Parametric global optimization; application to sailboat robotics

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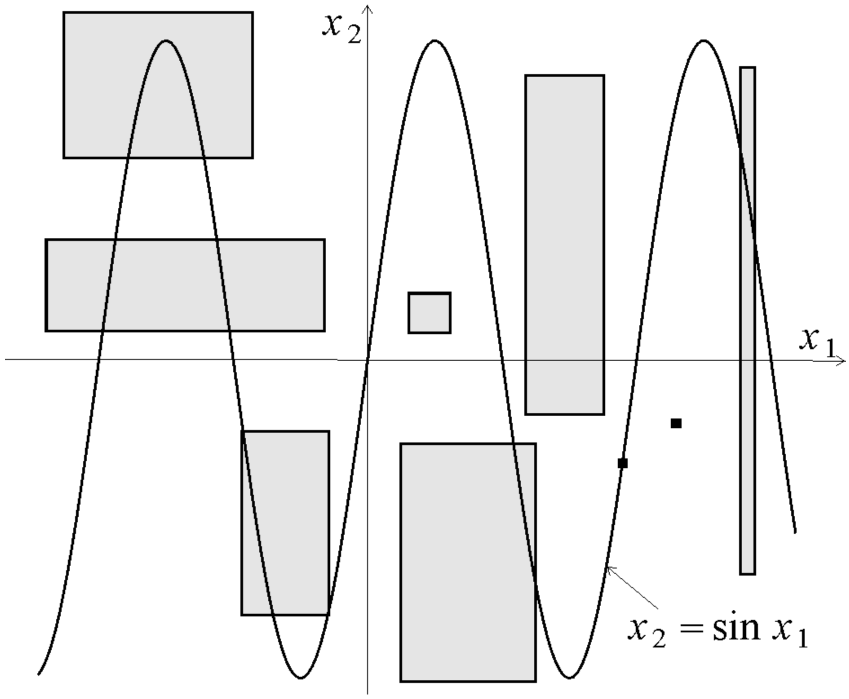
1 Contractors

The operator $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

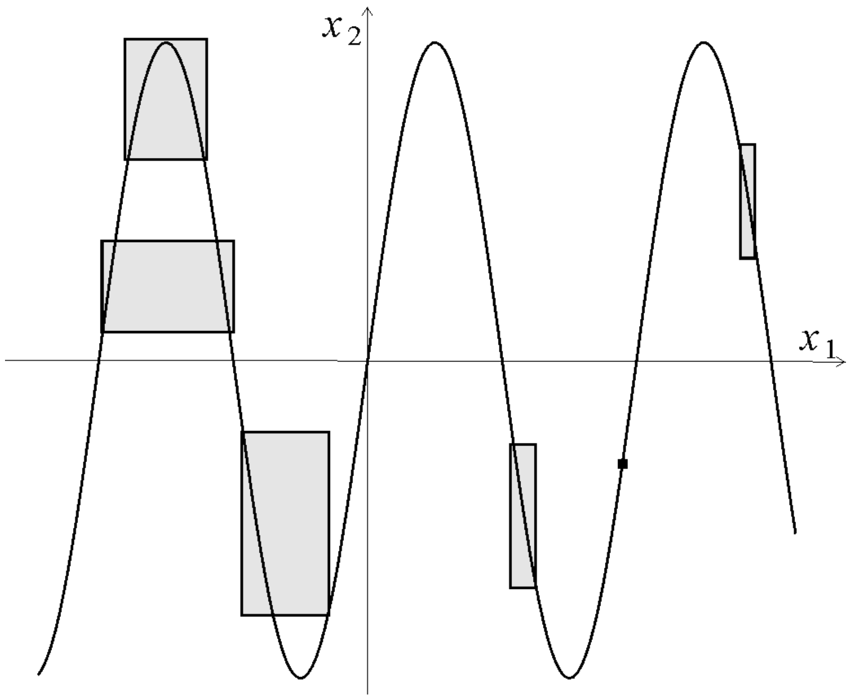
$$\left\{ \begin{array}{ll} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{array} \right.$$

Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$



x_2
x_1
$x_2 = \sin x_1$

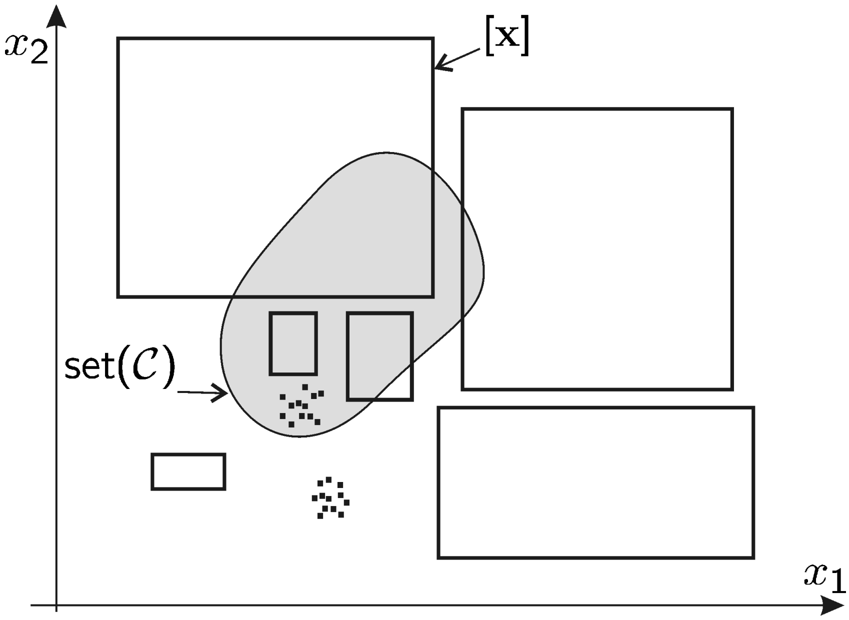

x_2
x_1

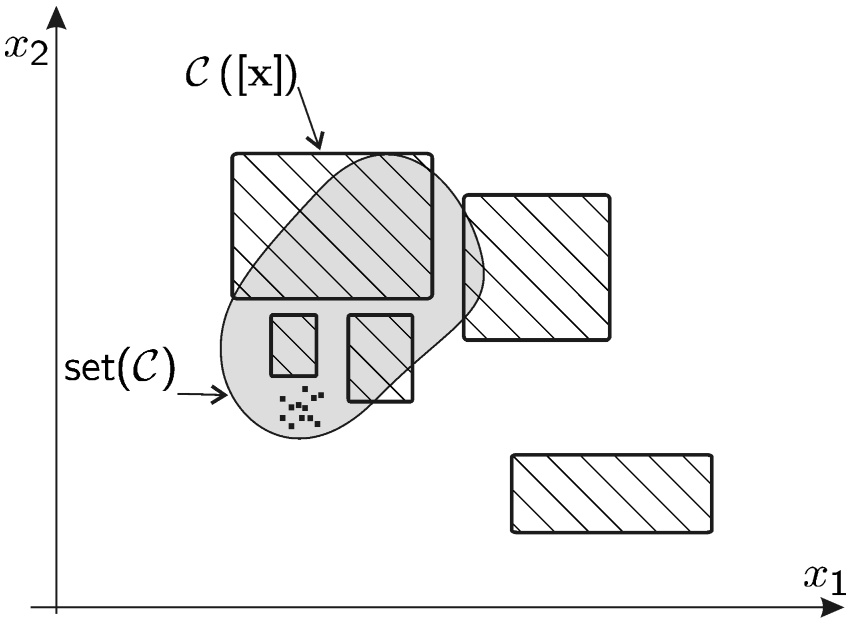
More generally, $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* if

(i)	$\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$	(contractance)
(ii)	$(\mathbf{a} \in [\mathbf{x}], \mathcal{C}(\{\mathbf{a}\}) = \{\mathbf{a}\}) \Rightarrow \mathbf{a} \in \mathcal{C}([\mathbf{x}])$	(consistence)

The set associated to $\mathcal{C}$ is

$$\text{set}(\mathcal{C}) = \{\mathbf{a} \in \mathbb{R}^n, \mathcal{C}(\{\mathbf{a}\}) = \{\mathbf{a}\}\}.$$


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
set($\mathcal{C}$)
x_1

$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *minimal* if	$\mathcal{C}([\mathbf{x}]) = [[\mathbf{x}] \cap \mathsf{set}(\mathcal{C})]$
$\mathcal{C}$ is *idempotent* if	$\mathcal{C}(\mathcal{C}([\mathbf{x}])) = \mathcal{C}([\mathbf{x}])$
$\mathcal{C}$ is *continuous* if	$\mathcal{C}(\mathcal{C}^{\infty}([\mathbf{x}])) = \mathcal{C}^{\infty}([\mathbf{x}])$.

Contractor algebra

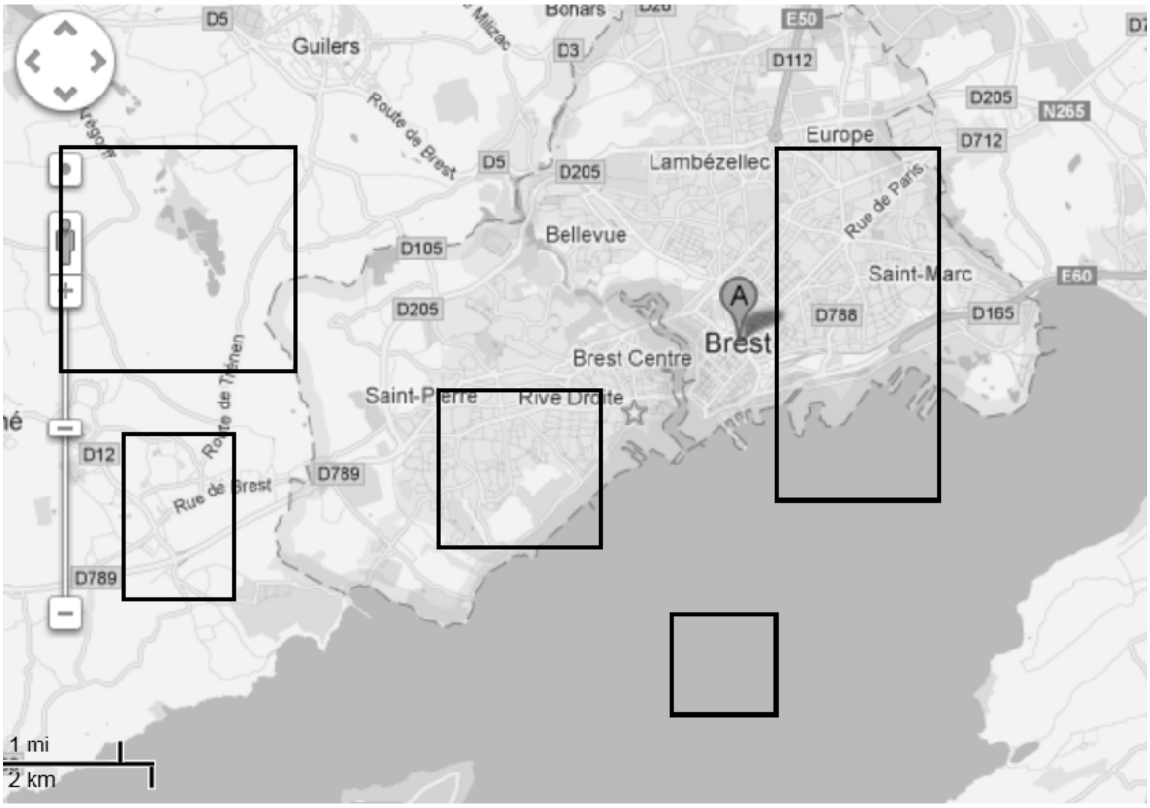
intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^{\infty} \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Dealing with outliers

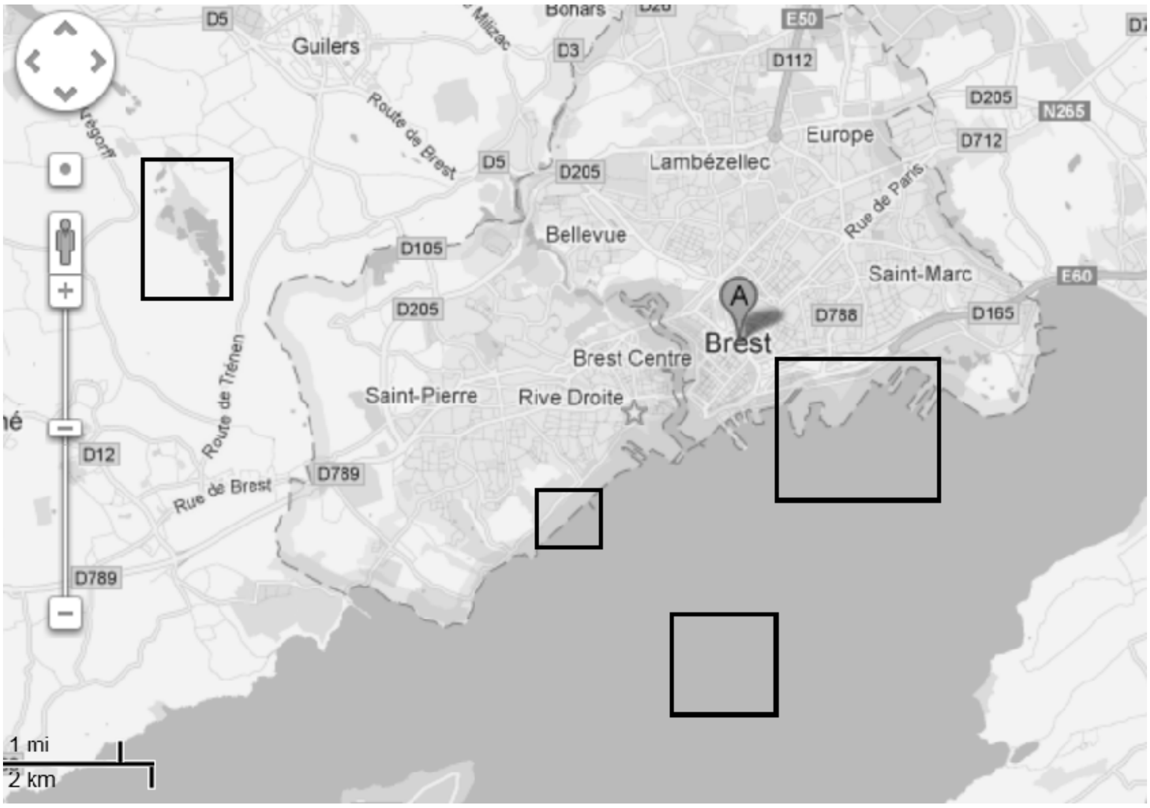
$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Bohars
D3
E50
D112
D205
N265
Route de Brest
Europe
D712
Lambézellec
D5
D205
Rue de Paris
Bellevue
D105
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
D12
Route de Trénen
Rue de Brest
D789
D789
1 mi
2 km



Guilers
Route de Brest
Lambézellec
Europe
Bellevue
Rue de Paris
Saint-Marc
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
Rue de Brest
D5
D3
D112
E50
D205
D712
N265
D105
E60
D788
D165
D12
D789
1 mi
2 km

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$
\begin{aligned}
x_3 = x_1 + x_2 \Rightarrow \quad & x_3 \in \quad [x_3] \cap ([x_1] + [x_2]) \quad // \text{ forward} \\
x_1 = x_3 - x_2 \Rightarrow \quad & x_1 \in \quad [x_1] \cap ([x_3] - [x_2]) \quad // \text{ backward} \\
x_2 = x_3 - x_1 \Rightarrow \quad & x_2 \in \quad [x_2] \cap ([x_3] - [x_1]) \quad // \text{ backward}
\end{aligned}
$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

Forward-backward contractor (HC4 revise)

