Programmation par contraintes appliquée à la robotique mobile.

Montpellier, November 21, 2013.
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1 Interval arithmetic

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$, a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}]\,, f\,(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Interval arithmetic

$$\begin{aligned}
[-1, 3] + [2, 5] &= [1, 8], \\
[-1, 3] \cdot [2, 5] &= [-5, 15], \\
\operatorname{abs}([-7, 1]) &= [0, 7]
\end{aligned}$$

If f is given

Algorithm f(in: $\mathbf{x} = (x_1, x_2, x_3)$, out: y)

$z := |x_3| + x_1$;
for $k := 0$ to 100
    $z := (\cos x_2) \cdot (\sin(z) + k x_3)$;
next;
$y := \sin(z \cdot x_1)$;

Its interval extension is

Algorithm $[f]$(in: $[\mathbf{x}] = ([x_1], [x_2], [x_3])$, out: $[y]$)

$[z] := |[x_3]| + [x_1]$;
for $k := 0$ to 100
    $[z] := (\cos [x_2]) \cdot (\sin ([z]) + k * [x_3])$;
next;
$[y] := \sin([z] \cdot [x_1])$;

Theorem (Moore, 1970)

$$[f]\left([\mathbf{x}]\right) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}]\,, f\left(\mathbf{x}\right) \geq 0$$

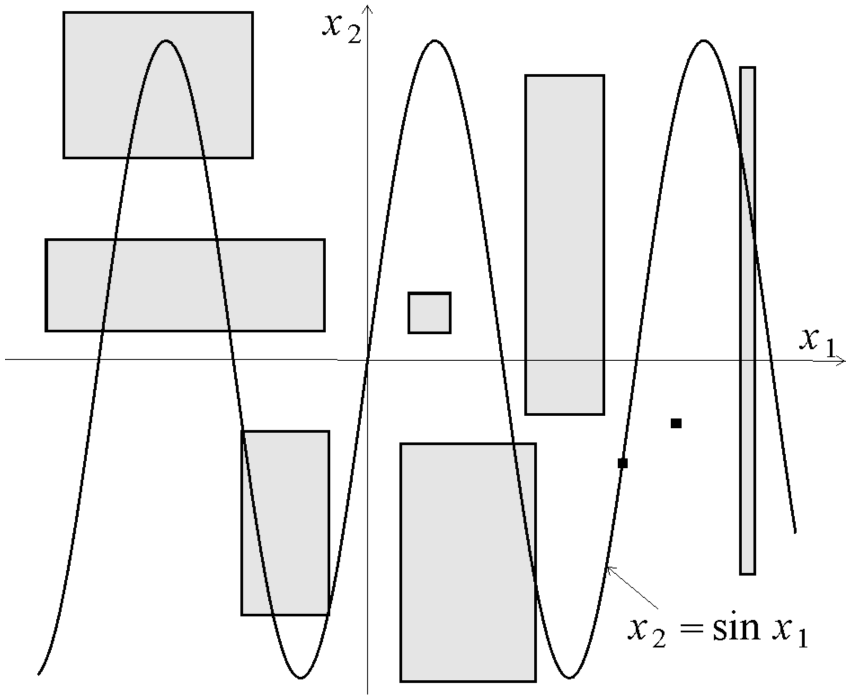
2 Contractors

The operator $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

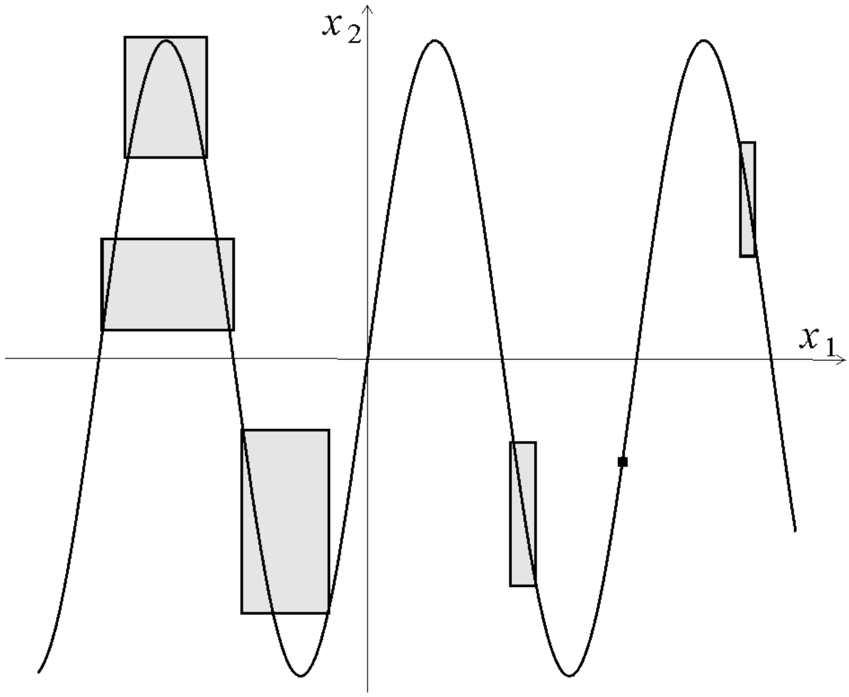
$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

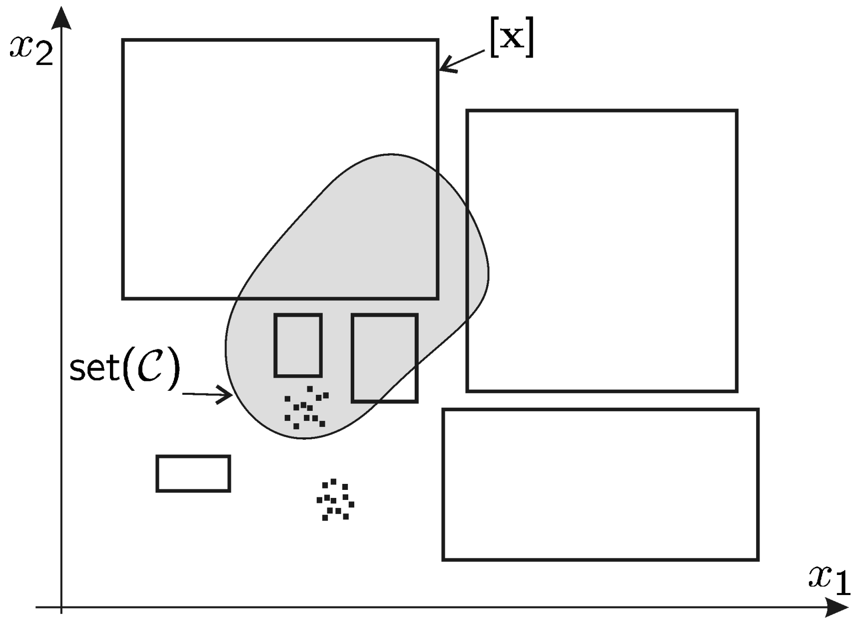
Example. Consider the primitive equation:

