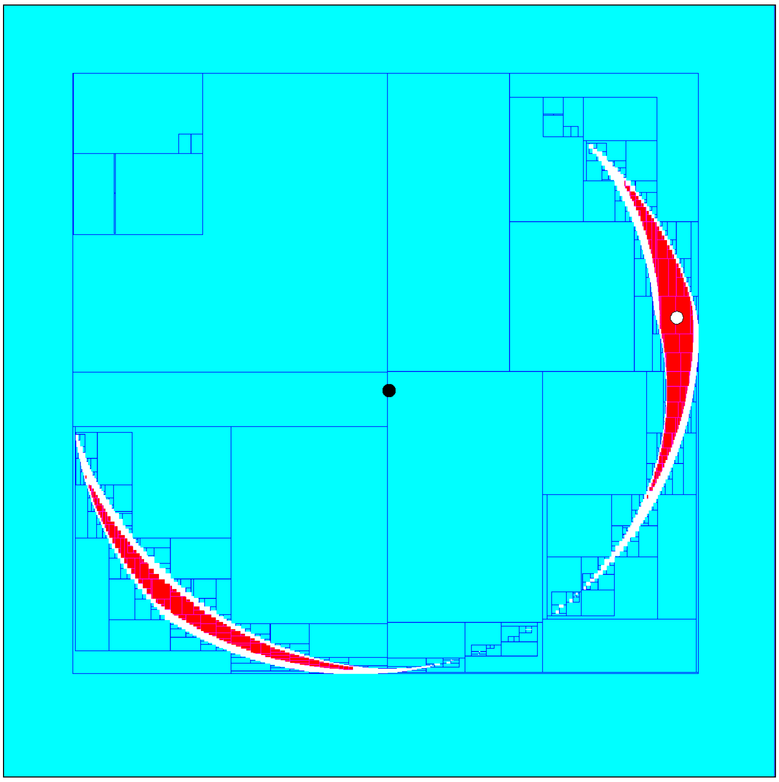
$t = 0.2$



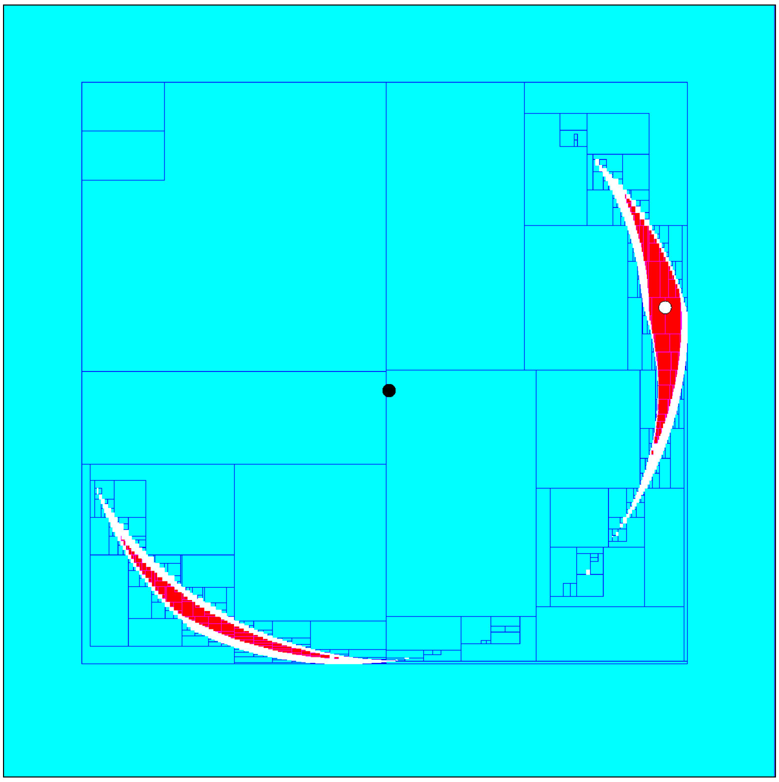
$t = 0.3$



$t = 0.4$



$t = 0.5$