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2]\,,$$

we have the following contractor:

algorithm $\mathcal{C}$ (inout $[x_1]\,, [x_2]\,, [x_3]$)	
$[a] = [x_1] + [x_2]$	$//\ a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	$//\ b = a \cdot x_3$
$[b] = [b] \cap [1, 2]$	$//\ b \in [1, 2]$
$[x_3] = [x_3] \ \cap \ \frac{[b]}{[a]}$	$//\ x_3 = \frac{b}{a}$
$[a] = [a] \ \cap \ \frac{[b]}{[x_3]}$	$//\ a = \frac{b}{x_3}$
$[x_1] = [x_1] \ \cap \ [a] - [x_2]$	$//\ x_1 = a - x_2$
$[x_2] = [x_2] \ \cap \ [a] - [x_1]$	$//\ x_2 = a - x_1$

2 Solver

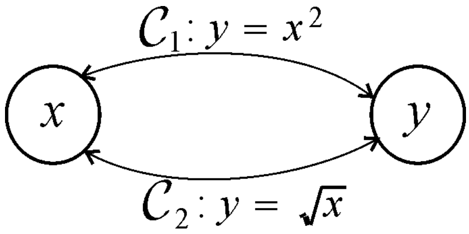
Example 1. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

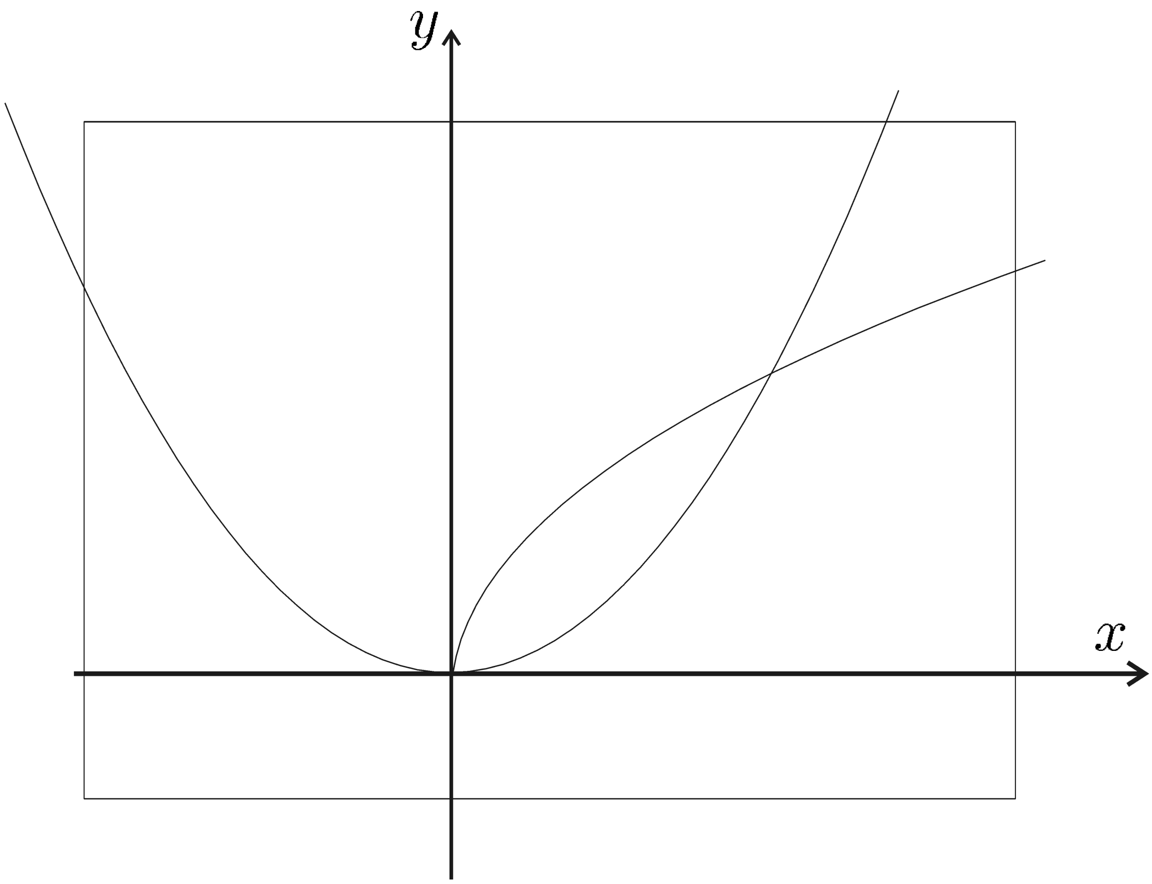
We build two contractors

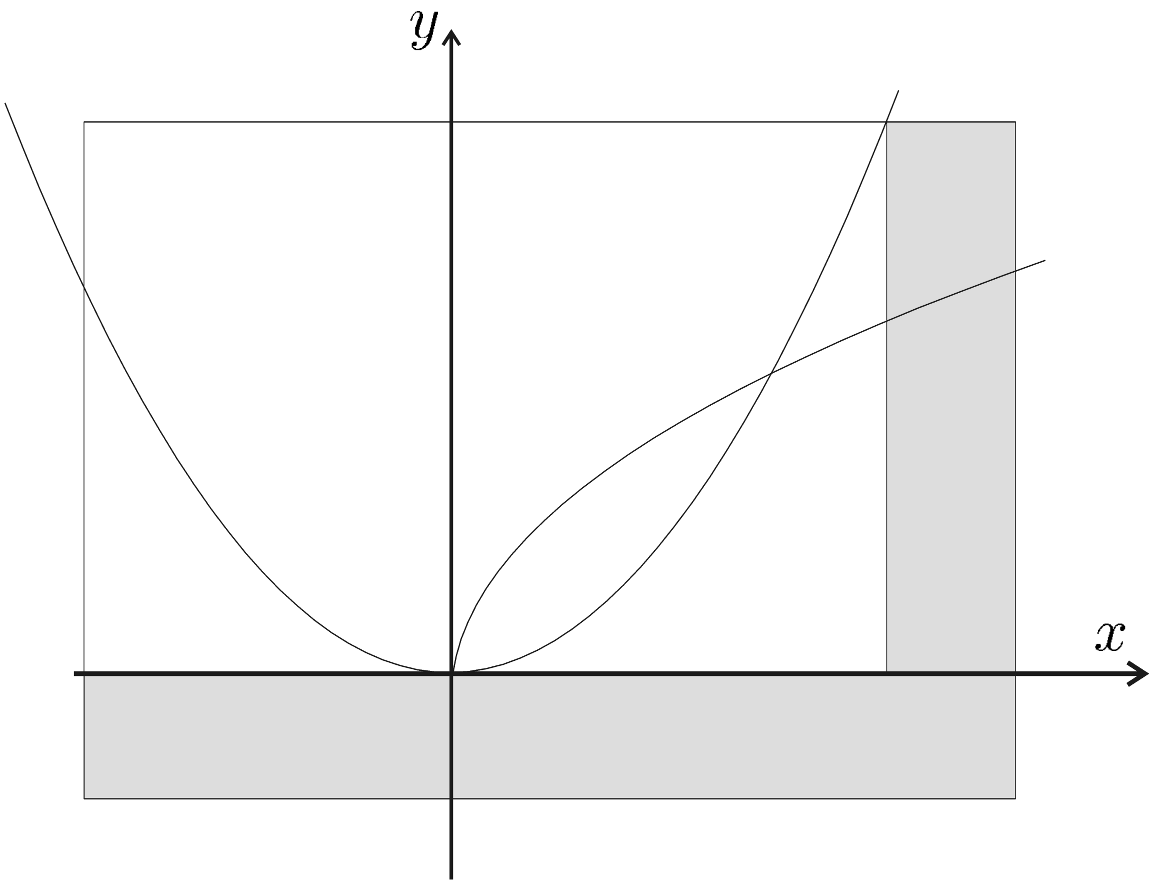
$$\mathcal{C}_1 : \left\{ \begin{array}{l} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{array} \right. \quad \text{associated to } y = x^2$$

$$\mathcal{C}_2 : \left\{ \begin{array}{l} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{array} \right. \quad \text{associated to } y = \sqrt{x}$$