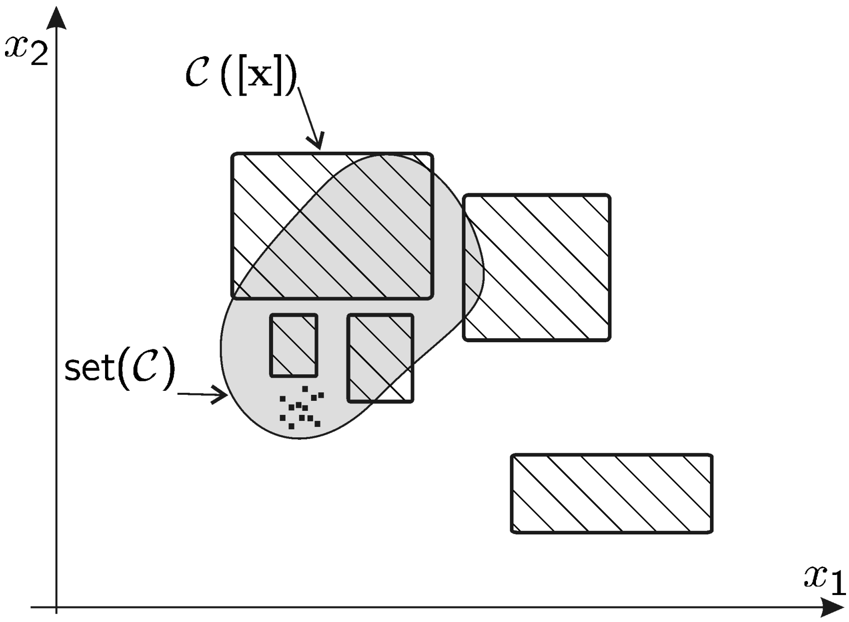
$$x_2 = \sin x_1.$$



x_2
x_1
$x_2 = \sin x_1$


x_2
x_1


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
set($\mathcal{C}$)
x_1

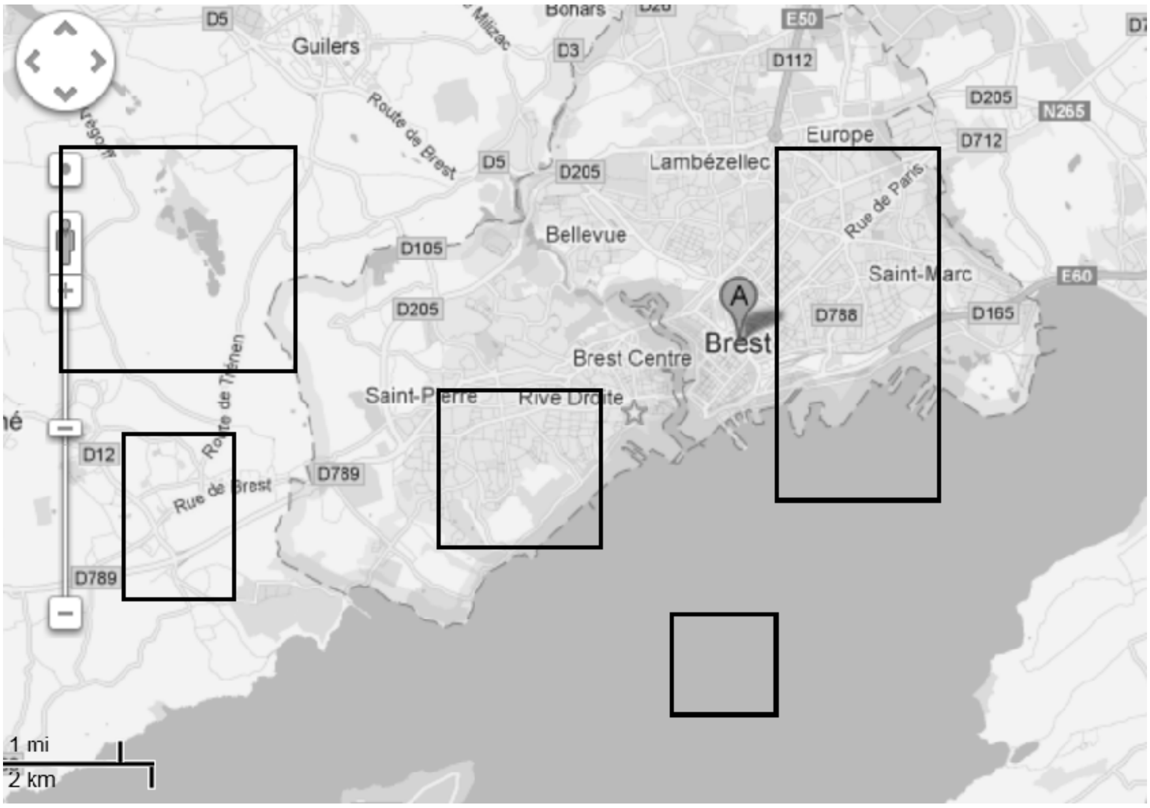
$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *idempotent* if	$\mathcal{C}\left(\mathcal{C}([\mathbf{x}])\right) = \mathcal{C}([\mathbf{x}])$

Contractor algebra

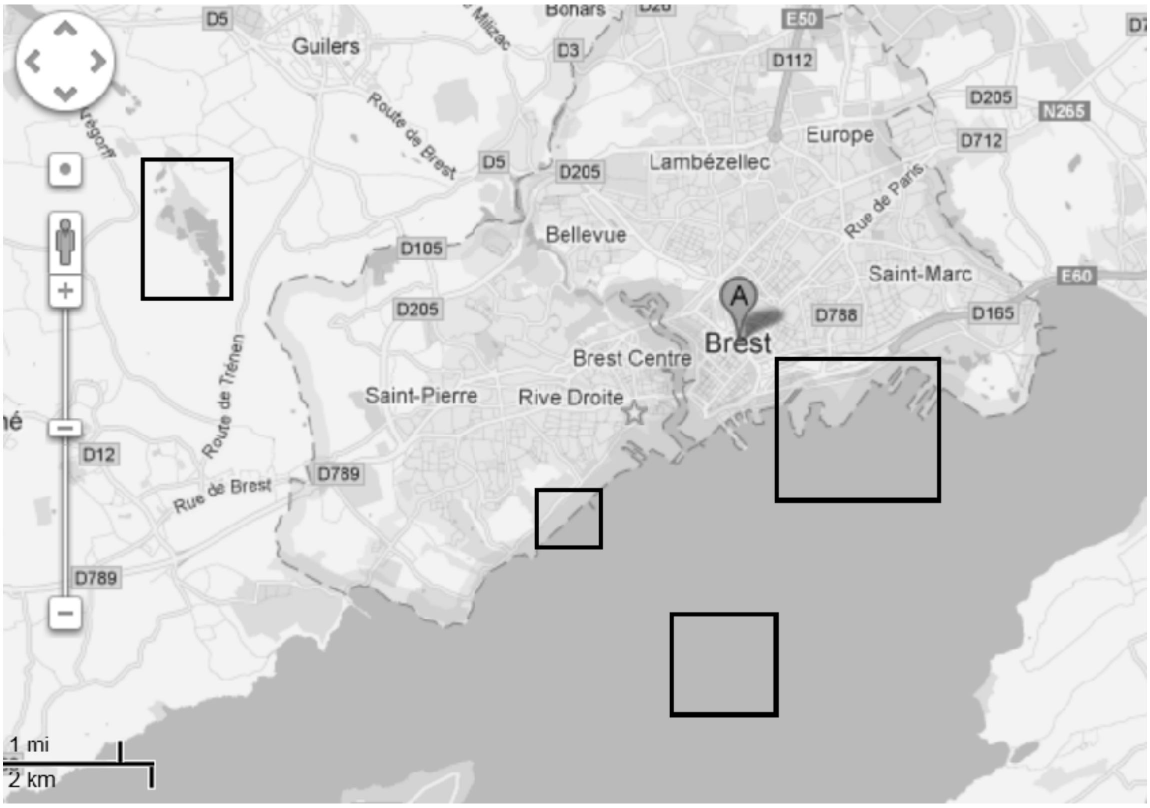
intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^{\infty} \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \ldots$

Contractor associated with a database

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Route de Brest
D3
D112
E50
D205
N265
D712
Europe
Lambézellec
D205
D5
Bellevue
D105
D205
Saint-Marc
E60
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
D12
Route de Trénen
Rue de Brest
D789
D789
Rue de Paris
1 mi
2 km



Guilers
Route de Brest
Lambézellec
Europe
Bellevue
Rue de Paris
Saint-Marc
Brest
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Rive Droite
Route de Trénen
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D5
D3
D112
E50
D205
D712
N265
D105
E60
D788
D165
D12
D789
1 mi
2 km

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$\begin{aligned}
x_3 = x_1 + x_2 \Rightarrow \quad & x_3 \in \quad [x_3] \cap ([x_1] + [x_2]) \quad // \text{ forward} \\
x_1 = x_3 - x_2 \Rightarrow \quad & x_1 \in \quad [x_1] \cap ([x_3] - [x_2]) \quad // \text{ backward} \\
x_2 = x_3 - x_1 \Rightarrow \quad & x_2 \in \quad [x_2] \cap ([x_3] - [x_1]) \quad // \text{ backward}
\end{aligned}$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

3 Solver

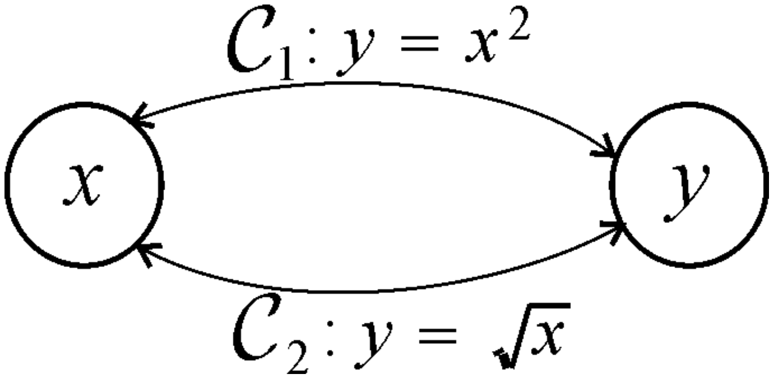
Example. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