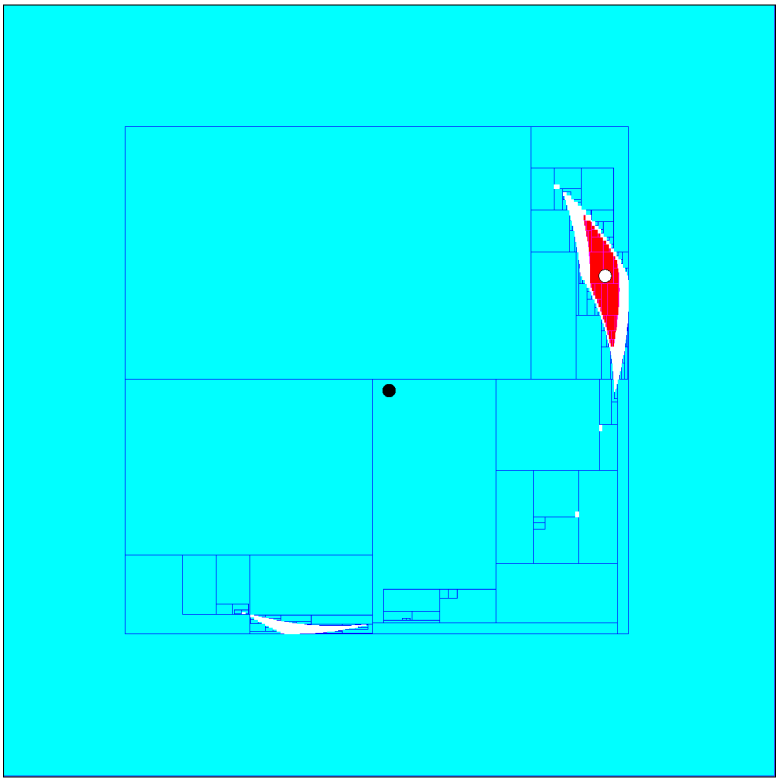


$t = 0.6$

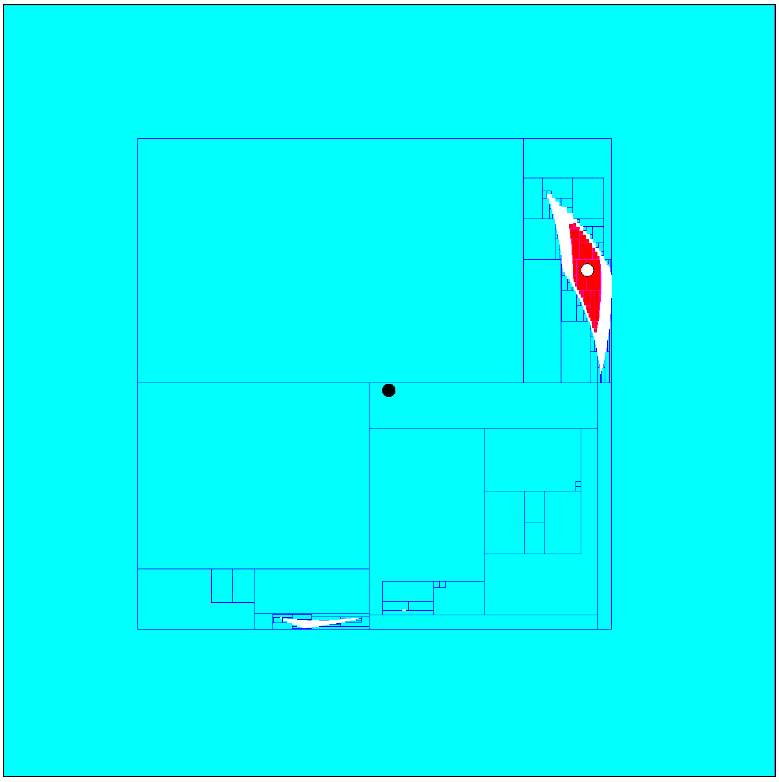
$t = 0.7$

$t = 0.8$

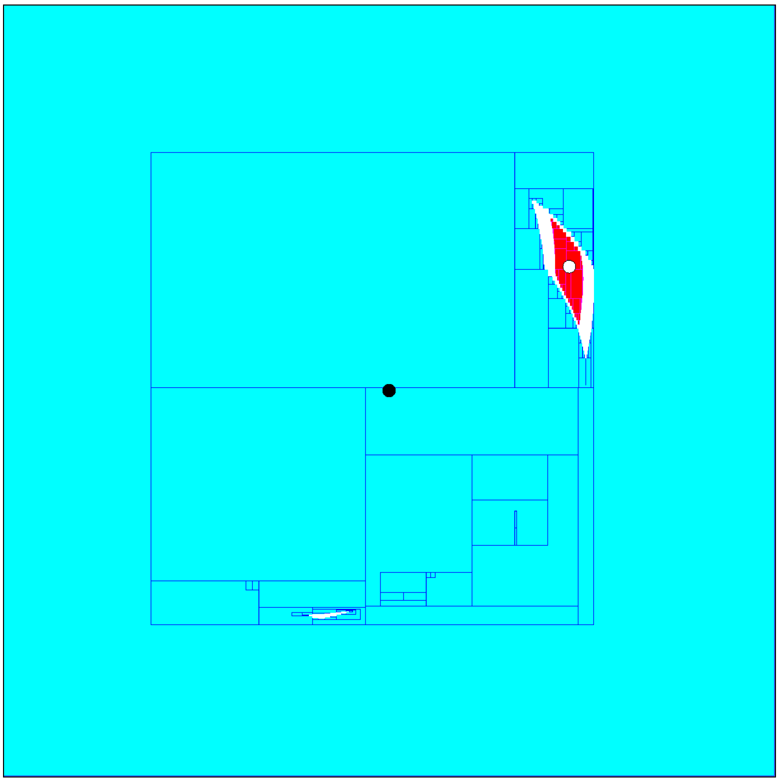
$t = 0.9$