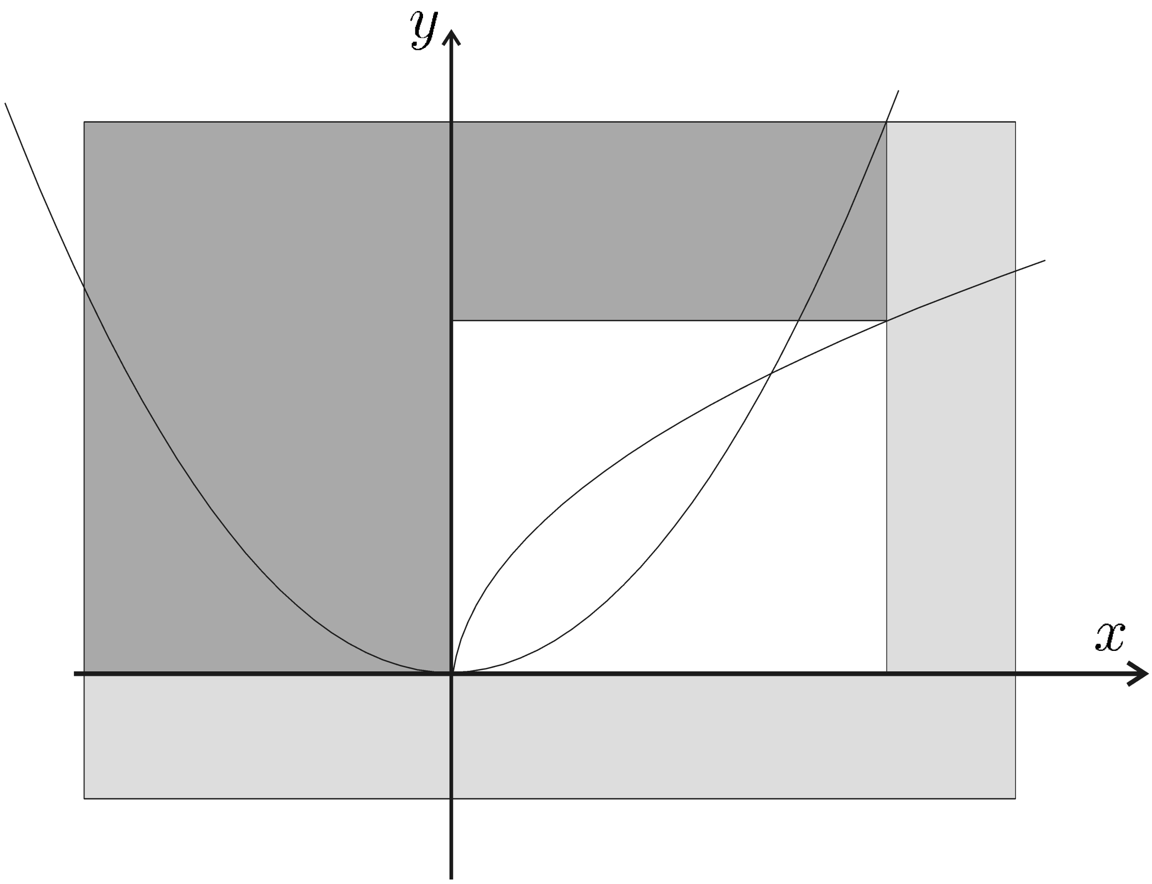


Contractor graph

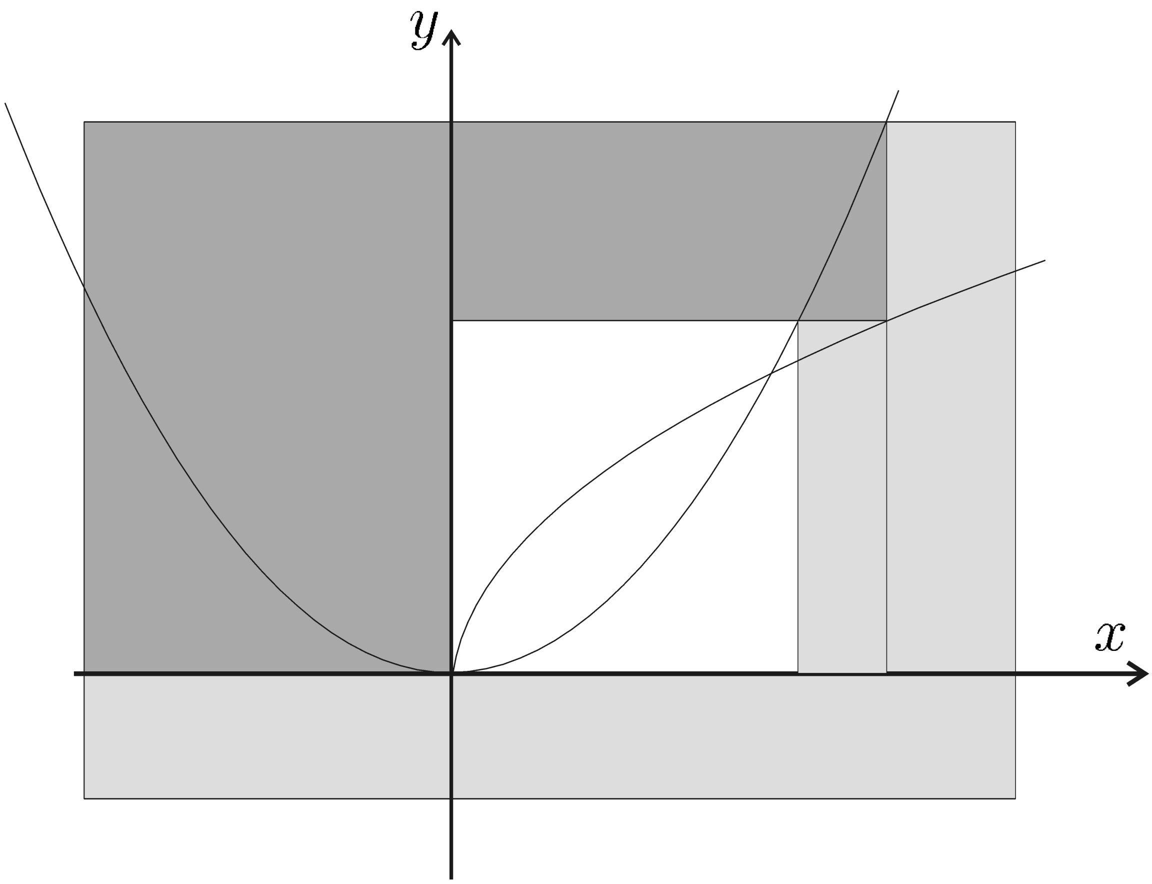

y
x



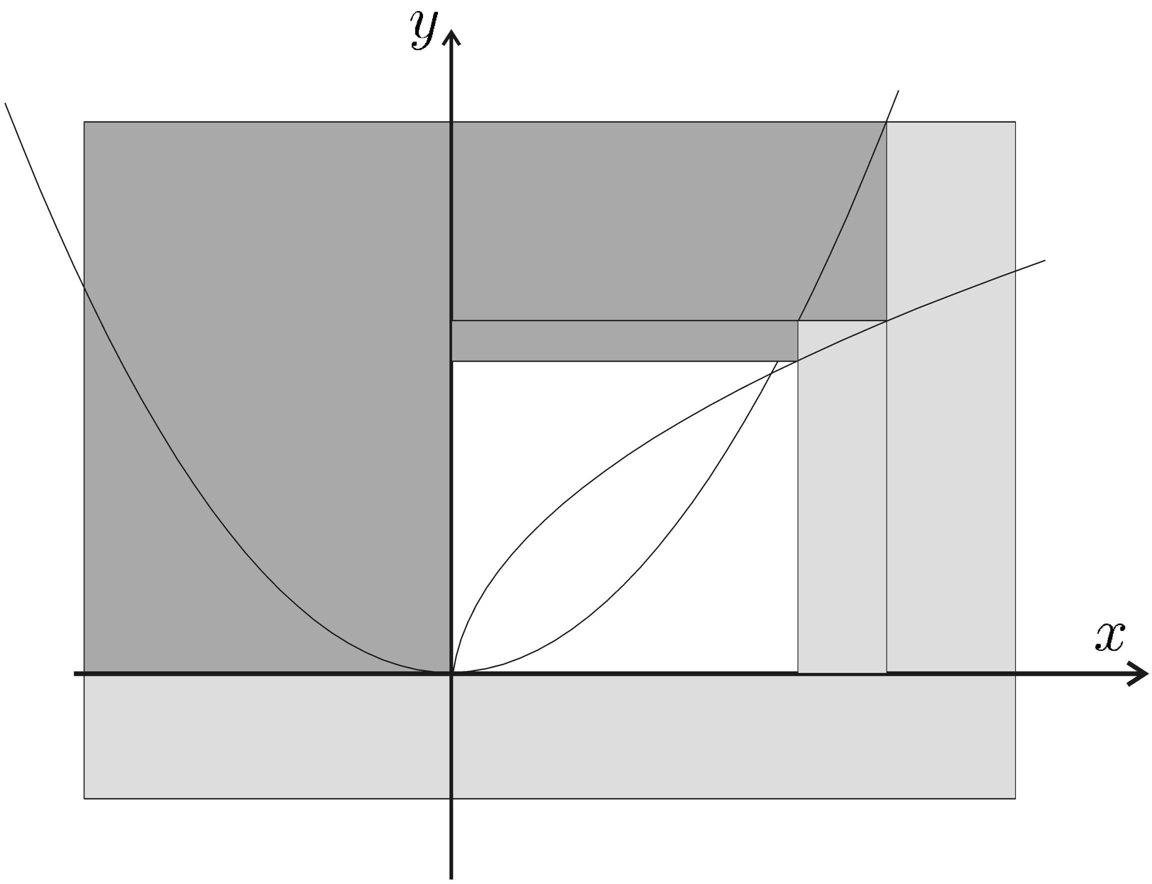
y
x



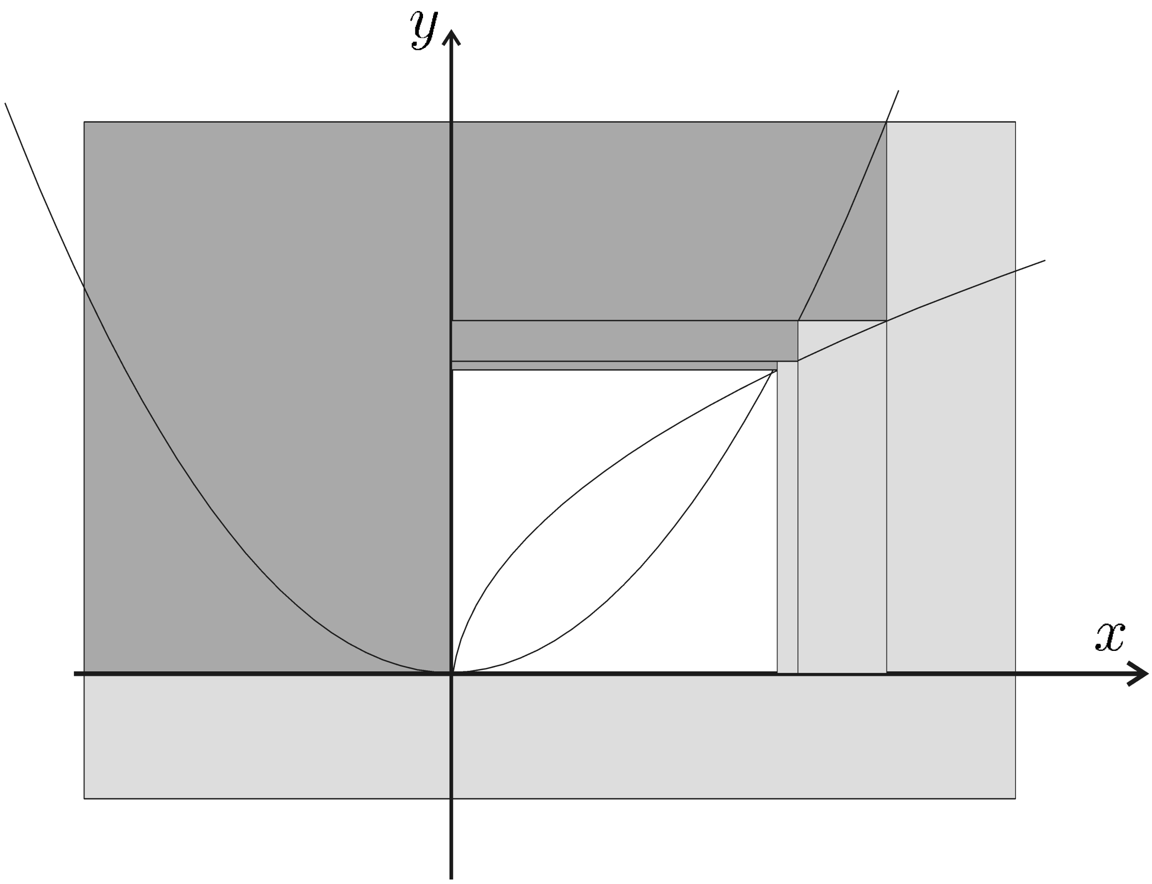
y
x

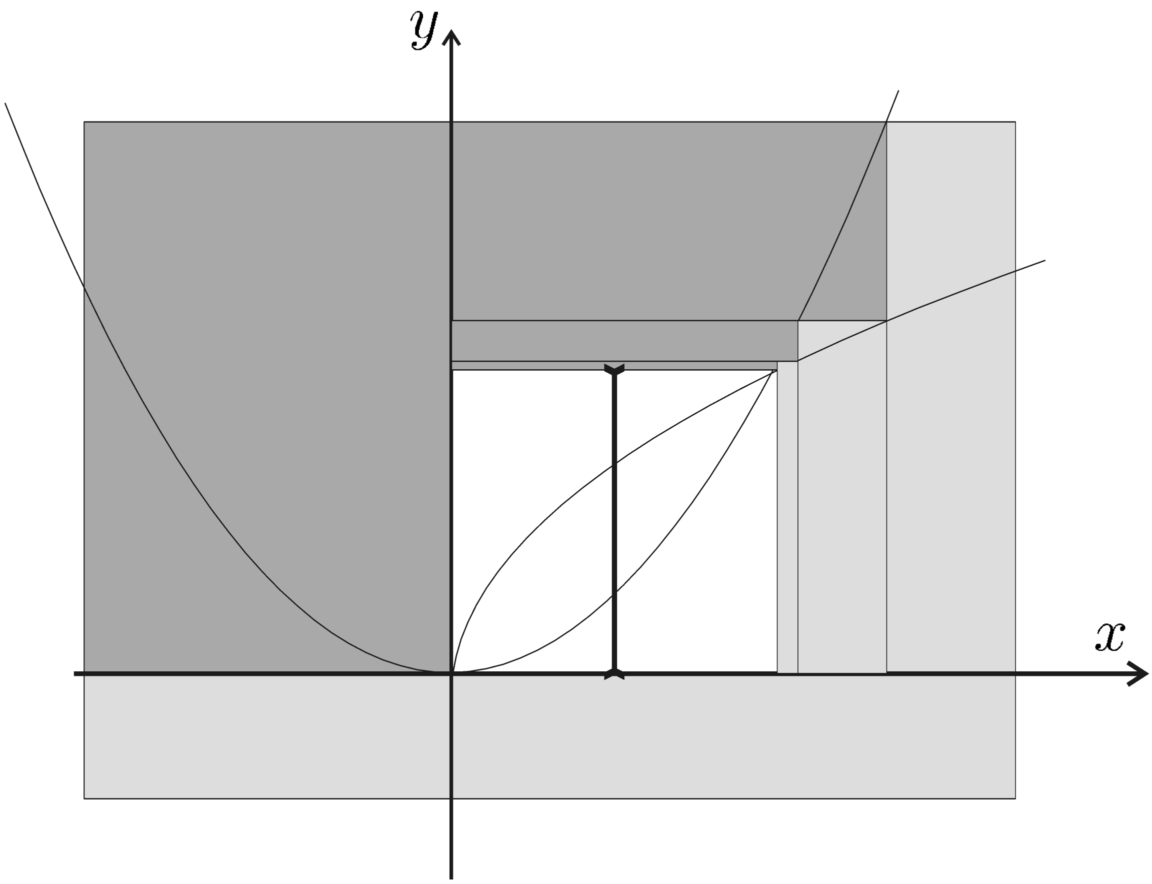


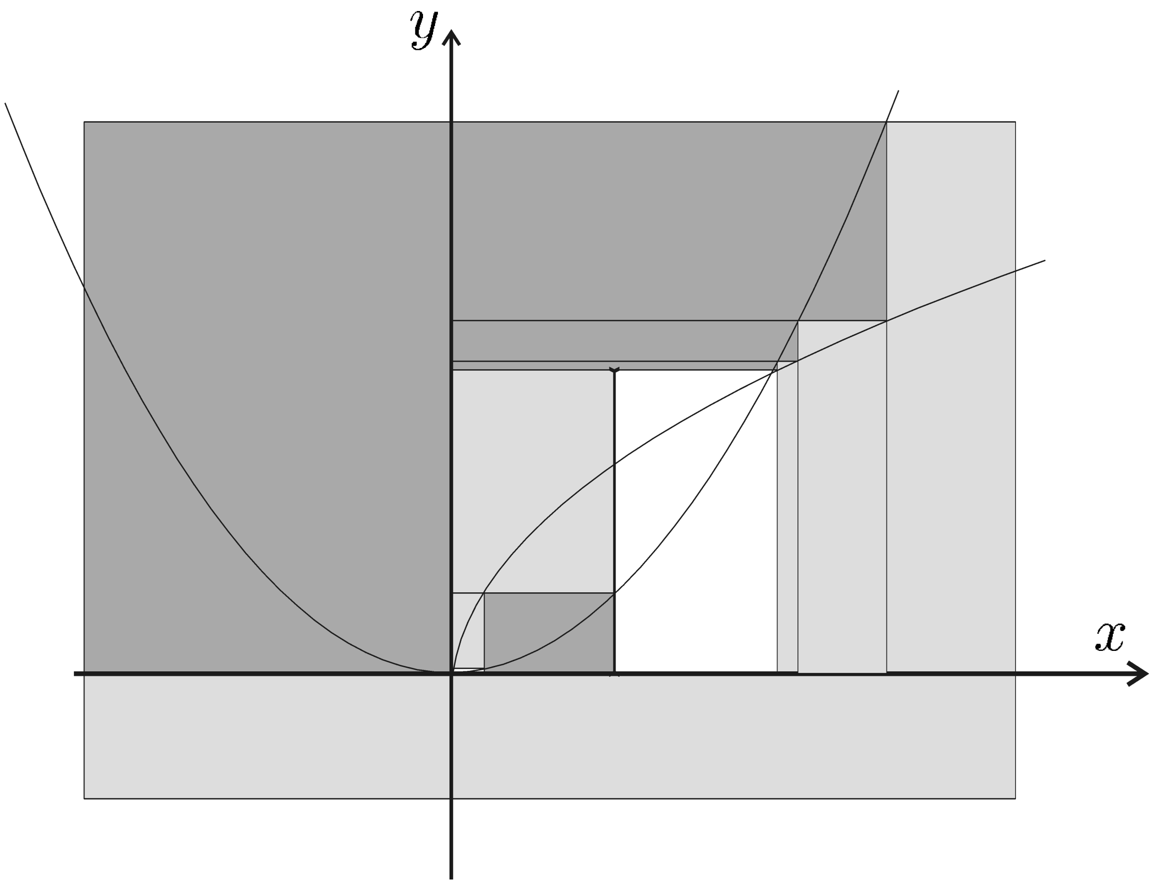
y
x

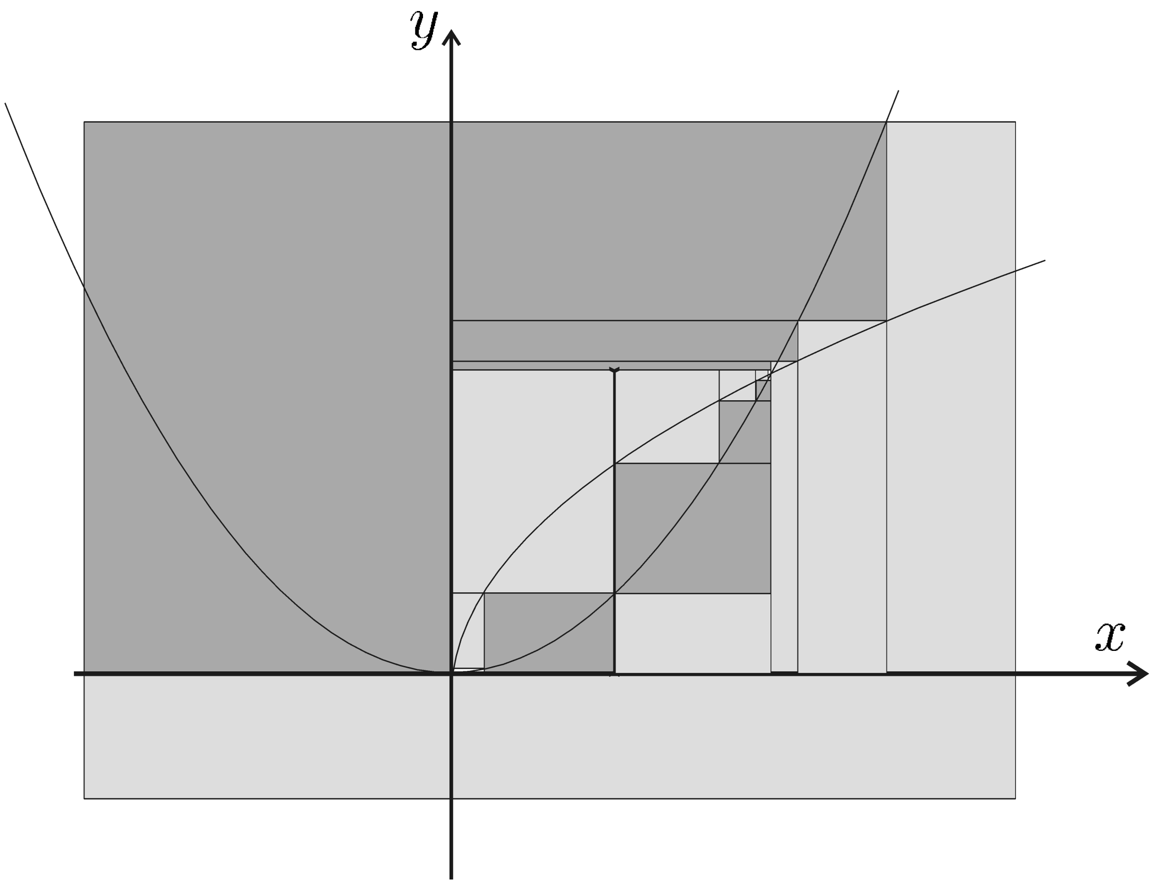


y
x


y
x


y
x


y
x



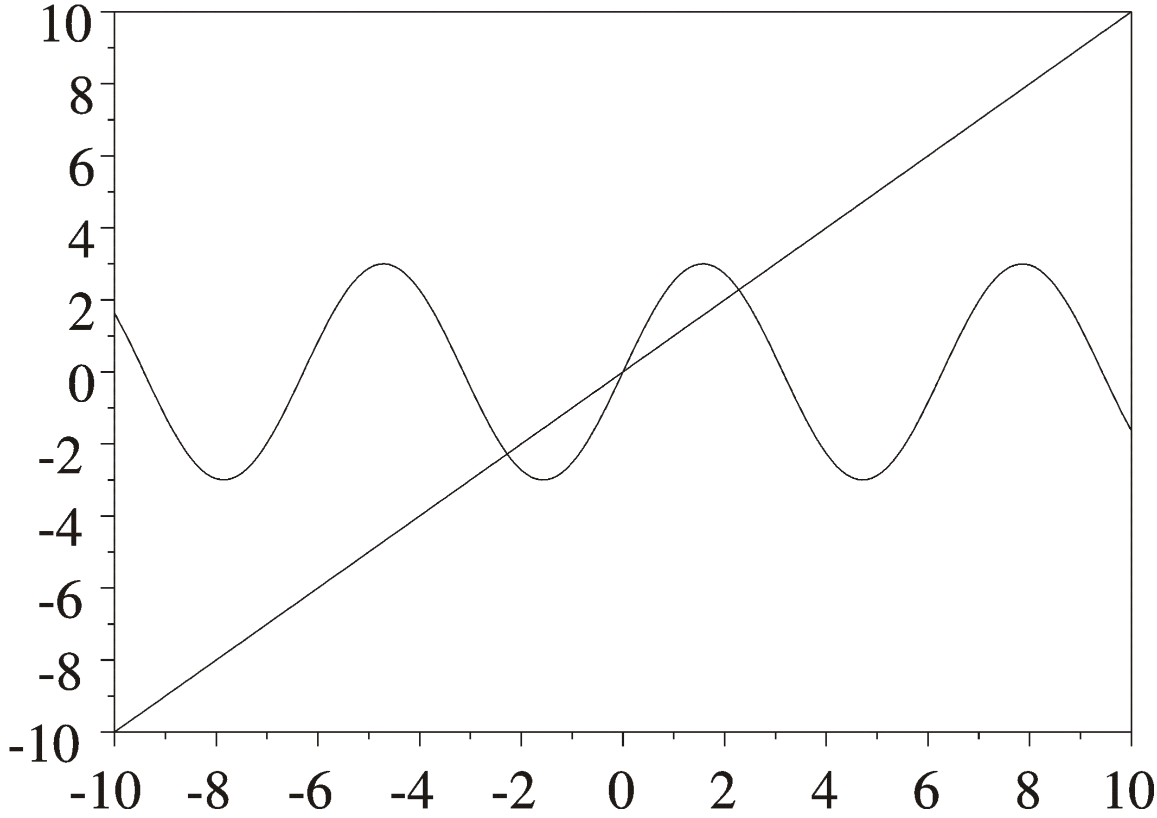
y
x

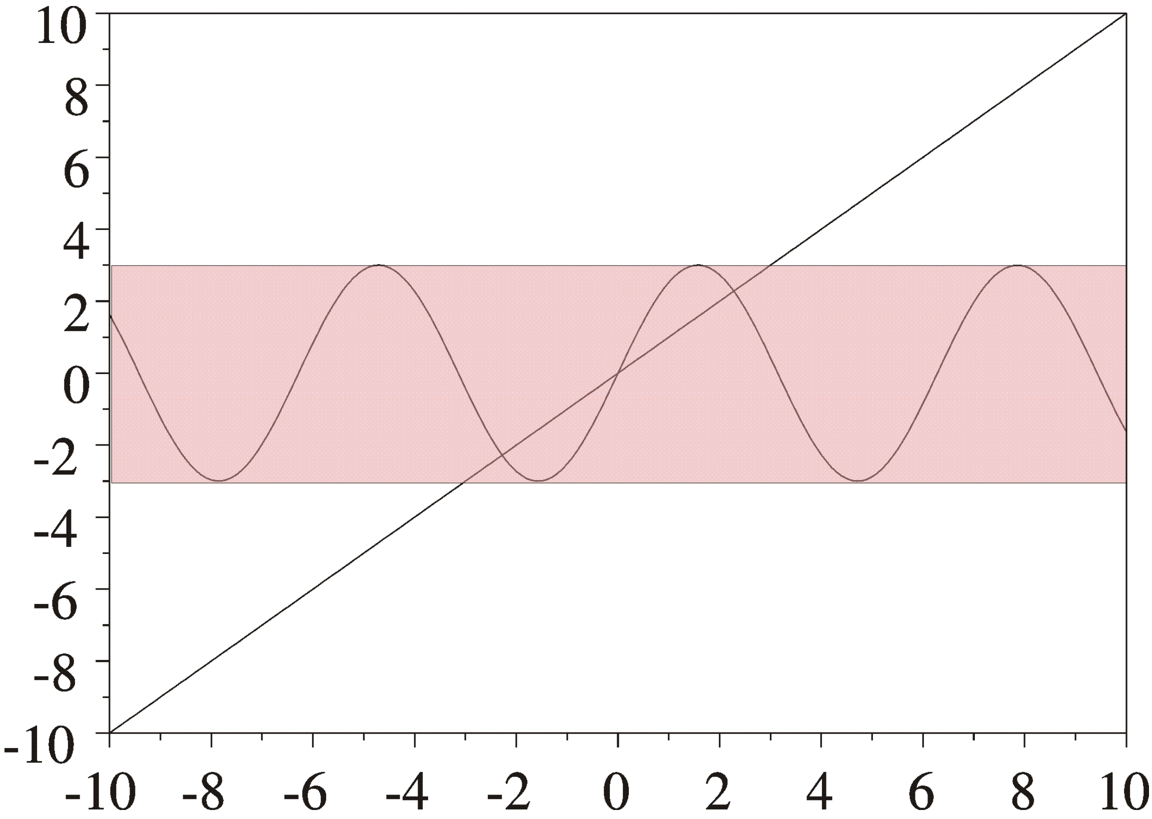
Note that

$$\begin{gathered}
\mathcal{C}_1 \text{ is optimal} \\
\mathcal{C}_2 \text{ is optimal} \\