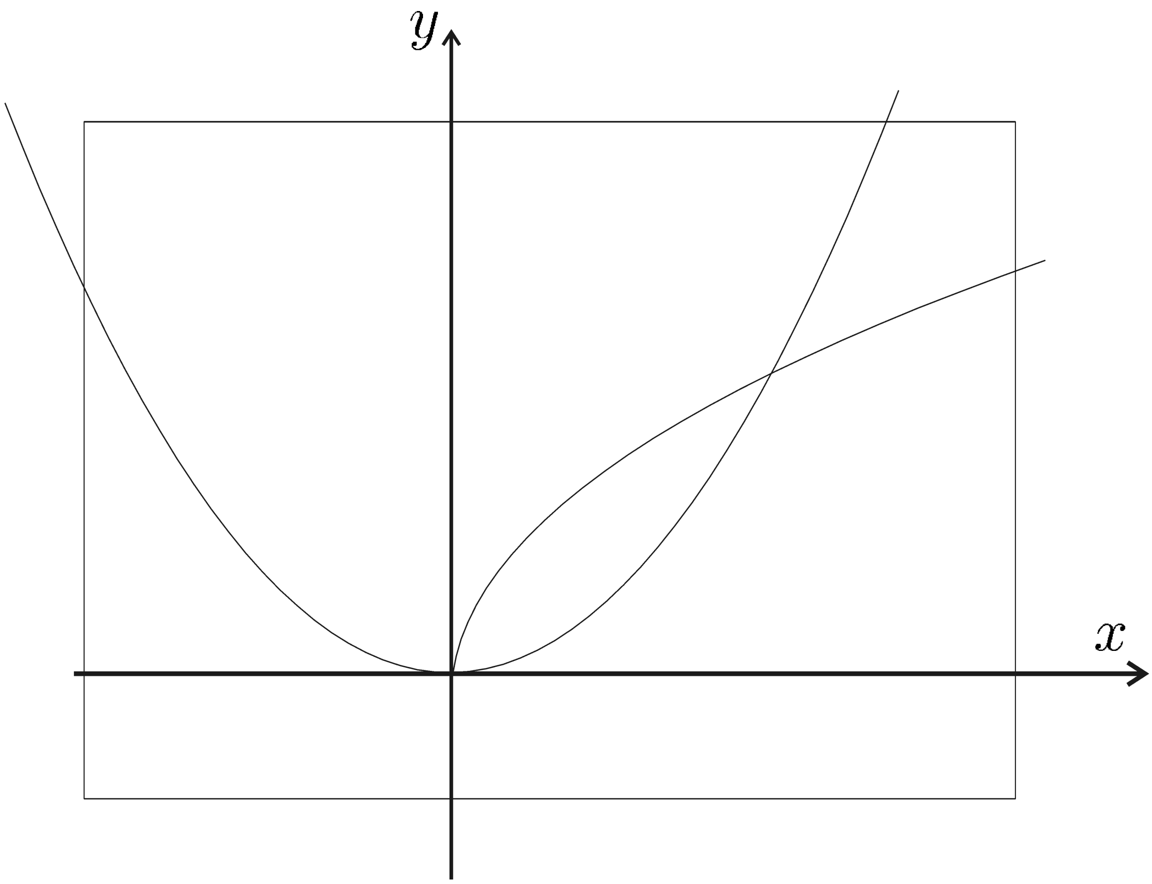
We build two contractors

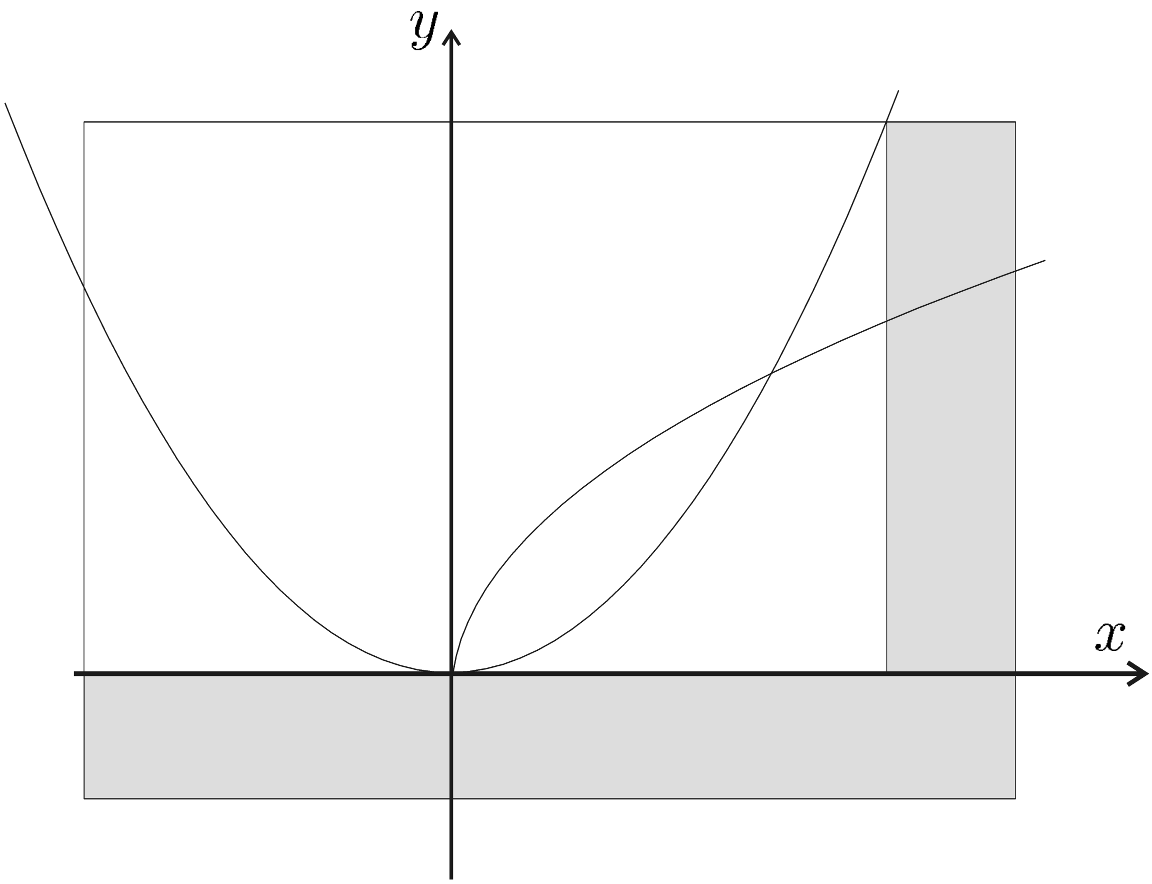
$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated with } y = x^2$$

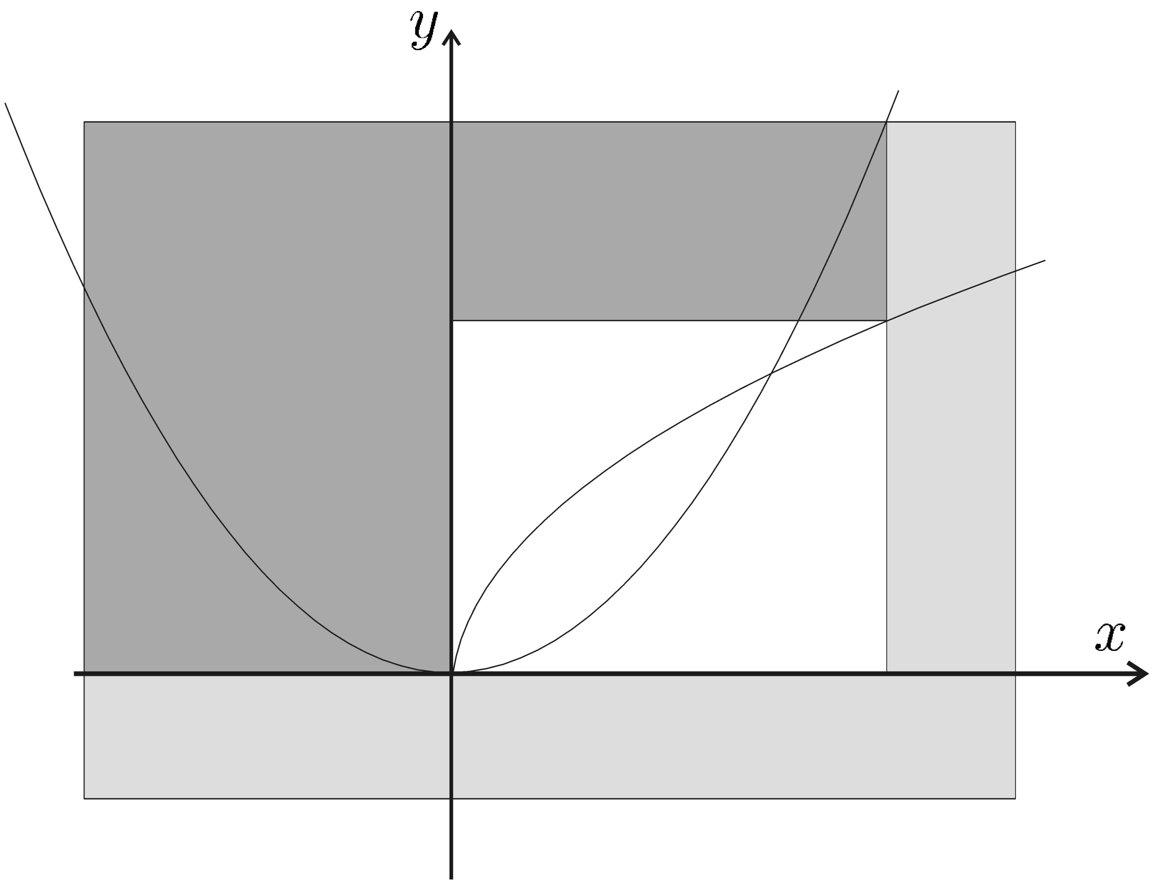
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated with } y = \sqrt{x}$$

