Computing a guaranteed approximation of the viability kernel
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What is viability?

System $S$ defined by:

$$\dot{x} = f(x, u),$$
$$f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$$

A state $x$ is viable if at least one evolution of $S$ from $x$ can stay indefinitely in a set of constraint $K$. The viability kernel of $K$ under $S$ noted $Viab_S(K)$ is the set that contains every viable state.
Why viability?

Example: management of renewable resources, economics, robotics,...
Is it possible to avoid the wall?
Why viability?

Example: management of renewable resources, economics, robotics,...

Is it possible to avoid the wall?
1. Attraction domains

2. Polygon expansion technique

3. Capture basin

4. Examples
   - Car on the hill
   - Double integrator

5. Conclusion
Plan

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Attraction domain of a system

Attraction domains of $S$ are interesting for viability, if they are located in $\mathbb{K}$.
We consider a dynamical system $\dot{x} = f(x, u)$, $U$ the set of possible control and $K$ a closed subset of $\mathbb{R}^n$.
Let $L \in C^1(K, \mathbb{R})$, and $B_L(r) = \{x \in \mathbb{R}^n| L(x) \leq r\}$, with $r \in \mathbb{R}^+$. If $B_L(r) \subseteq K$ and $\forall x \in B_L(r)$, $\exists u \in U$ such as $\langle f(x, u), \nabla L(x) \rangle \leq 0$, then $B_L(r) \subseteq \text{Viab}_S(K)$. 
Illustration of the theorem
Lyapunov function

**Definition**

A function $L: \mathbb{R}^n \to \mathbb{R}$ is said to be of Lyapunov for the dynamical system $\dot{x} = f(x)$ if:

1. $V(0) = 0$.
2. $\forall x \in \mathbb{R}^n \setminus \{0\}, \, V(x) > 0$.
3. $\forall x \in \mathbb{R}^n, \, \langle f(x), \nabla V(x) \rangle \leq 0$. 
How to find a lyapunov function

We choose a particular control $u \in \mathbb{U}$. $S_u$: $\dot{x} = f(x, u)$ is an autonomous system.

$x^*$ is an equilibrium point of $S_u \iff f(x^*, u) = 0$.

- Linearize $S_u$ around $x^*$, we get $S_u^{x^*}$ defined by $\dot{\tilde{x}} = A\tilde{x}$, $\tilde{x} = x - x^*$.
- Solve $A^T W + WA = -I$, where $W$ is the unknown amount.
- Check whether $W$ is positive definite.
- If $W$ is positive definite, then $\frac{1}{2} \tilde{x}^T W \tilde{x}$ is a Lyapunov function for the linear system, and $x^*$ is stable.

If we do not find a Lyapunov function for $S_u^{x^*}$, we compute the linear system $S_{ctrl}$ for which $x^*$ is a stable equilibrium point.
Lyapunov function and linearized system $S_{\mathbf{u}}^{x^*}$
Lyapunov function and autonomous system $S_u$
Lyapunov function and system $S$
Choose a control $u \in U$.
Find an equilibrium point $x^* \in K$.
Linearize $S_u$ around $x^*$.
Try to compute a Lyapunov function of $S_{x^*}^u$.
If no function found, compute $S_{ctrl}$.
Try to compute a Lyapunov function of $S_{ctrl}$.
Find $r \in \mathbb{R}^+$ such as conditions of the theorem are met.
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Theorem

Let \( P = \{p_1, p_2, \ldots, p_n\} \) be a polygon included in \( K \). We suppose \( \{p_1, p_2, \ldots, p_n\} \) sorted in clockwise order.

If

\[
\forall i, \forall x \in \text{segment } [p_i, p_{i+1}], \exists u \in U, \quad det(p_{i+1} - p_i, f(x, u)) \leq 0.
\]

Then \( P \subseteq Viab_S(K) \)
Theorem illustration
Theorem illustration

\[ \mathbf{p}_2 - \mathbf{p}_1 \]

\[ f(x, u) \]
Polygon expansion algorithm

1. Find a polygon $P \subseteq \text{Viab}_S(K)$.
2. Compute a larger polygon $P'$.
3. If $P' \subseteq \text{Viab}_S(K)$, $P = P'$, go to 1.
4. Else compute another polygon $P'$, go to 3.
Polygon expansion algorithm

Example polygon $P$
Polygon expansion algorithm
Polygon expansion algorithm

$P$
Polygon expansion algorithm
Polygon expansion algorithm
Polygon expansion algorithm
Plan

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2. Polygon expansion technique
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5. Conclusion
The capture basin of a set $T \subseteq K$ viable in $K$ noted $Capt_{S}(K, T)$ is composed of every states $x$ such as $S$ can reach $T$ from $x$ in a finite time without leaving $K$. 
Theorem on the viability of a capture basin

**Theorem**

Let $S$ a dynamical system, $\mathbb{K}$ a closed subset of the state space of $S$ and $T \subset \mathbb{K}$.

If $T$ is viable in $\mathbb{K}$, then $\text{Capt}_S(\mathbb{K}, T)$ is viable in $\mathbb{K}$.

The set $V_{in} = T \cup \text{Capt}_S(\mathbb{K}, T)$ is an inner approximation of $\text{Viab}_S(\mathbb{K})$. 
Over approximation of the viability kernel

- We try to find an over approximation of $\text{Viab}_S(K)$ to get an enclosure of $\text{Viab}_S(K)$.
- If $\forall u \in U$, $S$ cannot stay in $K$ from a state $x \in K$, then $x \notin \text{Viab}_S(K)$.
Guaranteed integration of a box [Chaputot, 2015]
Under approximation algorithm
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Plan

1. Attraction domains
2. Polygon expansion technique
3. Capture basin
4. Examples
   - Car on the hill
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The landscape is represented by the parametric function

\[ g : s \rightarrow \frac{-\frac{1.1}{1.2} \cos(1.2s) + \frac{1.2}{1.1} \cos(1.1s)}{2} \]

State vector: \( \mathbf{x} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \)

Evolution function:

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -9.81 \sin\left(\frac{dg}{dx_1}(x_1)\right) - 0.7x_2 + u
\end{cases}
\]

\( u \in [-2, 2] \)

The car must stay on the landscape, i.e. \( s \in [0, 12] \)
Viable sets computed with $u = 0$. Computation time: 25 sec.
Car on the hill

Result explained

Diagram showing attraction domains and polygon expansion technique.
Polygons are initialized with viable sets computed previously.

Computation time: 45 sec.
Car on the hill

Result of inner approximation algorithm

Computation time $\approx 60$ minutes
Car on the hill

Result of over approximation algorithm

Computation time ≈ 30 minutes
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Double integrator equations

Evolution function:

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{align*} \quad u \in [-1, 1] \]

Constraints:

- \( x_1 \in [-5, 5] \)
- \( x_2 \in [-5, 5] \)
Double integrator

Results of viable set characterization algorithm

Viable sets computed with $u = 0$. Computation time: 40 sec.
Result of inner approximation algorithm

Computation time: 5 minutes.
Result of over approximation algorithm

Computation time: 7 minutes.
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Conclusion

- We are able to deal with many viability problems in a guaranteed way.
- The system must have at least one equilibrium point.
- We can deal with 2D problems, but inner and over approximation algorithms are not efficient for higher dimensional problems.
- We approached viability problem with new methods based on the study of the frontier of closed sets.