\mathcal{C}_1 \circ \mathcal{C}_2 \text{ is not optimal} \\
(\mathcal{C}_1 \circ \mathcal{C}_2)^\infty \text{ is optimal.}
\end{gathered}$$

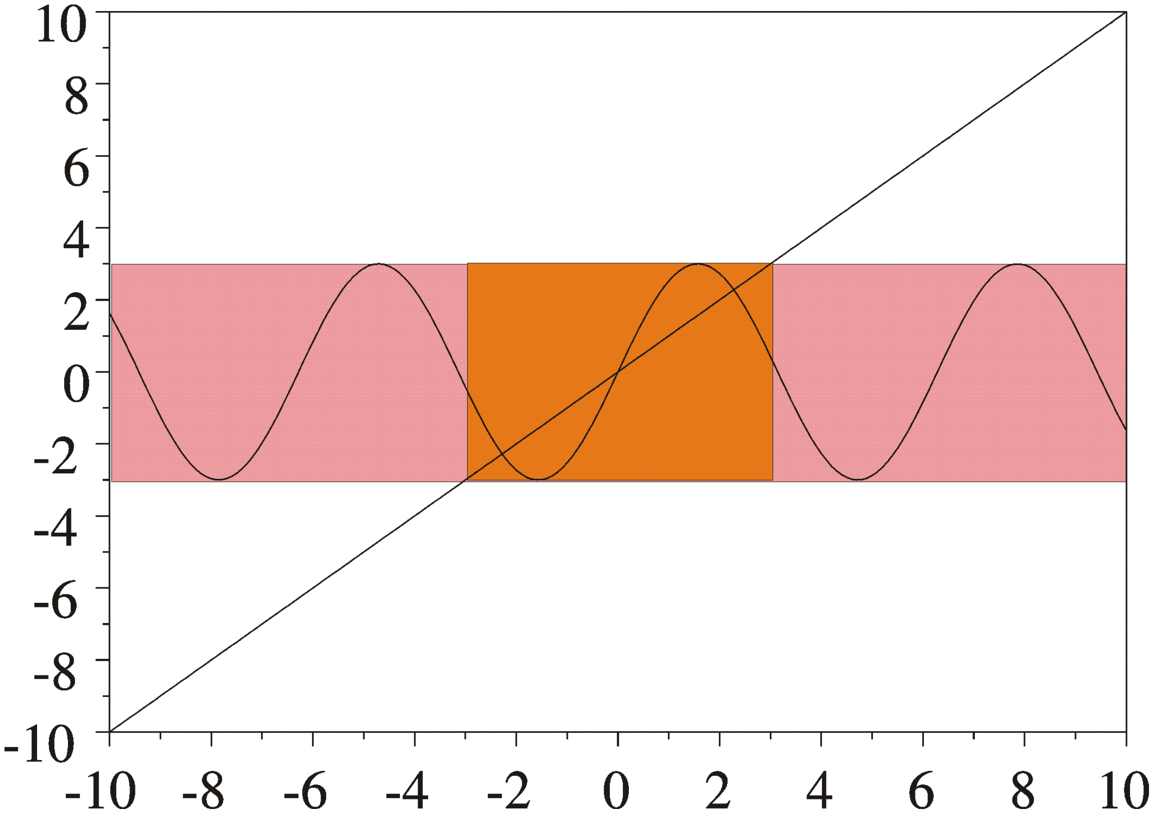
Example 2. Consider the system

$$\left\{ \begin{array}{lcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10



10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10

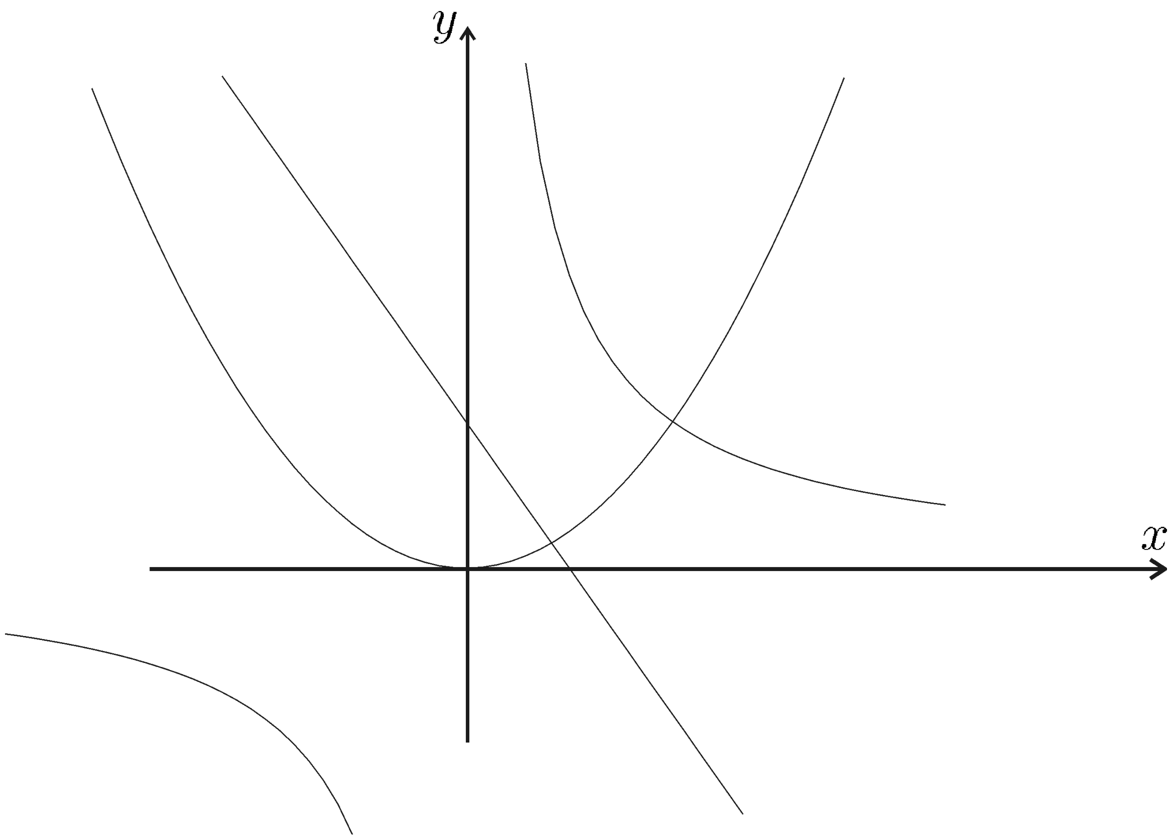


We converge the largest box $[\mathbf{x}]$ such that

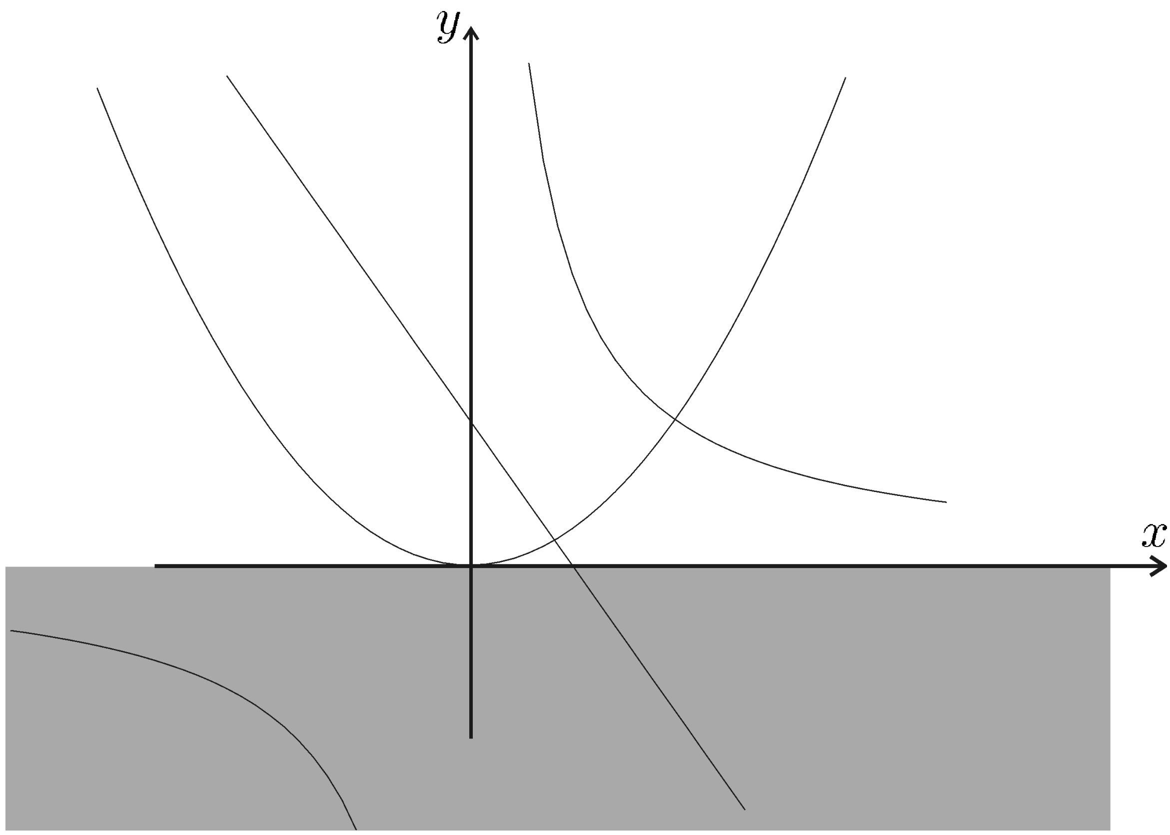
$$\mathcal{C}_1([\mathbf{x}]) = \mathcal{C}_2([\mathbf{x}]) = [\mathbf{x}].$$