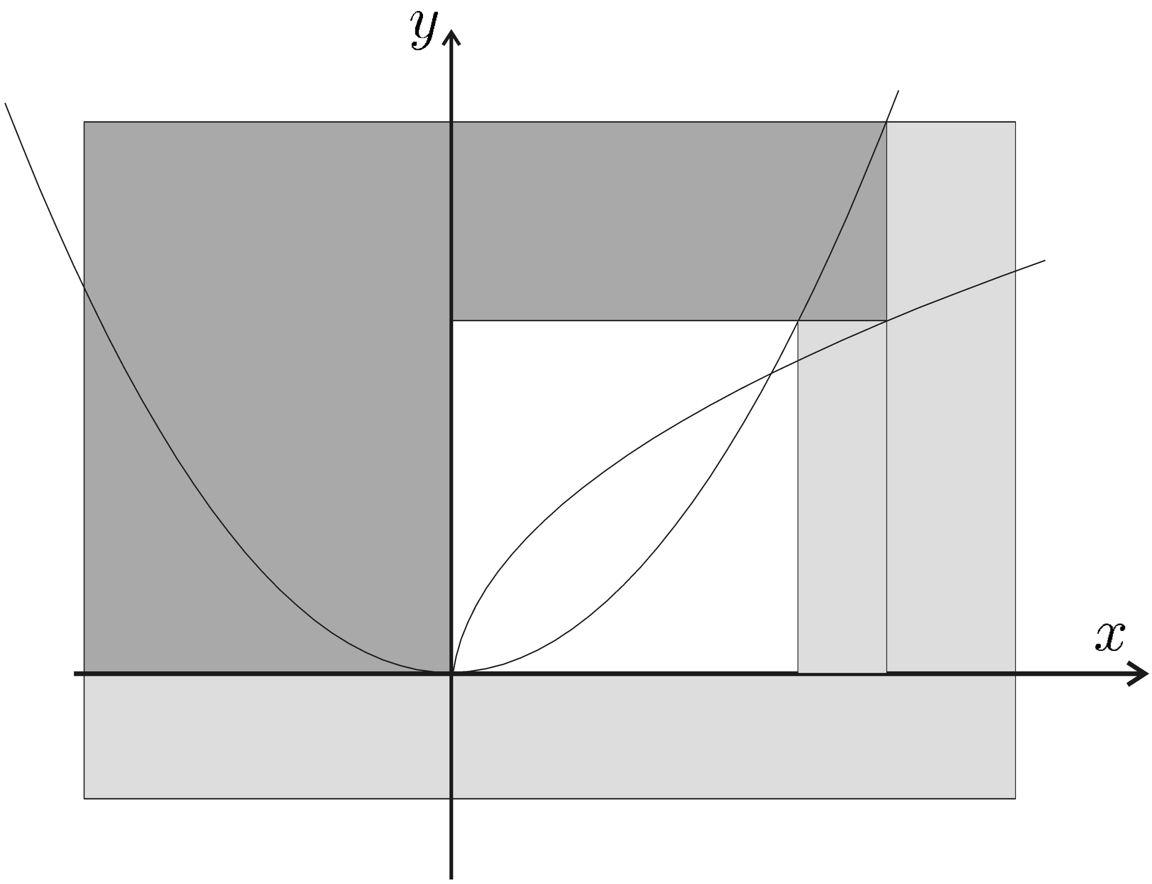


Contractor graph

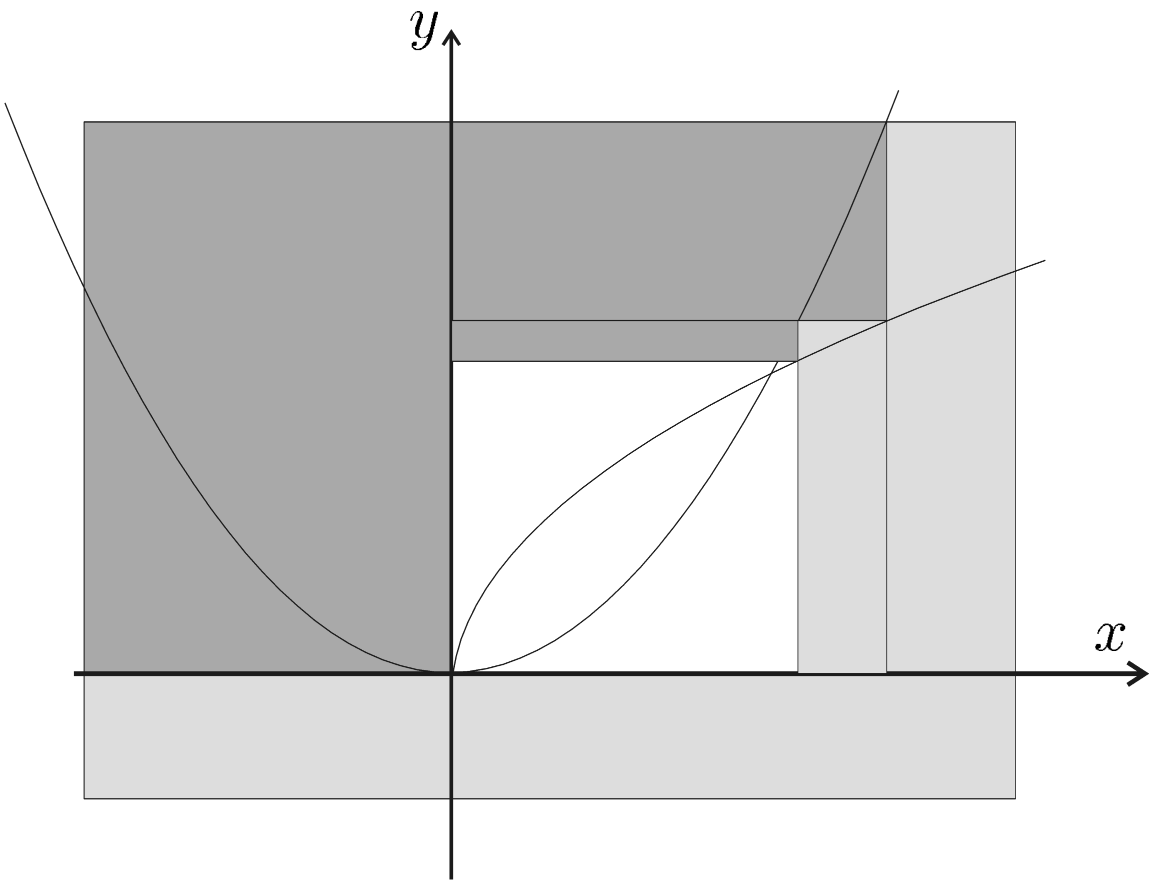

y
x

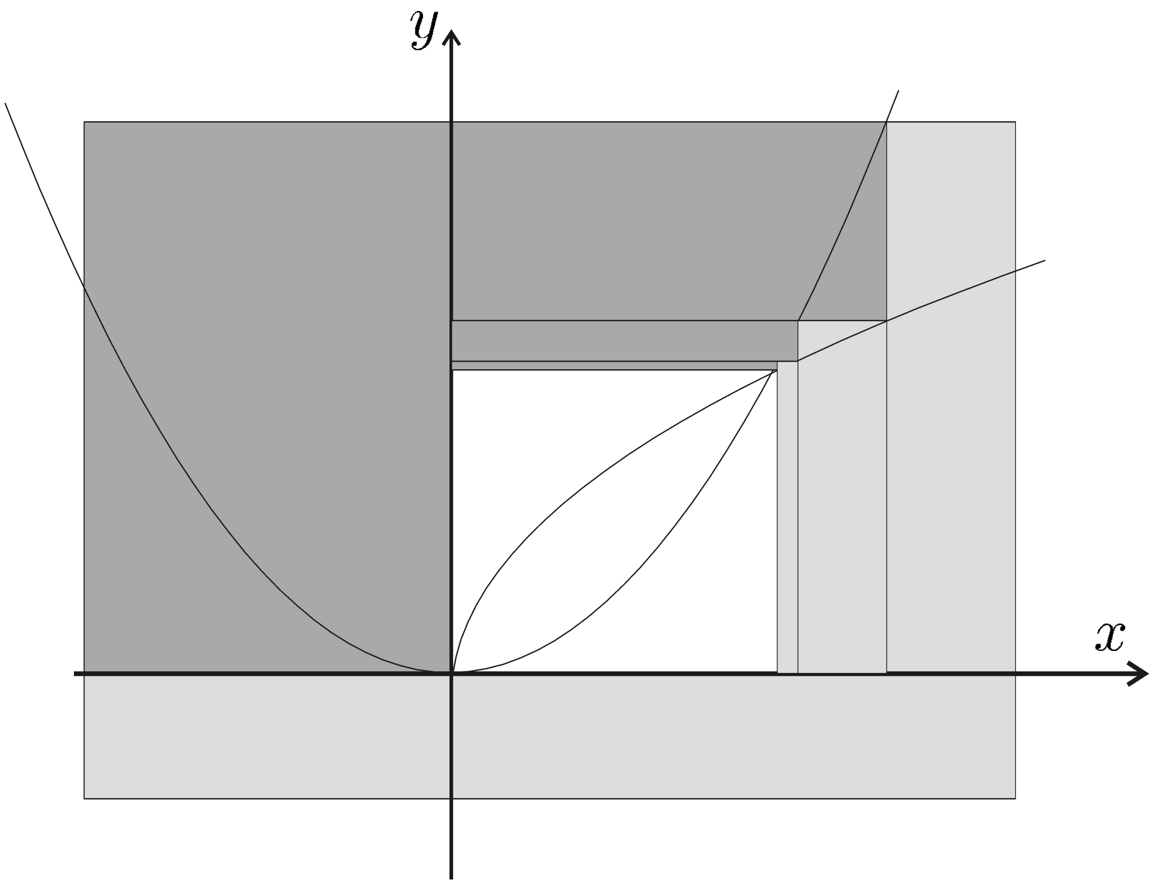

y
x

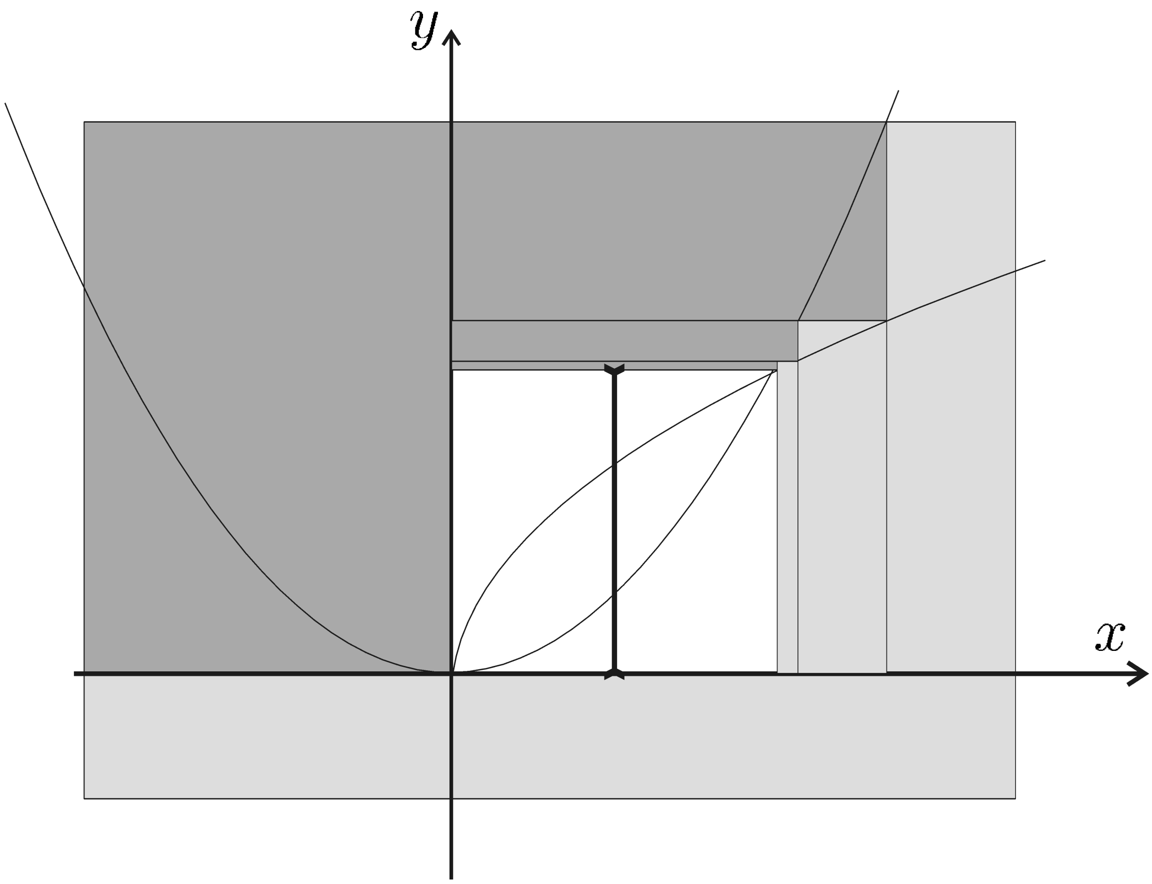

y