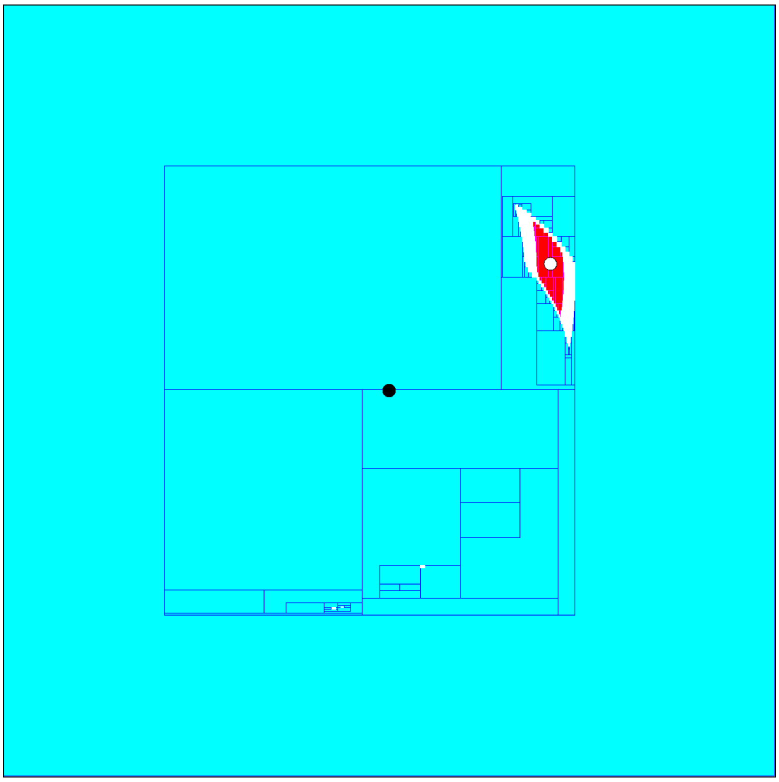
$t = 1.0$



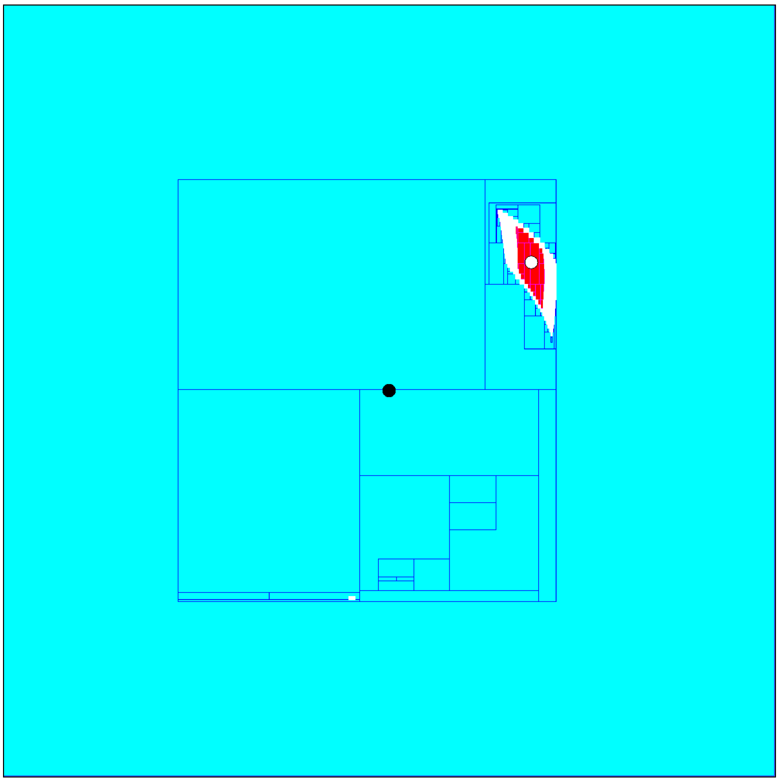
$t = 1.1$



$t = 1.2$



$t = 1.3$



$t = 1.4$