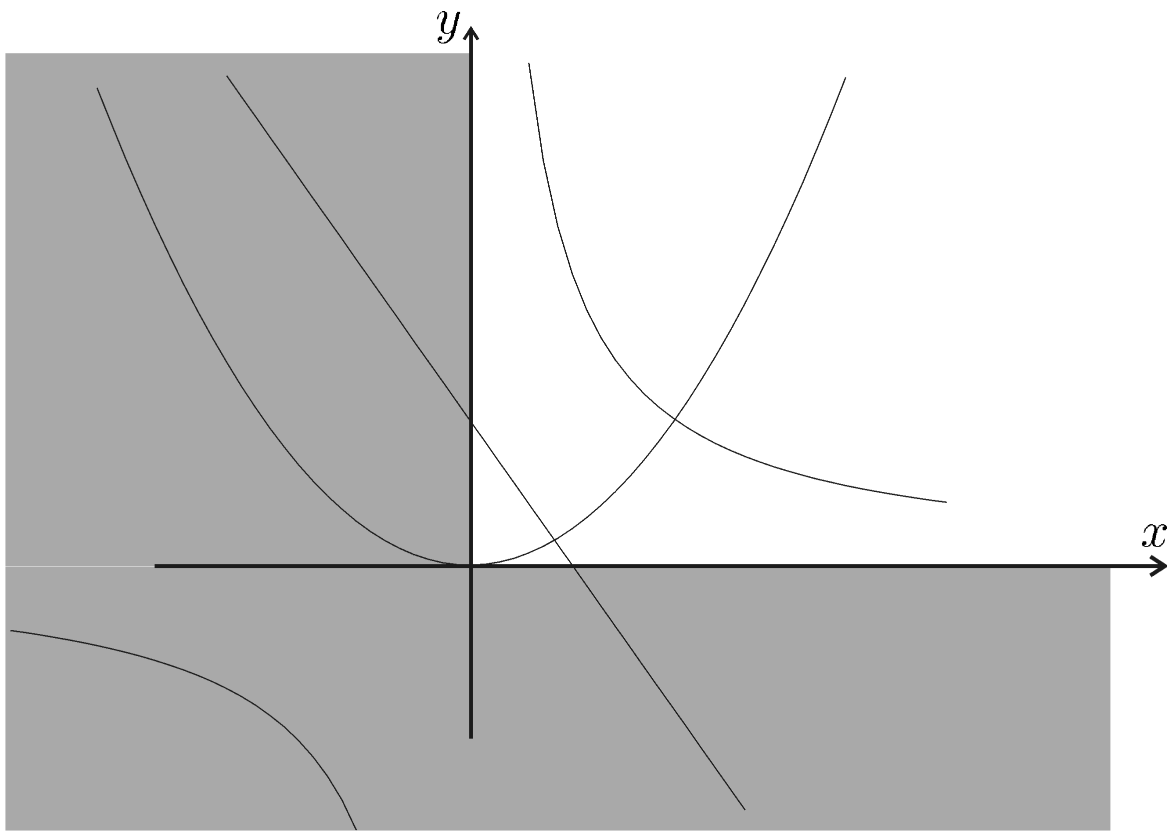
Example 3. Consider the following problem

$$\begin{cases} (C_1): & y = x^2 \\ (C_2): & xy = 1 \\ (C_3): & y = -2x + 1 \end{cases}$$

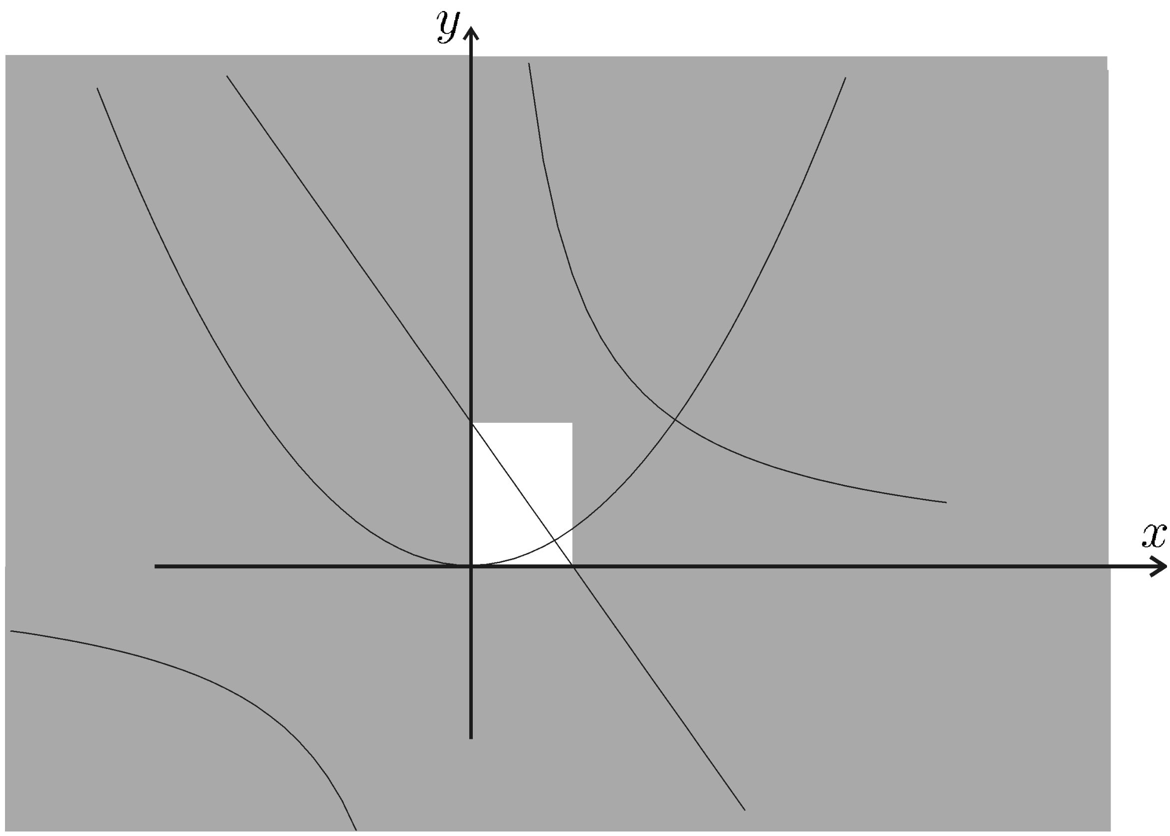


y
x

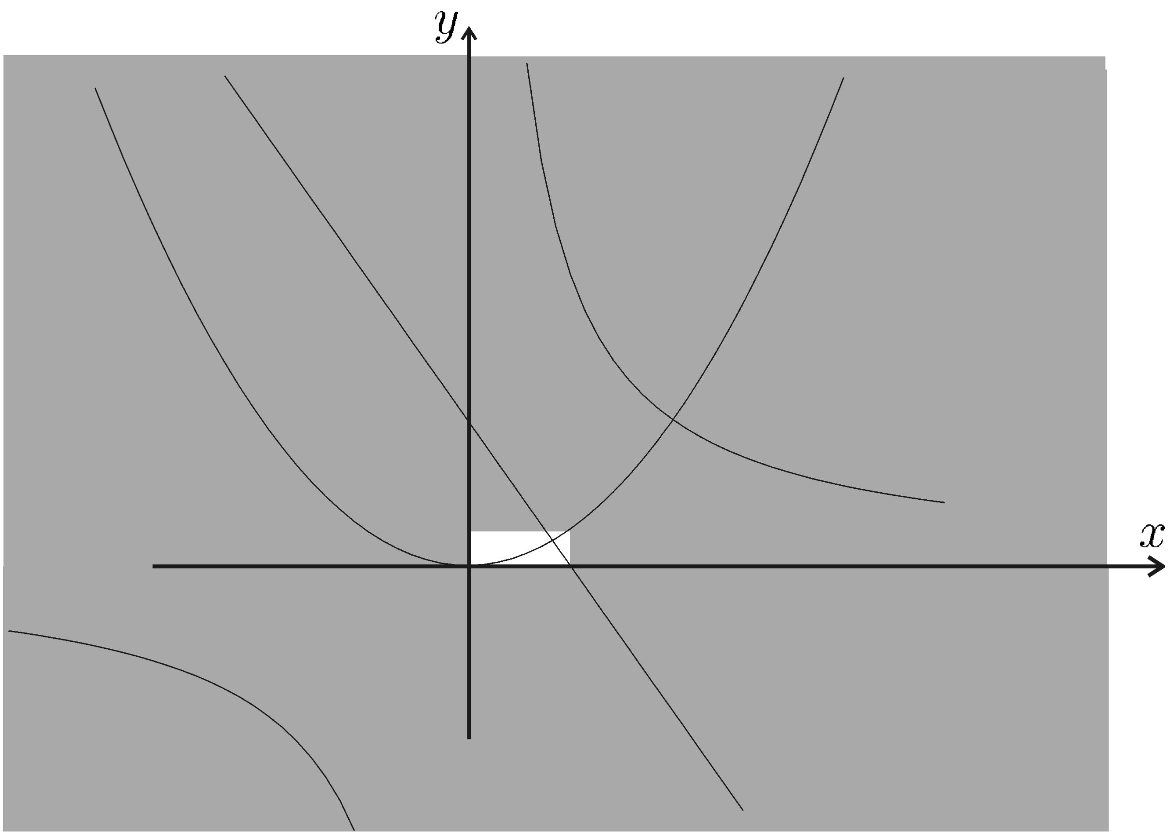

y
x

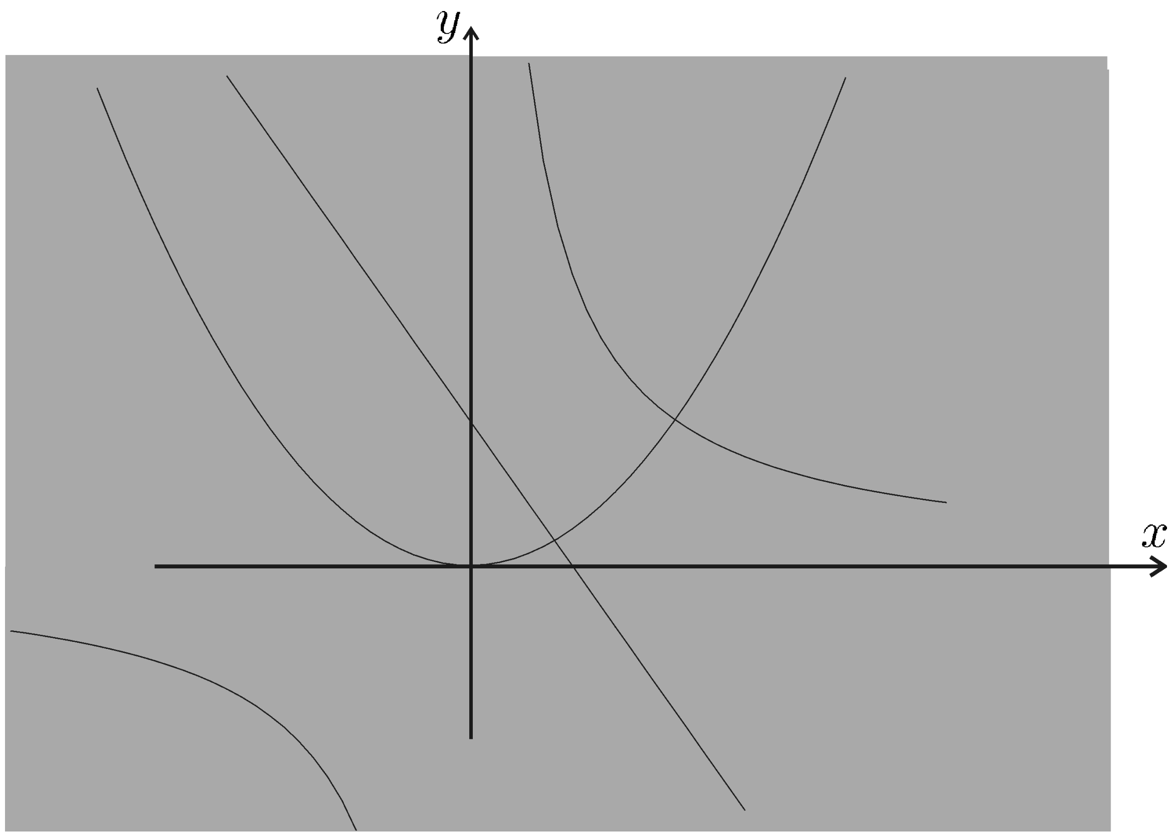


y
x



y
x


y
x



y
x

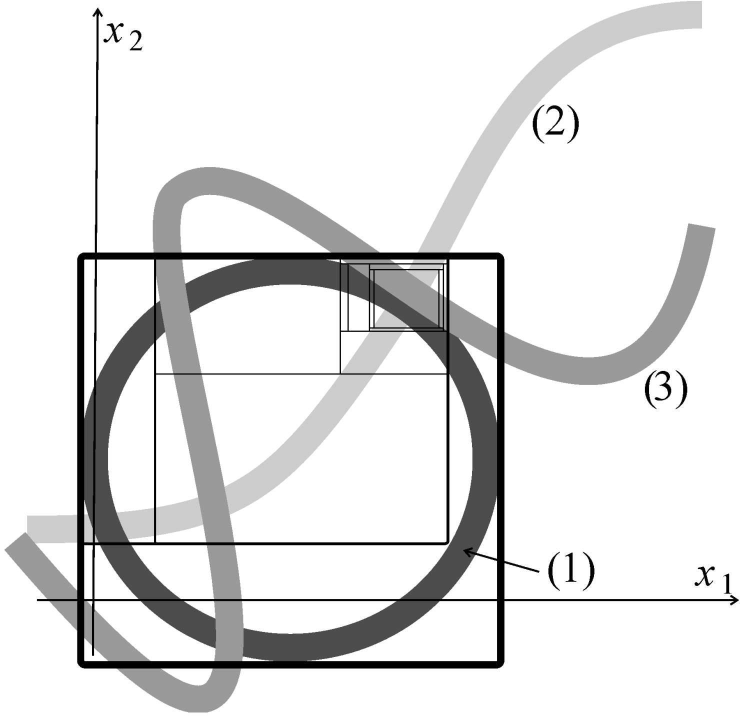
$$\begin{array}{lll}
(C_1) \Rightarrow & y \in & [-\infty, \infty]^2 \ = \ [0, \infty] \\
(C_2) \Rightarrow & x \in & 1/[0, \infty] \ = [0, \infty] \\
(C_3) \Rightarrow & y \in & [0, \infty] \ \cap \ ((-2)\,.[0, \infty] + 1) \\
 & & = [0, \infty] \ \cap \ ([-\infty, 1]) = [0, 1] \\
 & x \in & [0, \infty] \ \cap \ (-[0, 1]/2 + 1/2) = [0, \frac{1}{2}] \\
(C_1) \Rightarrow & y \in & [0, 1] \cap [0, 1/2]^2 = [0, 1/4] \\
(C_2) \Rightarrow & x \in & [0, 1/2] \ \cap \ 1/[0, 1/4] = \emptyset \\
 & y \in & [0, 1/4] \cap 1/\emptyset = \emptyset
\end{array}$$

3 Redundant system of equations

Problem

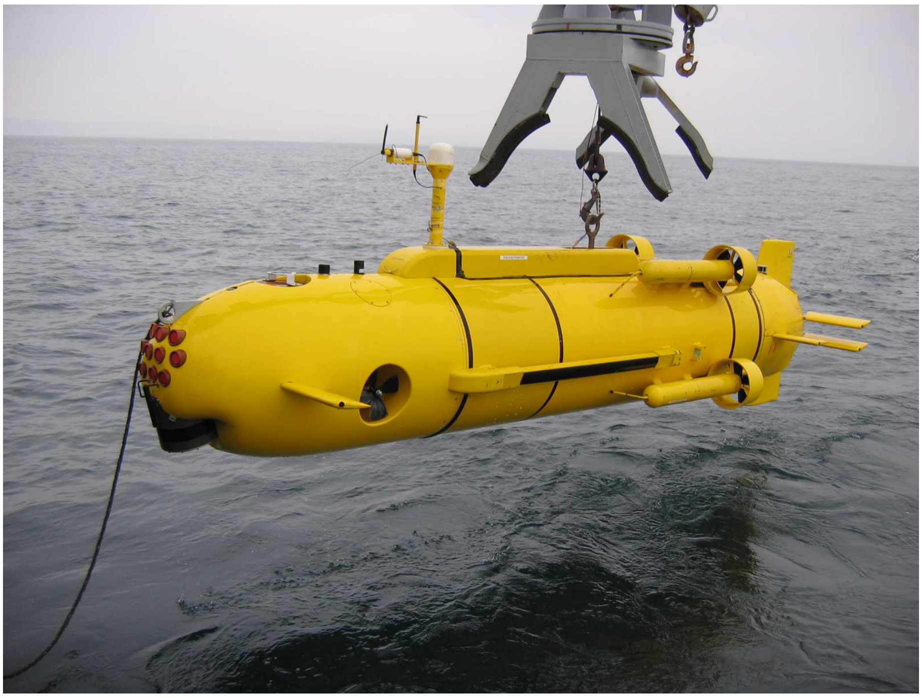
$$\left\{ \begin{array}{c} f_i\left(\mathbf{x}, \mathbf{y}_i\right) = 0, \\ \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y}_i \in [\mathbf{y}_i] \subset \mathbb{R}^{p_i} \\ i \in \{1, \ldots, m\} \end{array} \right. .$$

with $m \gg n \gg 1$.