x

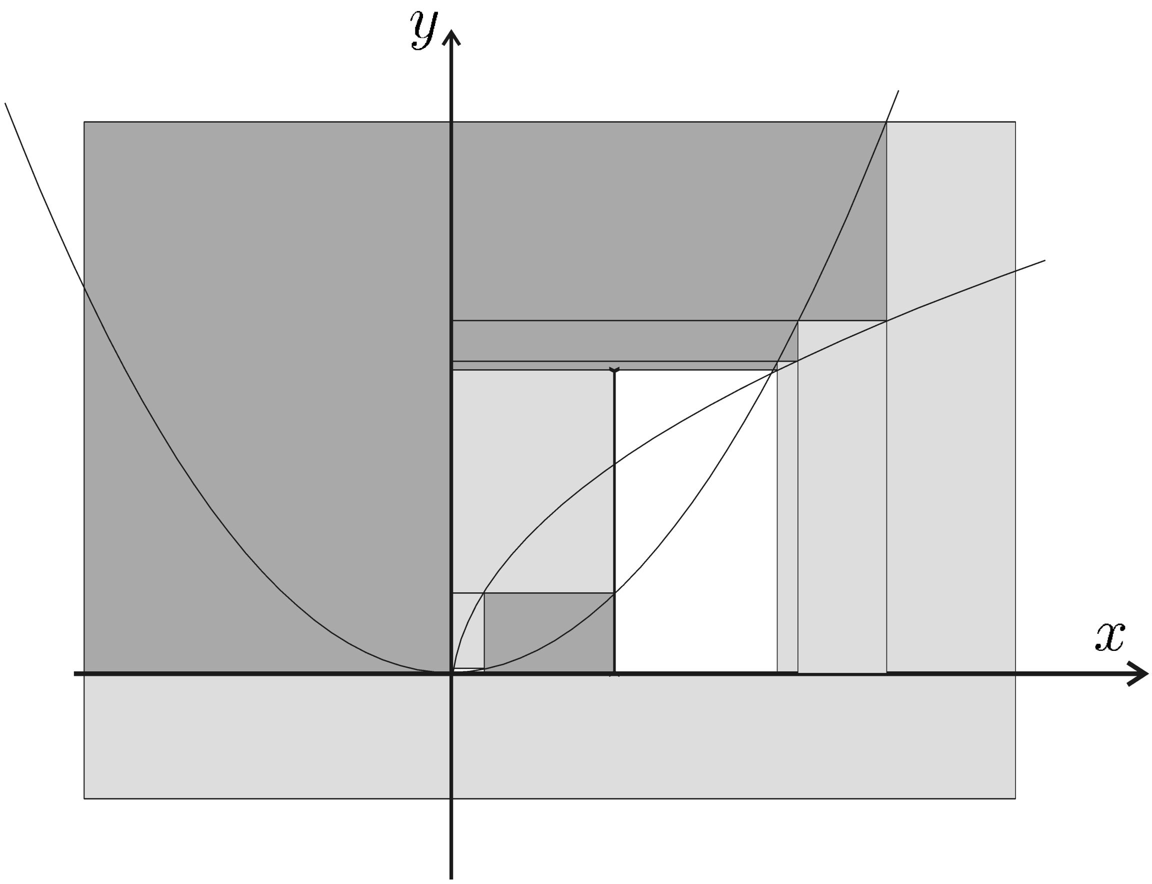


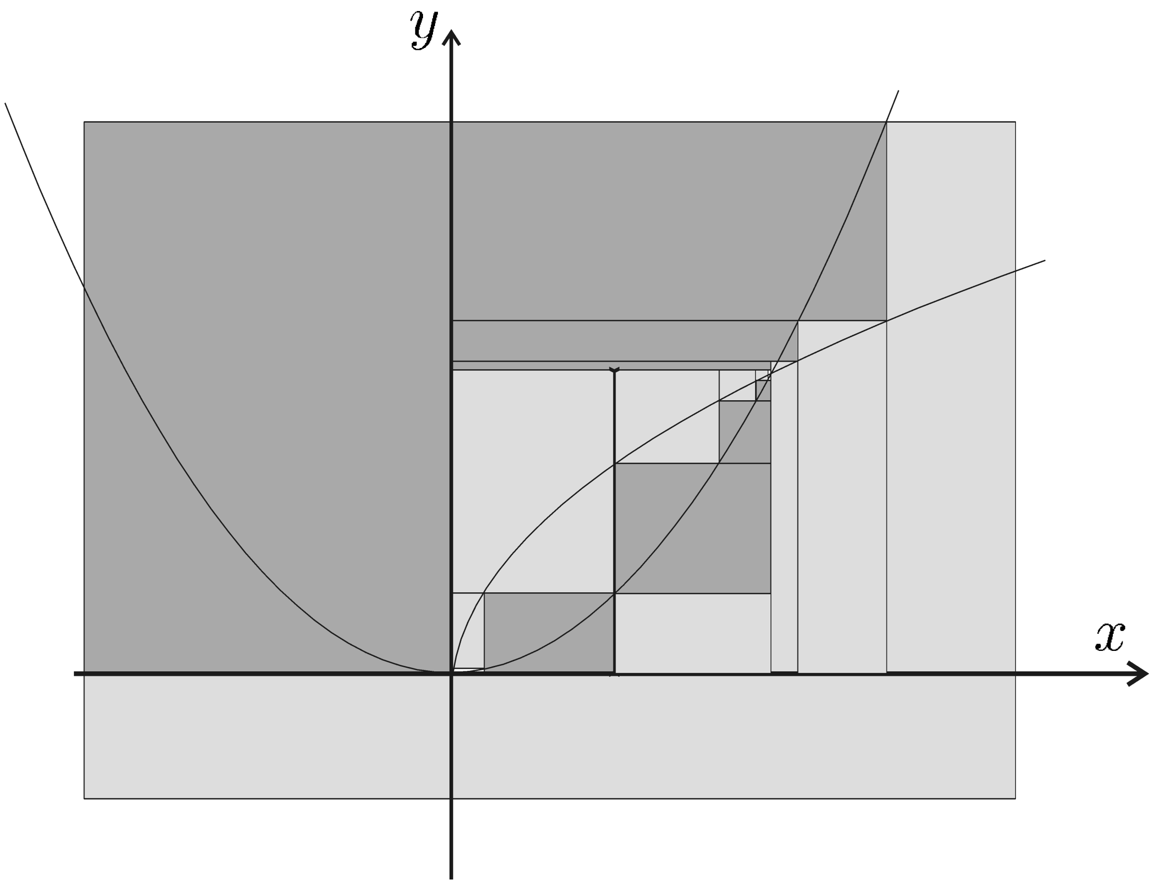
y
x


y
x


y
x


y
x


y
x



y
x

4 Vaimos

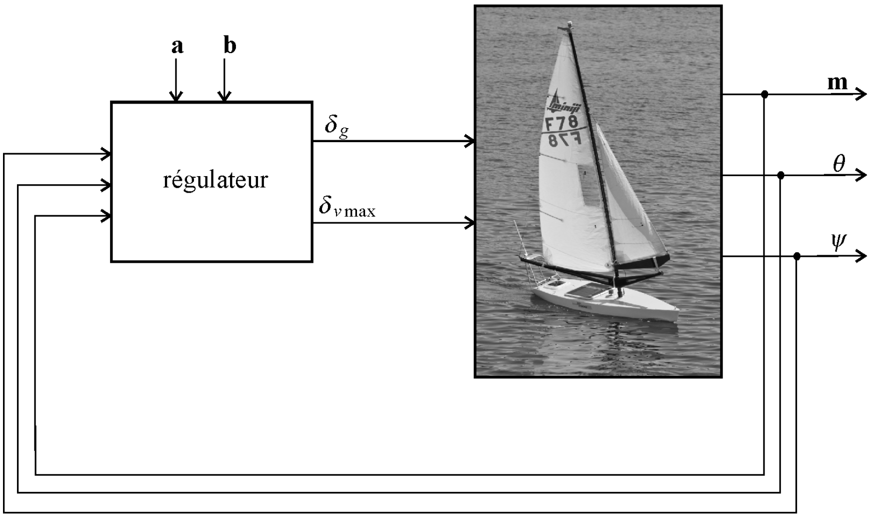
Vaimos (IFREMER and ENSTA)

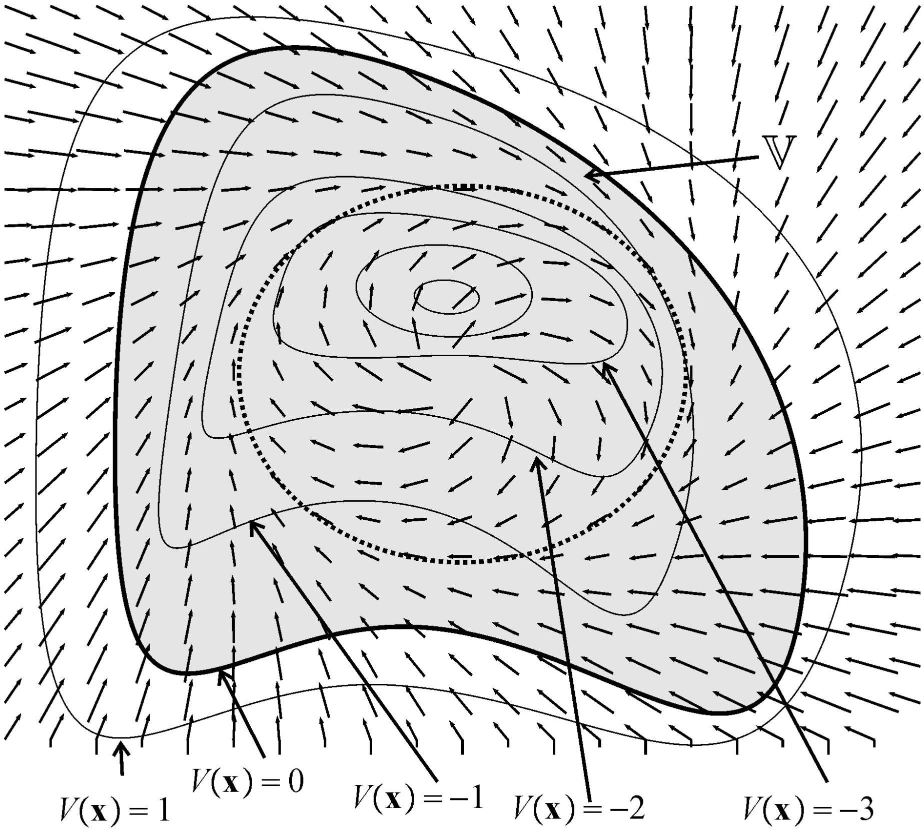
The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}(\mathbf{x})$, the robot satisfies

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$





𝕍
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

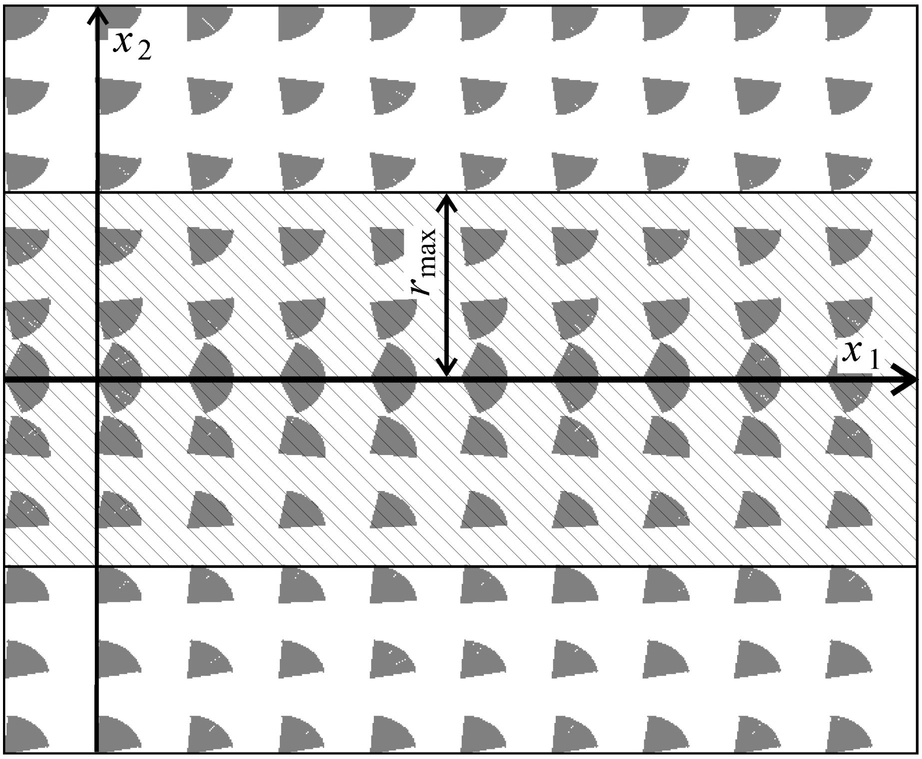
With uncertainty, the robot satisfies.

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

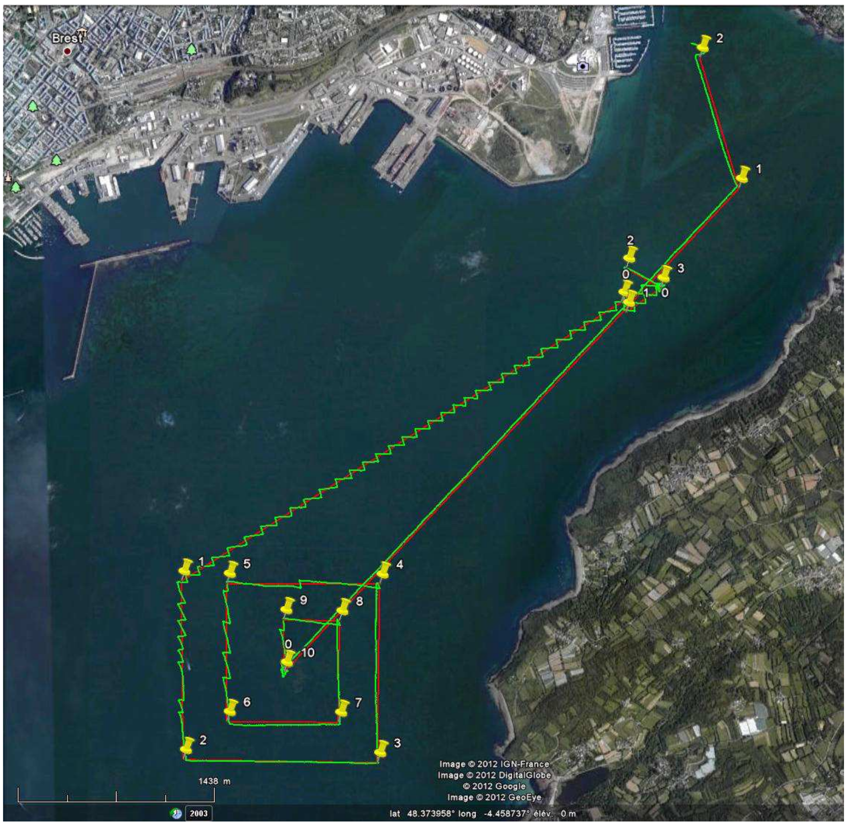
which is *a differential inclusion*.

Theorem. We have

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{a} \geq 0 \\ \quad \mathbf{a} \in \mathbf{F}\left(\mathbf{x}\right) \\ \quad V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} \in \mathbf{F}\left(\mathbf{x}\right) \text{ is } V\text{-stable}$$



x_2
r_{max}
x_1


Brest
1438 m
2003
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
lat 48.373958° long -4.458737° élev 0 m

Brest-Douarnenez. January 17, 2012, 8am


Brest
PRO
BR D98 505
YAMAHA

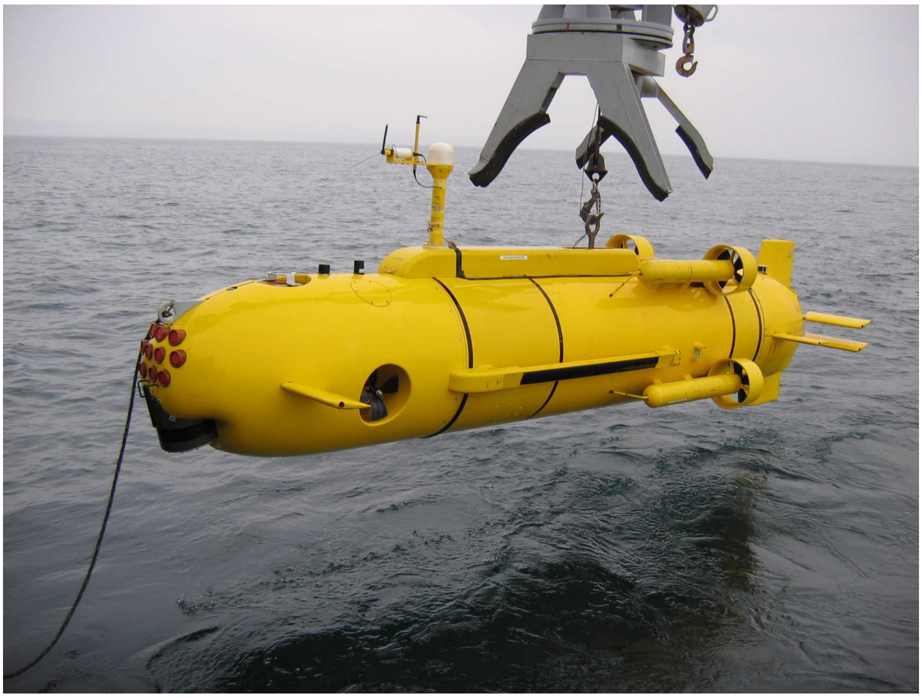

miniji
F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google

Middle of Atlantic ocean: 350 km in 53h, September 6-9, 2012.