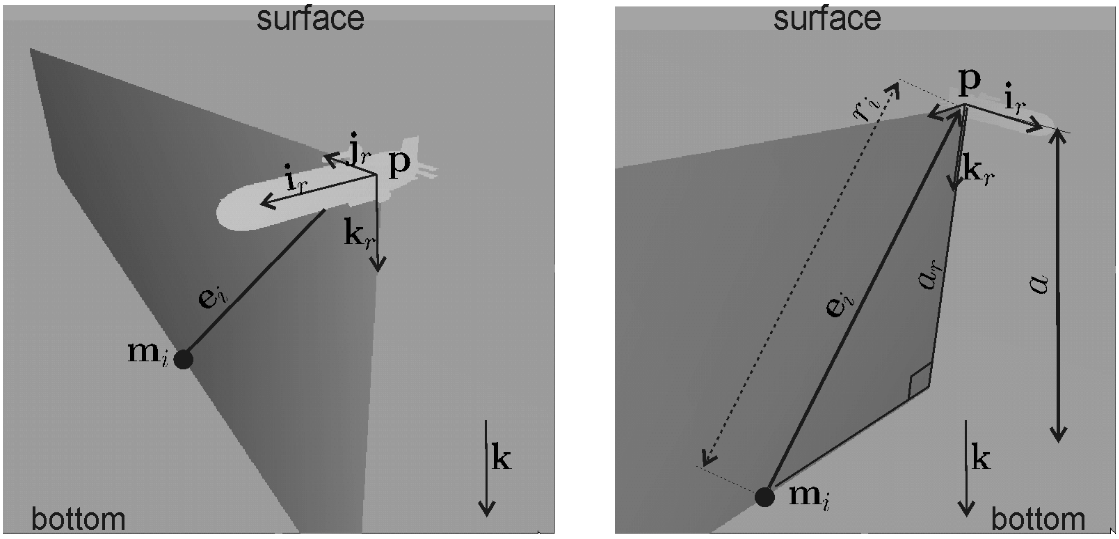


x_2
(2)
(3)
(1)
x_1

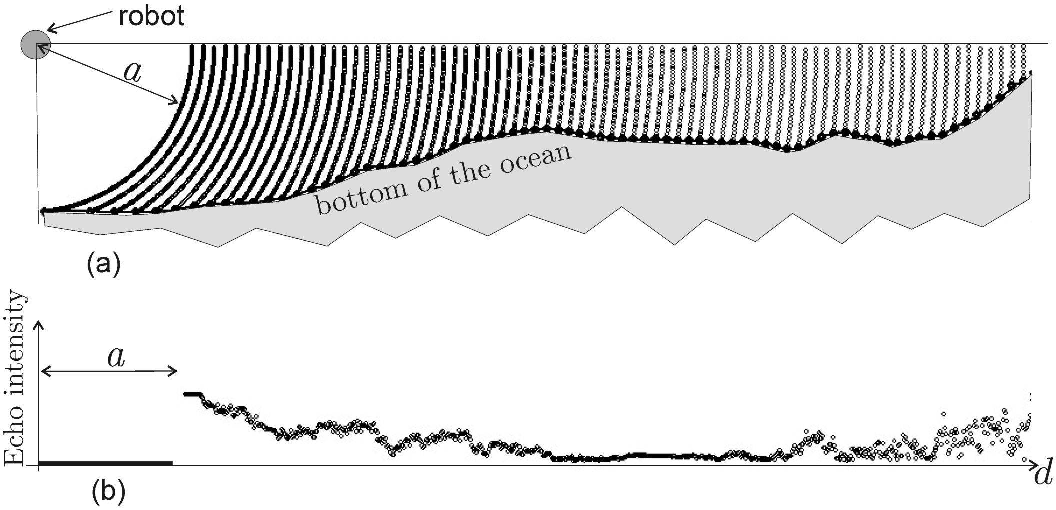
4 SLAM

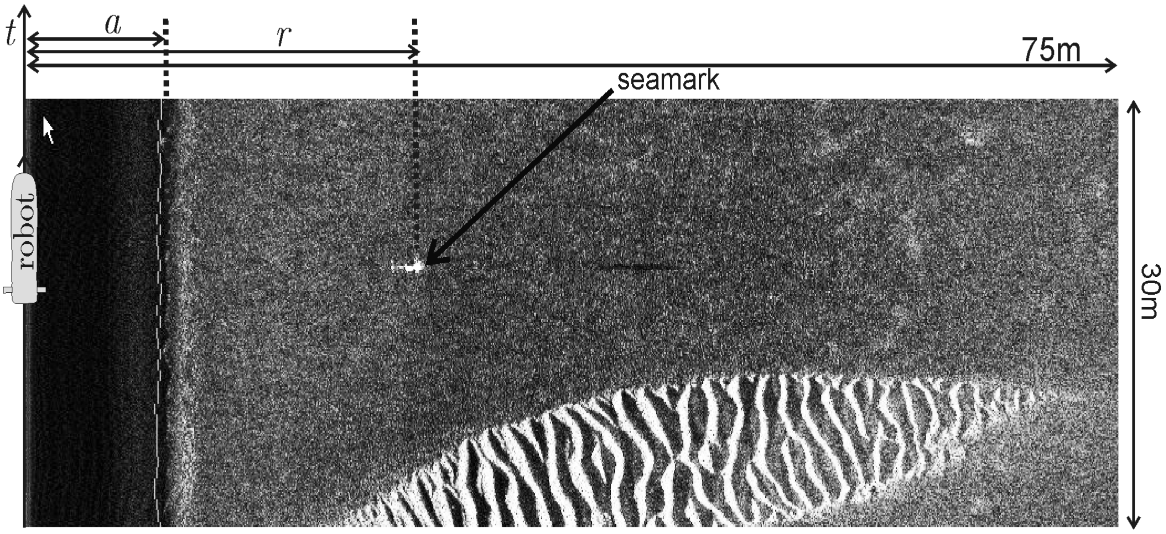


Show the video



surface
$\mathbf{j}_r$
$\mathbf{p}$
$\mathbf{i}_r$
$\mathbf{k}_r$
$\mathbf{e}_i$
$\mathbf{m}_i$
$\mathbf{k}$
bottom
surface
$\mathbf{p}$
$\mathbf{i}_r$
r_i
$\mathbf{k}_r$
$\mathbf{e}_i$
a_r
a
$\mathbf{k}$
$\mathbf{m}_i$
bottom


robot
a
bottom of the ocean
(a)
Echo intensity
a
d
(b)



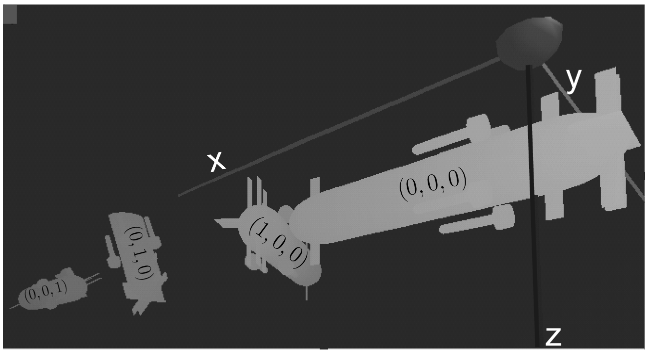
Mine detection with SonarPro

Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \tilde{\mathbf{v}}_r + 0.004 * [-1, 1] \, .\tilde{\mathbf{v}}_r + 0.004 * [-1, 1]$$

Inertial central (Octans III from IXSEA).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.\,[-1,1] \\ 1.75 \times 10^{-4}.\,[-1,1] \\ 5.27 \times 10^{-3}.\,[-1,1] \end{pmatrix}.$$



Six mines have been detected.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	7054	7092	7374	7748	9038	9688	10024	10817	11172	11232	11279	11688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

4.1 Constraints

$$t \in \{6000.0, 6000.1, 6000.2, \ldots, 11999.4\},$$

$$i \in \{0, 1, \ldots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \cdot \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) \cdot \frac{\pi}{180}\right) & 0 \end{pmatrix} \cdot \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix},$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix},$$

$$\mathbf{R}_\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{pmatrix},$$

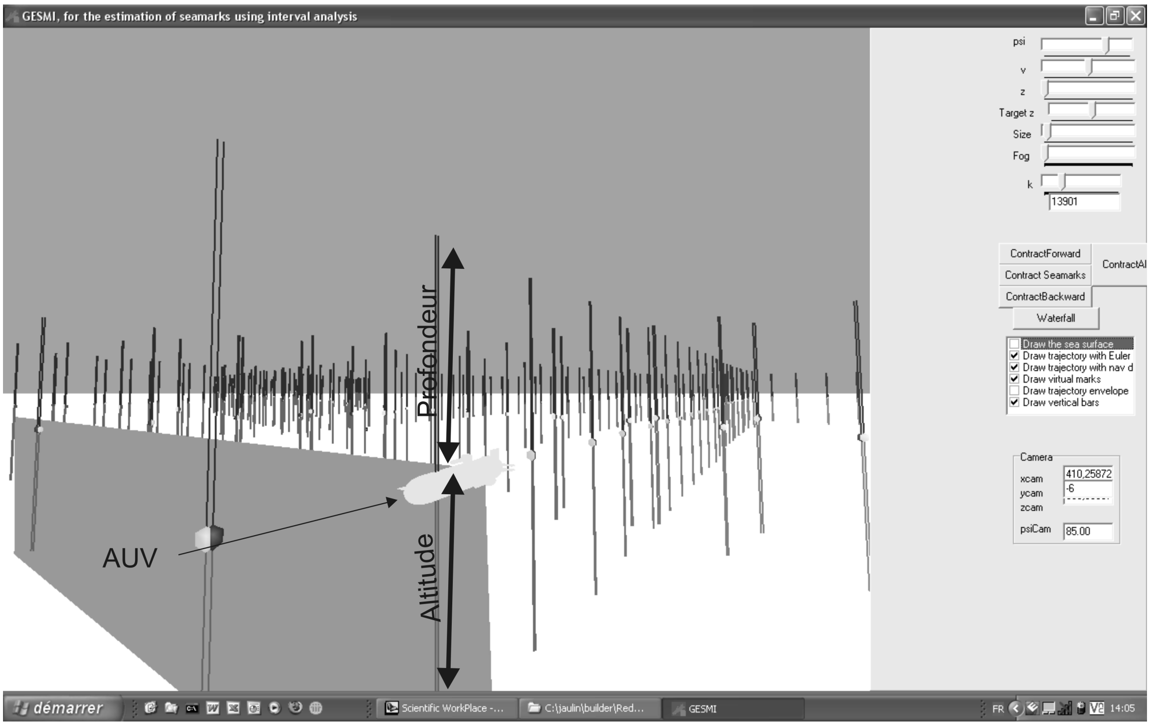
$$\mathbf{R}(t) = \mathbf{R}_\psi(t) \cdot \mathbf{R}_\theta(t) \cdot \mathbf{R}_\varphi(t),$$

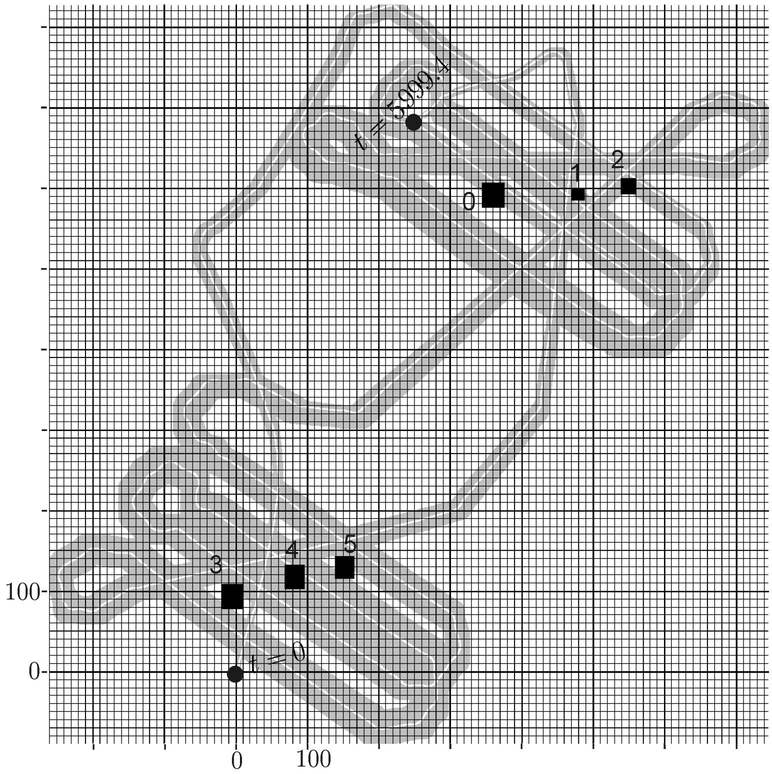
$$\dot{\mathbf{p}}(t) = \mathbf{R}(t) \cdot \mathbf{v}_r(t),$$

$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| \ = r(i),$$

$$\mathbf{R}^\mathsf{T}(\tau(i)) \cdot (\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^{\times 2}.$$

4.2 GESMI


GESMI, for the estimation of seamarks using interval analysis
Profondeur
Altitude
AUV
psi
v
z
Target z
Size
Fog
k
13901
ContractForward
Contract Seamarks
ContractBackward
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
Camera
xcam
410,25872
ycam
-6
zcam
psiCam
85.00
démarrer
GESMI
14:05



$t = 5999.4$
0
1
2
3
4
5
$t = 0$
100
0
0
100

5 Sailboat robotics

5.1 Vaimos

Vaimos (IFREMER and ENSTA)

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}\left(\mathbf{x}\right)$, the robot satisfies an equation of the form

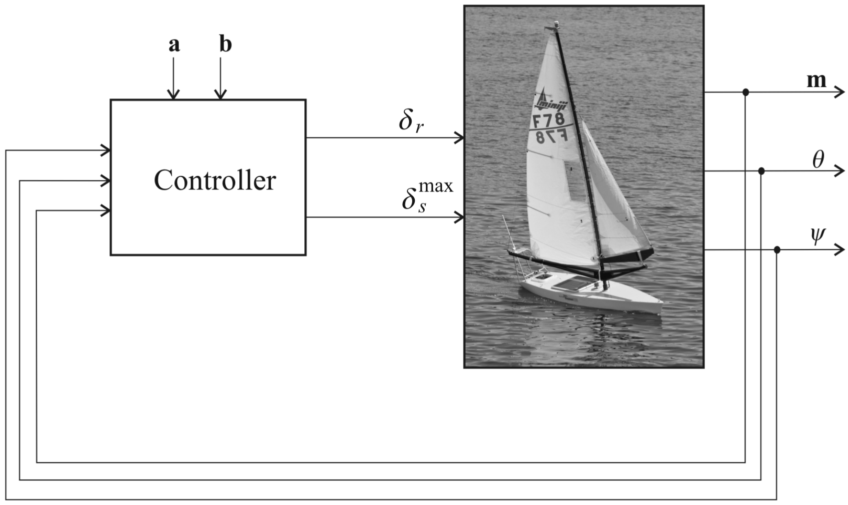
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right).$$

With all uncertainties, the robot satisfies.