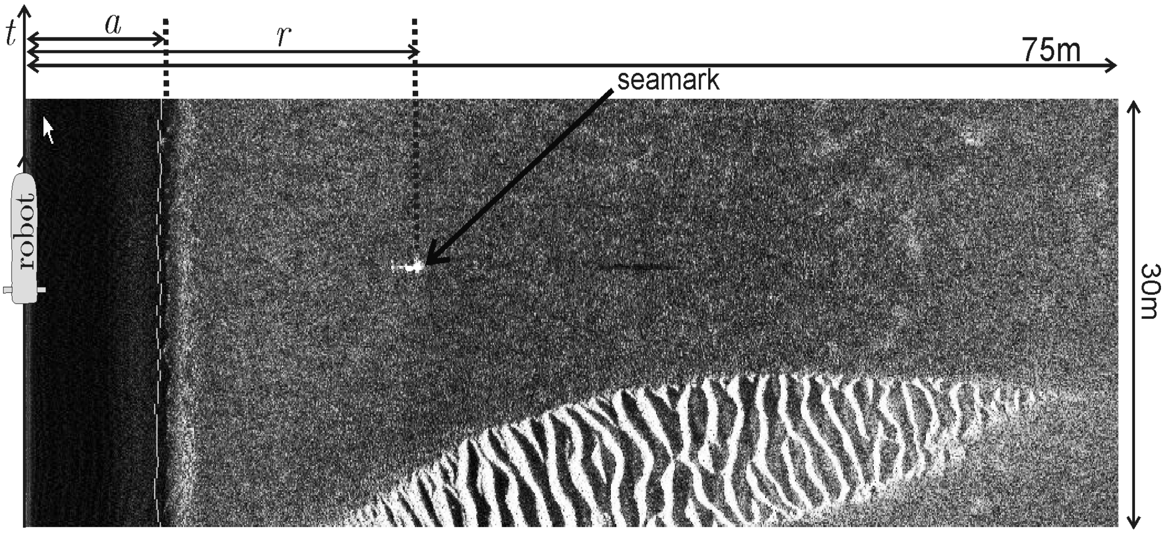
5 Underwater SLAM

Context : Mines detection.



Dans le monde sous marin, les amers sont souvent indistingables et partiellement observables.

Voir film CLAPOT.



Mine detection with SonarPro

6 Formalization

Robot: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}),\ \mathbf{x}(0) = \mathbf{0}$.
Marks $\mathcal{M} = \{\mathbf{m}(1), \mathbf{m}(2), \dots\} \subset \mathbb{R}^2$.

Context: indistinguishable point marks that are partially observable

Our SLAM is a *chicken and egg* problem of cardinality three:
(i) if the map and the associations are known, we have localization problem,
(ii) if the trajectory and the associations are known, we have a mapping problem
(iii) if the trajectory and the map are known we have an association problem.

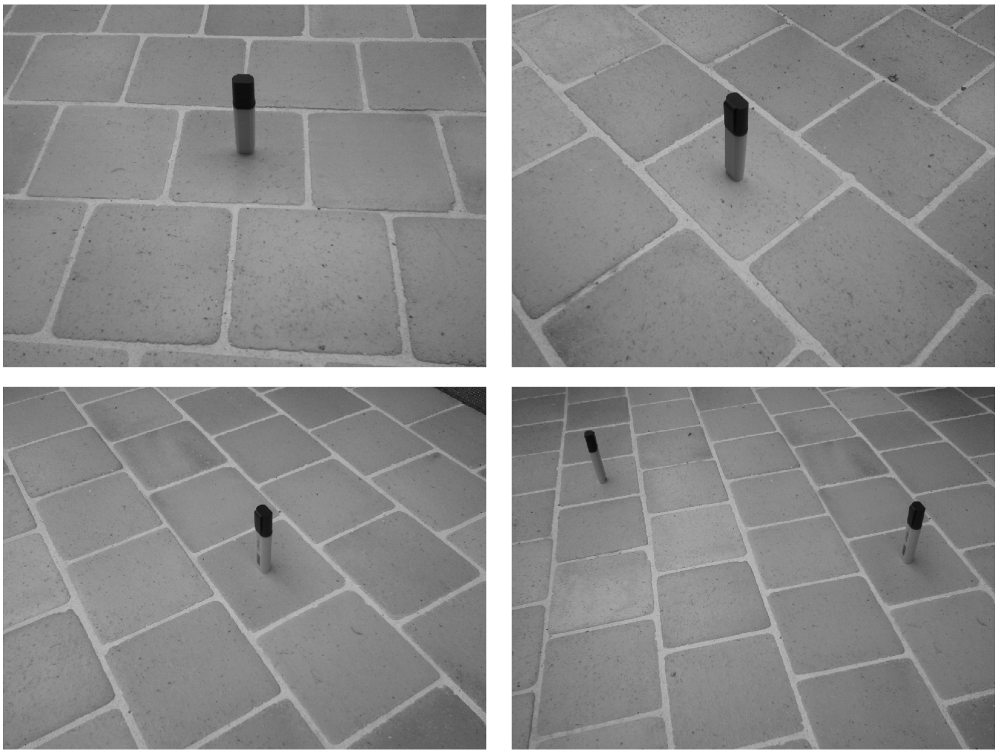
The unknown variables have an heterogenous nature:

(i) marks $\mathbf{m}(j) \in \mathbb{R}^2$

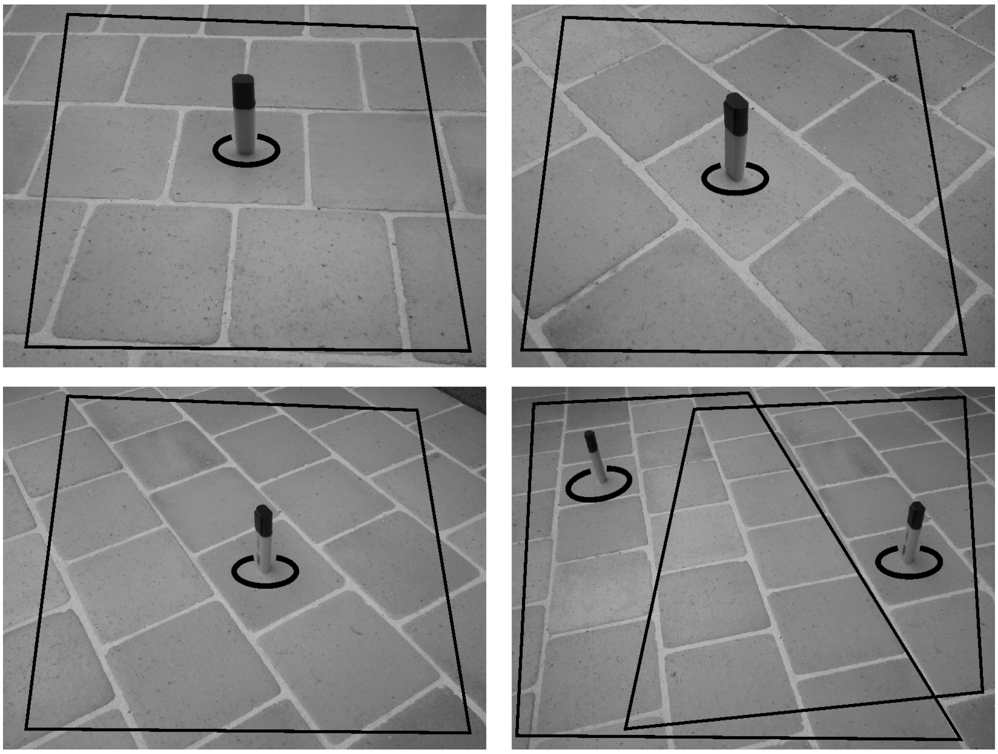
(ii) trajectory $\mathbf{x}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$,

(iii) the free space $\mathbb{F} \in \mathcal{P}\left(\mathbb{R}^2\right)$

(iv) the data associations is a graph $\mathcal{G}$.



A *sector* $\mathbb{H}$ is a subset of $\mathbb{R}^2$ which contains a single mark.



Our SLAM problem:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) & \text{(evolution equation)} \\ (t_i, \mathcal{H}_i(\mathbf{x})) & \text{(sector list)} \end{cases}$$

where $t \in [0, t_{\text{max}}]$, $\mathbf{u}(t) \in [\mathbf{u}](t)$.

Each set $\mathcal{H}_i(\mathbf{x}(t_i)) \subset \mathbb{R}^2$ contains a unique mark.

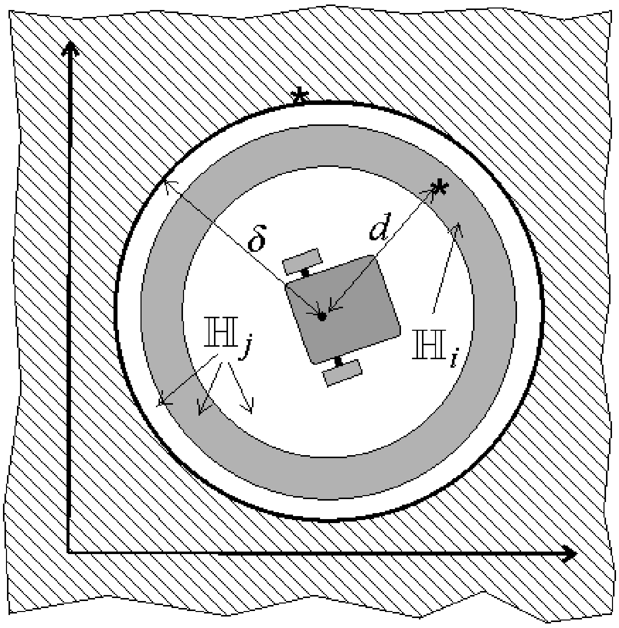
We have an egocentric representation.

We define $\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$.

Example 1. A robot moving in a plane and located at (x_1, x_2) . At t_3 the robot detects a unique mark at a distance $d \in [4, 5]$. We have

$$\mathcal{H}_3(\mathbf{x}) = \left\{\mathbf{a} \in \mathbb{R}^2 | (x_1 - a_1)^2 + (x_2 - a_2)^2 \in [16, 25]\right\}.$$

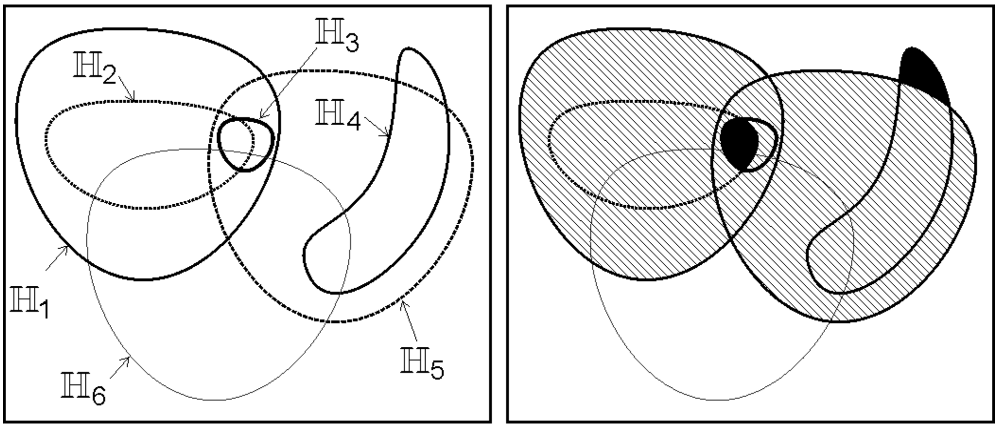
Example 2. We have two sectors $\mathbb{H}_i$ and $\mathbb{H}_j$.
Since $\mathbb{H}_i \subset \mathbb{H}_j$, $\mathbb{H}_j \backslash \mathbb{H}_i$ has no mark. Thus we can associate $\mathbb{H}_i$ with $\mathbb{H}_j$.