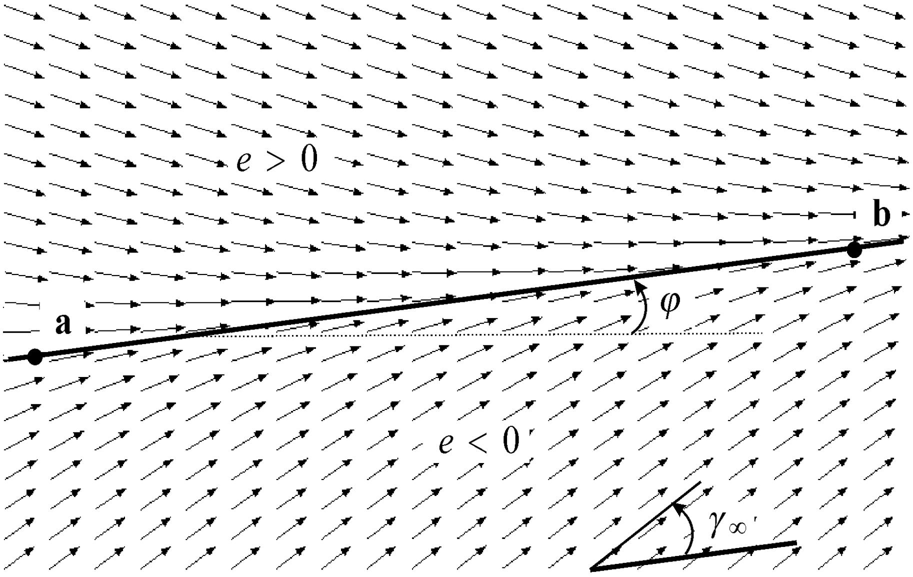
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

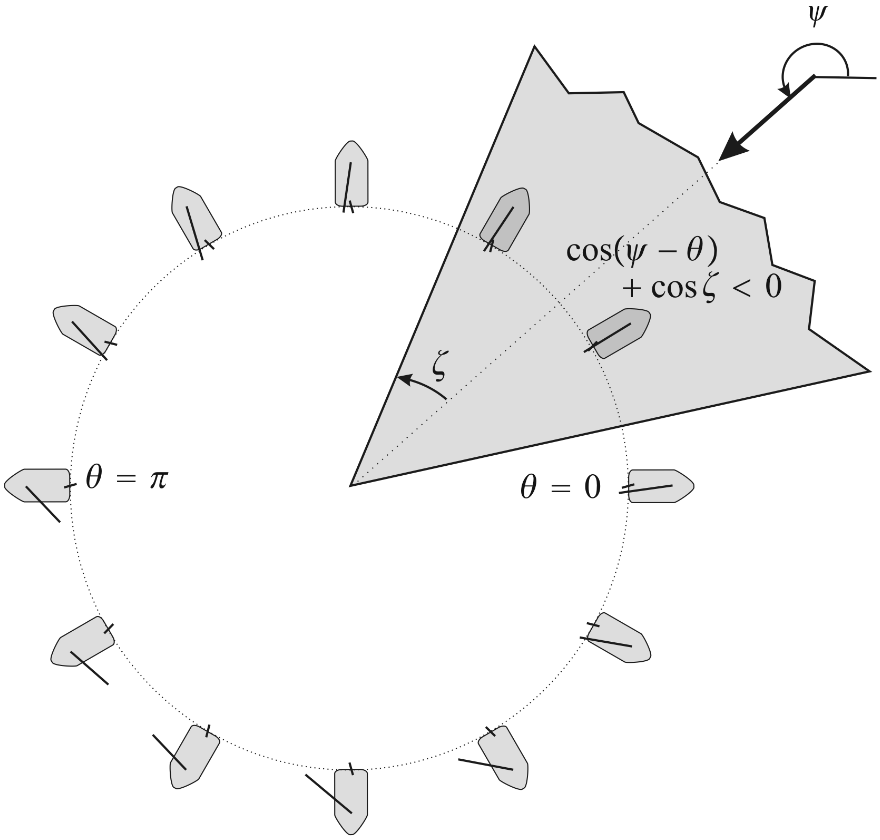
5.2 Line following

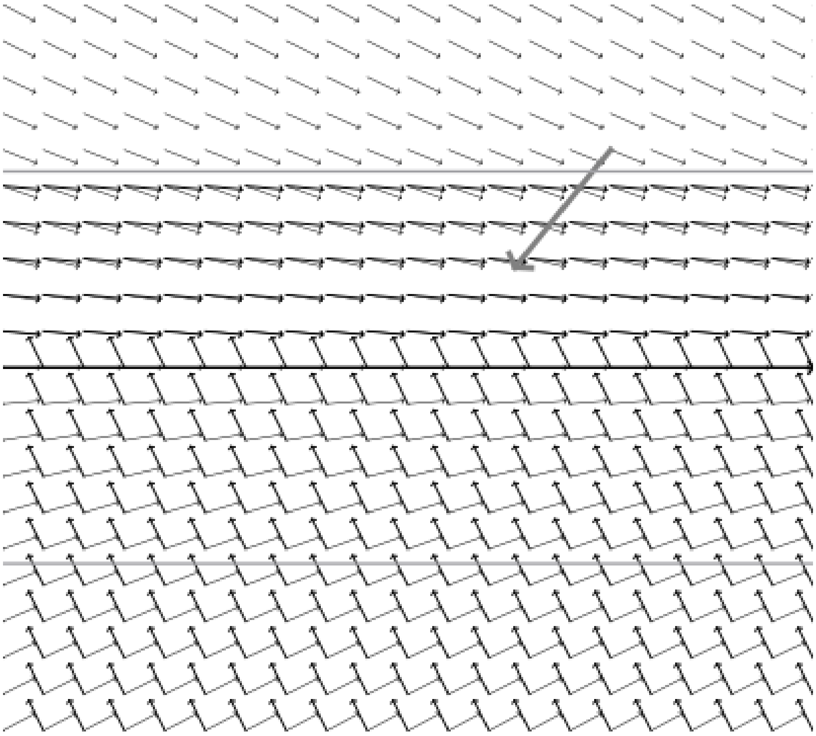


Controller of a sailboat robot



Nominal vector field θ^*


ψ
$\cos(\psi - \theta)$
$+ \cos\zeta < 0$
ζ
$\theta = \pi$
$\theta = 0$



Keep close hauled strategy

5.3 V-stability

The system

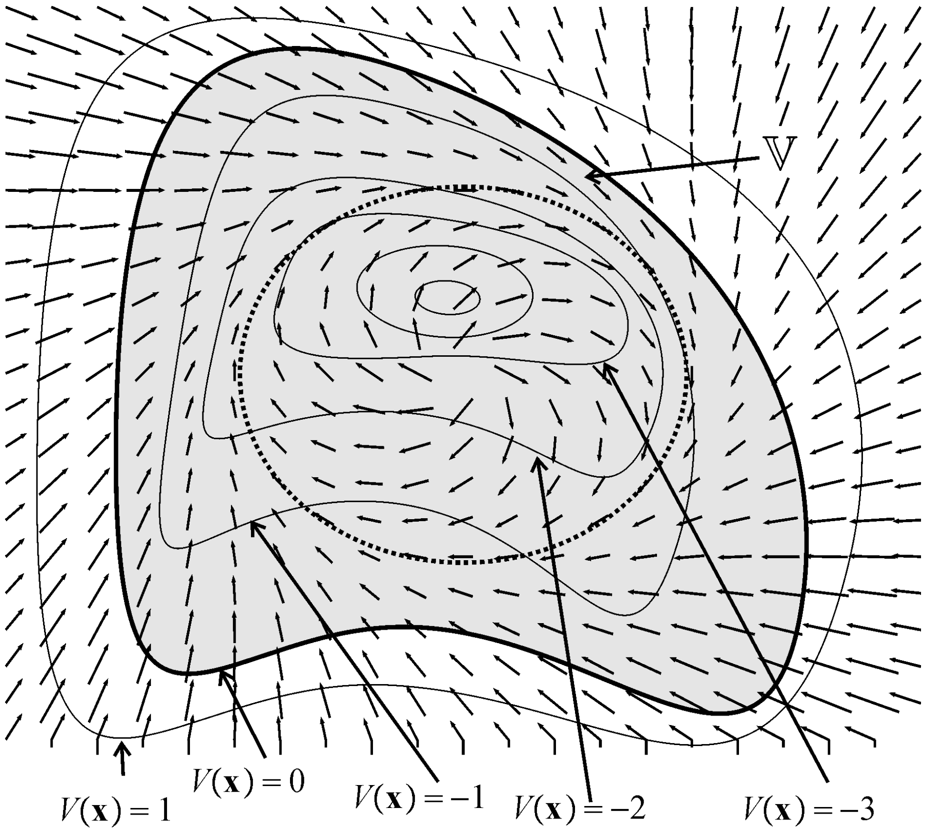
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0} \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0}. \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x})$. The system is V-stable if we have

$$\dot{V}(\mathbf{x}) < 0 \text{ if } V(\mathbf{x}) \geq 0.$$


$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

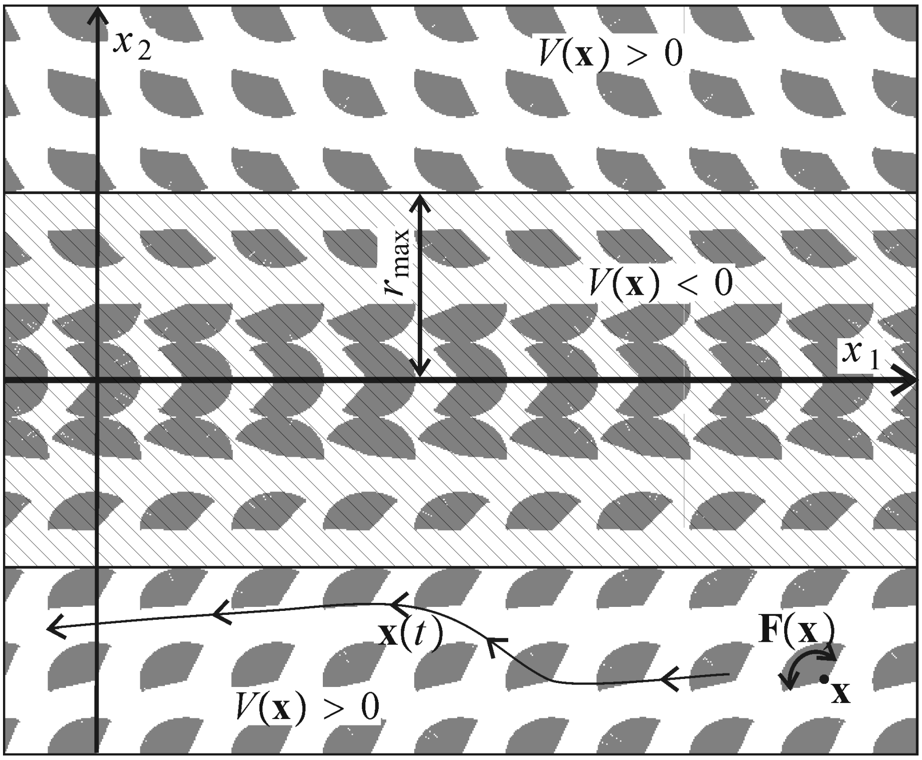
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

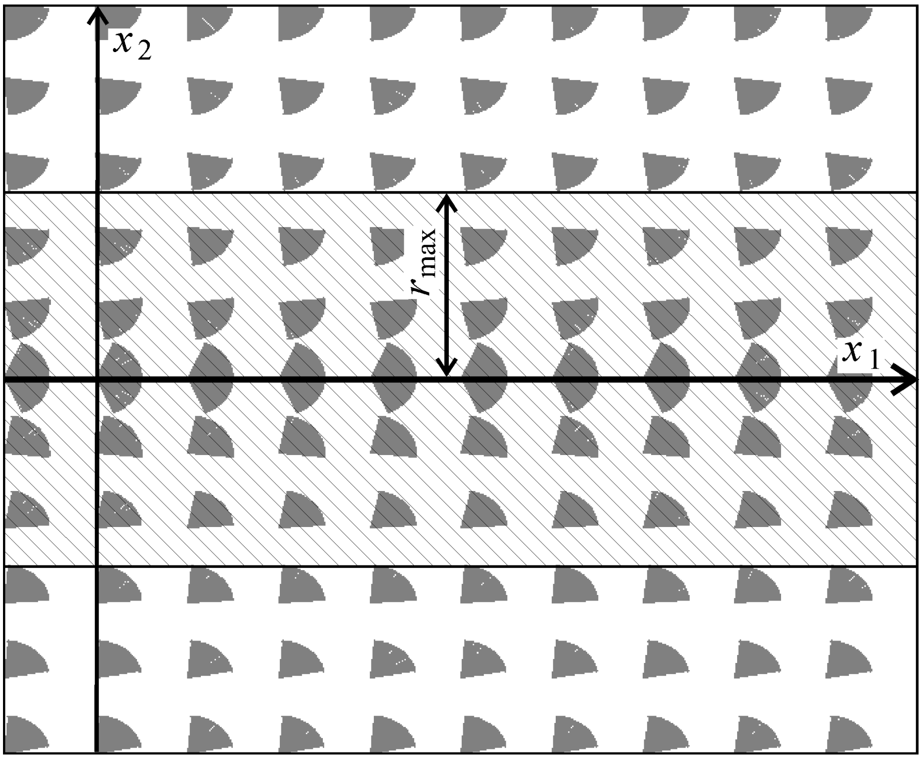
Theorem. We have

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x})\,.\mathbf{a} \geq 0 \\ \quad \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ \quad V(\mathbf{x}) \geq 0 \end{array} \right. \quad \text{inconsistent} \;\; \Leftrightarrow \;\; \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) \text{ is } V\text{-stable}$$



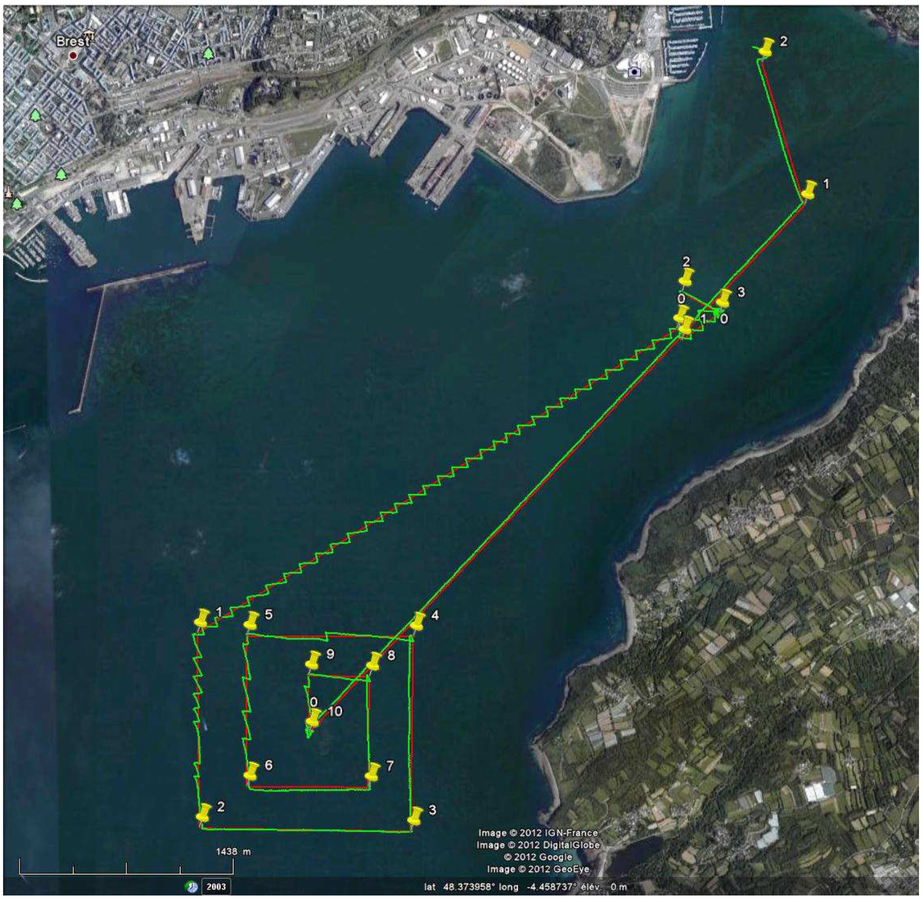
Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$


x_2
r_{max}
x_1

5.4 Experimental validation

Brest


Brest
2
1
2
0
3
1
0
1
5
4
9
8
0
10
6
7
2
3
1438 m
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
2003
lat 48.373958° long -4.458737° élév. 0 m

Brest-Douarnenez. January 17, 2012, 8am


Brest
BR D98 505
YAMAHA


miniji
F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google
Start
vaimos.txt -...
Logs
sftp://root...
Command Pr...
Command Pr...
data
Interface po...
VCtrl (Runni...
VAIMOS aut...
Google Earth
FR
98%
01:42

Middle of Atlantic ocean

350 km made by Vaimos in 53h, September 6-9, 2012.

Consequence.

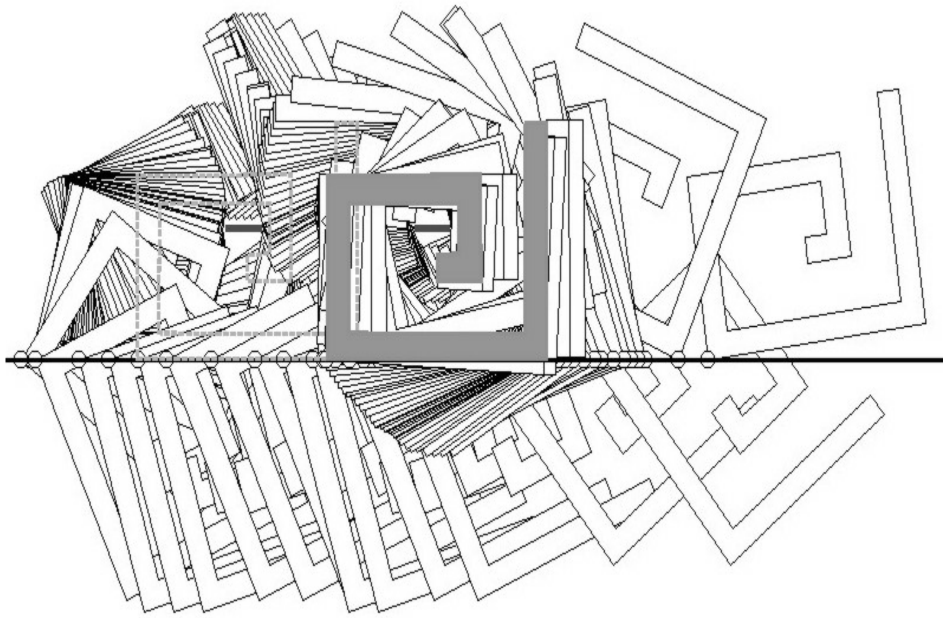
It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.

6 Optimization in robotics

6.1 Path planning



6.2 Control

A mobile robot is described by

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} & = & \mathbf{g}(\mathbf{x}) \end{cases}$$

where $\mathbf{u} \in \mathbb{R}^{\dim \mathbf{u}}$ are the inputs, $\mathbf{x} \in \mathbb{R}^{\dim \mathbf{x}}$ the state, $\mathbf{y} \in \mathbb{R}^{\dim \mathbf{y}}$ are the variables to be controlled.

In operating conditions we have $\dot{\mathbf{x}} = \mathbf{0}$.

$$
\left\{
\begin{array}{ccc}
\dot{x} & = & v\cos\theta + p_1 a \cos\psi \\
\dot{y} & = & v\sin\theta + p_1 a \sin\psi \\
\dot{\theta} & = & \omega \\
\dot{v} & = & \frac{f_s \sin\delta_s - f_r \sin u_1 - p_2 v}{p_9} \\
\dot{\omega} & = & \frac{f_s(p_6 - p_7\cos\delta_s) - p_8 f_r \cos u_1 - p_3\omega}{p_{10}} \\
f_s & = & p_4 a \sin\left(\theta - \psi + \delta_s\right) \\
f_r & = & p_5 v \sin u_1 \\
\gamma & = & \cos\left(\theta - \psi\right) + \cos\left(u_2\right) \\
\delta_s & = & \left\{ \begin{array}{cc} \pi - \theta + \psi & \text{if } \gamma \leq 0 \\ sign\left(\sin\left(\theta - \psi\right)\right).u_2 & \text{otherwise.} \end{array} \right.
\end{array}
\right.
$$

If $\dim \mathbf{u} > \dim \mathbf{y}$ the robot is overactuated.

We then want to maximize some performance criteria $\mathbf{h}(\mathbf{x})$.

In operating conditions ($\dot{\mathbf{x}} = \mathbf{0}$), the optimization problem is

$$\hat{\mathbf{h}}\left(\bar{\mathbf{y}}\right) = \max_{\bar{\mathbf{u}},\bar{\mathbf{x}}} \mathbf{h}\left(\bar{\mathbf{x}}\right) \qquad \text{s.t.} \quad \left\{ \begin{array}{rcl} \mathbf{0} & = & \mathbf{f}\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}\right) \\ \bar{\mathbf{y}} & = & \mathbf{g}\left(\bar{\mathbf{x}}\right). \end{array} \right.$$

Or equivalently

$$\hat{\mathbf{h}}\left(\bar{\mathbf{y}}\right) = \max_{\bar{\mathbf{u}},\bar{\mathbf{x}}} \mathbf{v} \qquad \text{s.t.} \quad \left\{ \begin{array}{rcl} \mathbf{0} & = & \mathbf{f}\left(\bar{\mathbf{x}},\bar{\mathbf{u}}\right) \\ \bar{\mathbf{y}} & = & \mathbf{g}\left(\bar{\mathbf{x}}\right) \\ \mathbf{v} & = & \mathbf{h}\left(\bar{\mathbf{x}}\right) \end{array} \right.$$

with $\dim \mathbf{v} = \dim \mathbf{u} - \dim \mathbf{y}$.

Often $\mathbf{x}$ can be eliminated symbolically:

$$\underbrace{\left\{\begin{array}{lcl} \mathbf{0} & = & \mathbf{f}\left(\bar{\mathbf{x}}, \bar{\mathbf{u}}\right) \\ \bar{\mathbf{y}} & = & \mathbf{g}\left(\bar{\mathbf{x}}\right) \\ \mathbf{v} & = & \mathbf{h}\left(\bar{\mathbf{x}}\right) \end{array}\right.}_{\substack{\dim \mathbf{x} + \dim \mathbf{u} \text{ equations} \\ \dim \mathbf{x} + 2\dim \mathbf{u} \text{ variables}}} \quad \Leftrightarrow \quad \underbrace{\boldsymbol{\psi}\left(\bar{\mathbf{v}}, \bar{\mathbf{u}}, \bar{\mathbf{y}}\right) = \mathbf{0}}_{\substack{\dim \mathbf{u} \text{ equations} \\ 2\dim \mathbf{u} \text{ variables}}}$$

We get

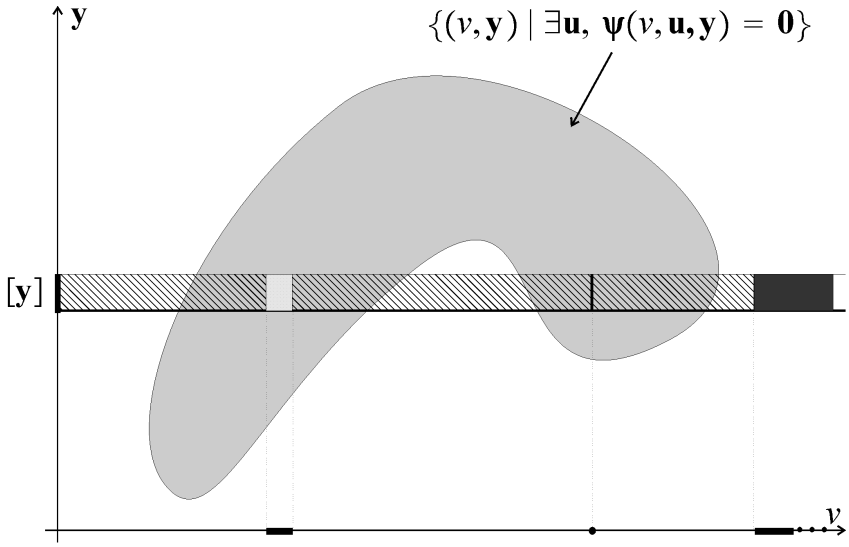
$$\hat{\mathbf{h}}\left(\bar{\mathbf{y}}\right) = \max_{\bar{\mathbf{u}},\bar{\mathbf{v}}} \bar{\mathbf{v}} \qquad \text{s.t. } \boldsymbol{\psi}\left(\bar{\mathbf{v}}, \bar{\mathbf{u}}, \bar{\mathbf{y}}\right) = \mathbf{0}.$$

6.3 Resolution

We assume that $\dim v = 1$ (mono-objective case).

$$\hat{h}(\mathbf{y}) = \max_{\mathbf{u}\in\mathbb{R}^{\dim\mathbf{u}}, v\in\mathbb{R}} v$$
$$\text{s.t. } \boldsymbol{\psi}(v, \mathbf{u}, \mathbf{y}) = \mathbf{0}$$

with $\dim\boldsymbol{\psi} = \dim\mathbf{u}$.



$\{(v,\mathbf{y}) \mid \exists \mathbf{u},\ \boldsymbol{\psi}(v,\mathbf{u},\mathbf{y}) = \mathbf{0}\}$
$\mathbf{y}$
$[\mathbf{y}]$
v

We need an inner test to prove that

$$[v] \times [\mathbf{y}] \subset \underbrace{\{(v, \mathbf{y}), \exists \mathbf{u}, \boldsymbol{\psi}(v, \mathbf{u}, \mathbf{y}) = \mathbf{0}\}}_{\mathbb{S}_{v\mathbf{y}}}.$$

6.4 Newton inner test

Given a box $[\mathbf{p}]$, we need to be able to prove that

$$\forall \mathbf{p} \in [\mathbf{p}], \exists \mathbf{u} \in [\mathbf{u}], \boldsymbol{\psi}(\mathbf{u}, \mathbf{p}) = \mathbf{0}.$$

with $\dim \boldsymbol{\psi} = \dim \mathbf{u}$.

Parametric interval Newton method

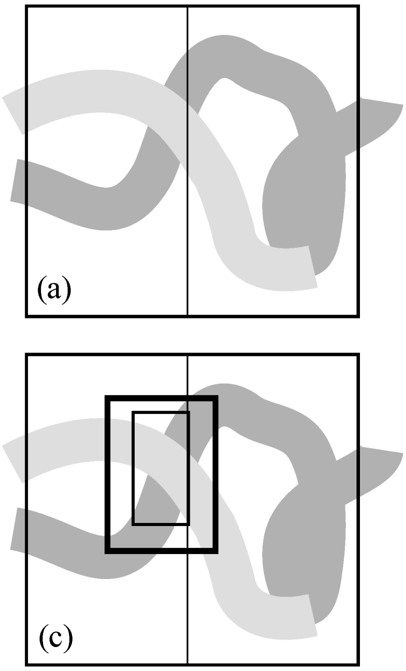
Define

$$\mathcal{N}_{\boldsymbol{\psi}}\left([\mathbf{u}]\,,[\mathbf{p}]\right)=\widehat{\mathbf{u}}-\left[\frac{\partial\boldsymbol{\psi}}{\partial\mathbf{u}}\left([\mathbf{u}]\,,[\mathbf{p}]\right)\right]^{-1}\cdot[\boldsymbol{\psi}]\left(\widehat{\mathbf{u}},[\mathbf{p}]\right).$$

We have

$$\mathcal{N}_{\boldsymbol{\psi}}\left([\mathbf{u}],[\mathbf{p}]\right)\subset[\mathbf{u}]\;\Rightarrow\;\forall\mathbf{p}\in[\mathbf{p}]\,,\;\exists!\mathbf{u}\in[\mathbf{u}]\,,\boldsymbol{\psi}(\mathbf{u},\mathbf{p})=\mathbf{0}.$$

Epsilon inflation



6.5 Sailboat

Two inputs: the sail angle u_1 and the rudder angle u_2.

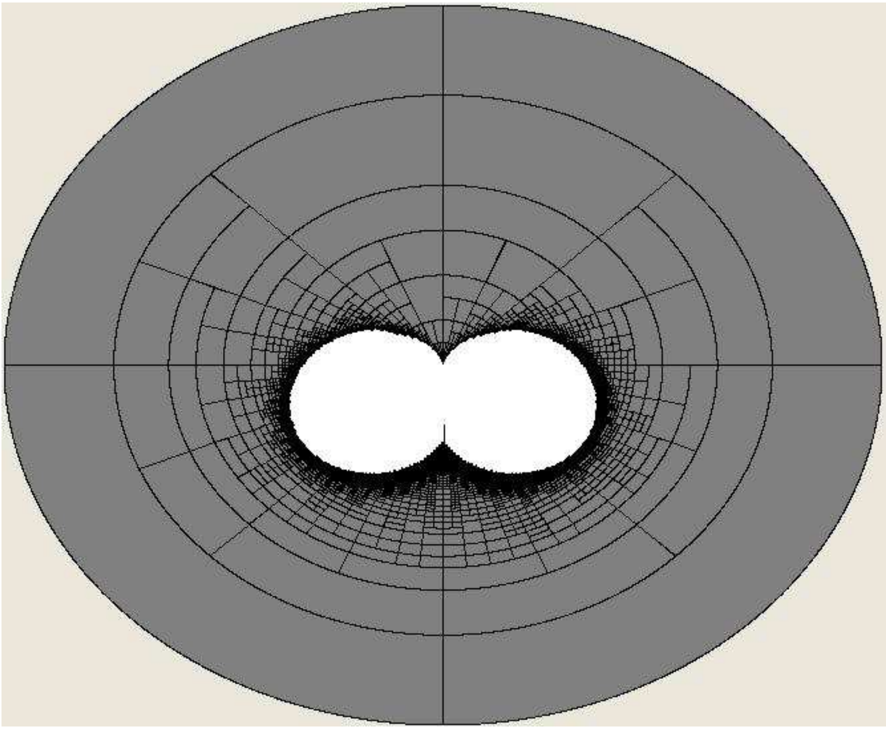
The output is the heading θ.

The variable to be maximized is v.

The optimization problem is

$$\hat{v}(\theta) = \max_{\mathbf{u}\in\mathbb{R}^2, v\in\mathbb{R}} \quad v$$

$$\text{s.t.} \quad \begin{cases} 0 &= \quad \sin u_1 (\cos(\theta + u_1) - v \sin u_1) - v \sin^2 u_2 - v \\ 0 &= (1 - \cos u_1)(\cos(\theta + u_1) - v \sin u_1) - v \frac{\sin 2u_2}{2}. \end{cases}$$



References.

L. Jaulin (2009), A nonlinear set-membership approach for the localization and map building of an underwater robot using interval constraint propagation, IEEE Transactions on Robotics.

L. Jaulin and F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE Transaction on Robotics.

G. Chabert and L. Jaulin (2009), Contractor programming. Artificial Intelligence. Vol. 173.