Theorem. Define the free space as $\mathbb{F} = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \mathbf{p} \notin \mathcal{M} \right\}$. Consider m sectors $\mathbb{H}_1, \ldots, \mathbb{H}_m$. Denote by $\mathbf{a}(i)$ the mark in $\mathbb{H}_i$. We have

(i) $\mathbb{H}_i \subset \mathbb{H}_j \Rightarrow \mathbf{a}(i) = \mathbf{a}(j)$
(ii) $\mathbb{H}_i \cap \mathbb{H}_j = \emptyset \Rightarrow \mathbf{a}(i) \neq \mathbf{a}(j)$
(iii) $\mathbb{H}_i \subset \mathbb{H}_j \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}$.

Example.



The two black zones contain a single mark and no mark exists in the hatched area.

Association graph. Consider m detections $\mathbf{a}(1), \ldots, \mathbf{a}(m)$. The *association graph* is the graph with nodes $\mathbf{a}(i)$ such that $\mathbf{a}(i) \to \mathbf{a}(j)$ means that $\mathbf{a}(i) = \mathbf{a}(j)$.

7 Generalized contractors

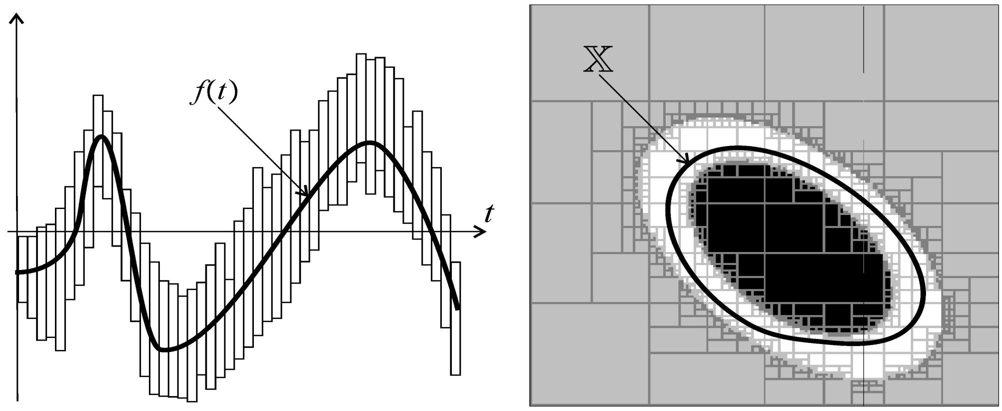
7.1 Lattices

A *lattice* $(\mathcal{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds.
The *join*: $x \vee y$.
The *meet*: $x \wedge y$.

An *interval* $[x]$ of a complete lattice $\mathcal{E}$ is a subset of $\mathcal{E}$ which satisfies

$$[x] = \{x \in \mathcal{E} \mid \wedge [x] \leq x \leq \vee [x]\} \, .$$

Both $\emptyset$ and $\mathcal{E}$ are intervals of $\mathcal{E}$.



An interval function (or tube) and a set interval

7.2 Contractors

A CSP is composed of
variables $\{x_1, \ldots, x_n\}$,
constraints $\{c_1, \ldots, c_m\}$ and
domains $\{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$.

The domains $\mathbb{X}_i$ should belong to a lattice $(\mathcal{L}_i, \subset)$.

Here domains are
(i) subsets of $\mathbb{R}^n$ for the location of the marks,
(ii) tubes for the unknown trajectory and
(iii) intervals of subsets of $\mathbb{R}^n$ for the sectors and the free space.

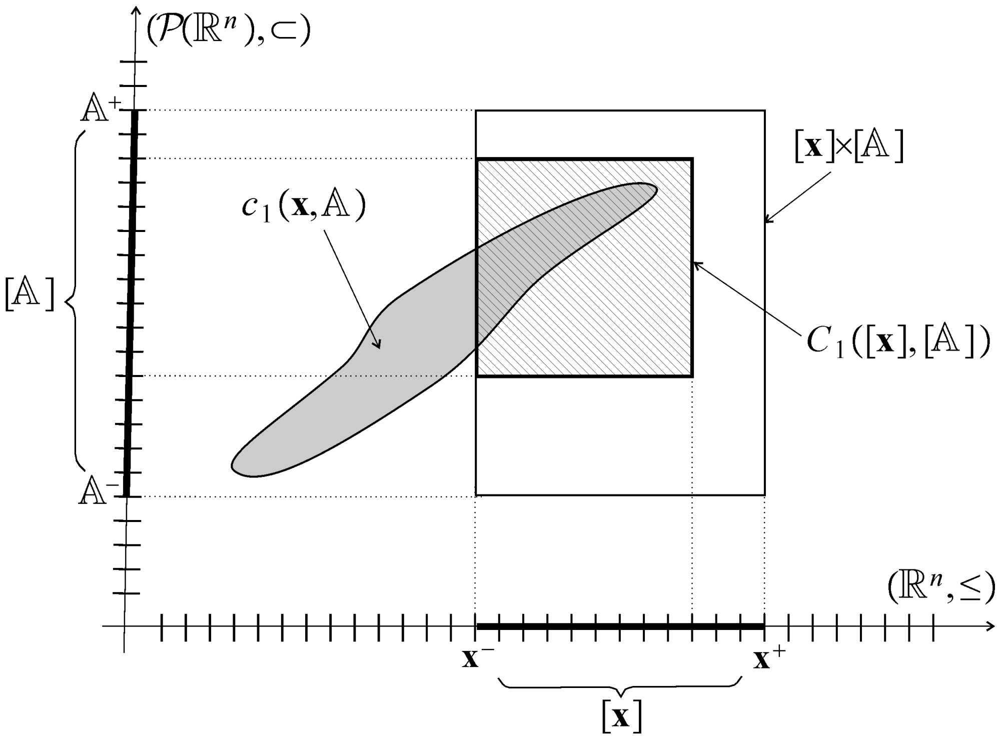
Define $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$.
An element $\mathbb{X}$ of $\mathcal{L}$ is the Cartesian product of n elements of $\mathcal{L}_i$: $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$.
The set $\mathbb{X}$ will be called *hyperdomain*.

A *contractor* is an operator

$$\mathcal{C}: \begin{array}{ccc} \mathcal{L} & \rightarrow & \mathcal{L} \\ \mathbb{X} & \rightarrow & \mathcal{C}(\mathbb{X}) \end{array}$$

which satisfies

$$\begin{array}{ll} \mathbb{X} \subset \mathbb{Y} \Rightarrow \mathcal{C}(\mathbb{X}) \subset \mathcal{C}(\mathbb{Y}) & \text{(monotonicity)} \\ \mathcal{C}(\mathbb{X}) \subset \mathbb{X} & \text{(contractance)} \end{array}$$



(𝒫(ℝⁿ),⊂)
𝔸⁺
𝔸⁻
[𝔸]
c₁(x,𝔸)
[x]×[𝔸]
C₁([x],[𝔸])
(ℝⁿ,≤)
x⁻
x⁺
[x]

7.3 Graph intervals

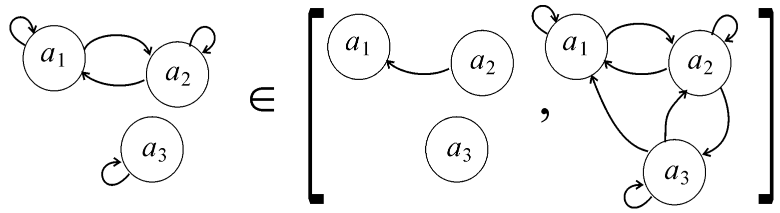
The set of graphs of $\mathcal{A}$ with the relation

$$\mathcal{G} \leq \mathcal{H} \Leftrightarrow \forall i, j \in \{1, \ldots, m\}, \; g_{ij} \leq h_{ij},$$

corresponds to a complete lattice. Intervals of graphs of $\mathcal{A}$ can thus be defined.

Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} [0,1] & [0,1] & 0 \\ 1 & [0,1] & [0,1] \\ [0,1] & [0,1] & [0,1] \end{pmatrix}$$



8 SLAM as a CSP

Variables

(i) the trajectory of the robot $\mathbf{x}$.

(ii) the sectors $\mathbb{H}_i$

(iii) the location of the mark $\mathbf{a}\,(i)$ detected at time t_i

(iv) the association graph $\mathcal{G}$

(v) the free space $\mathbb{F}$.

Domains

$$
\begin{aligned}
&\mathbf{x} \in [\mathbf{x}] = \left[\mathbf{x}^{-}, \mathbf{x}^{+}\right] \\
&\mathbf{a}\,(i) \in \mathbb{A}\,(i) \\
&\mathbb{H}_i \in [\mathbb{H}_i] = \left[\mathbb{H}_i^{-}, \mathbb{H}_i^{+}\right] \\
&\mathbb{F} \in [\mathbb{F}] = \left[\mathbb{F}^{-}, \mathbb{F}^{+}\right] \\
&\mathcal{G} \in [\mathcal{G}] = \left[\mathcal{G}^{-}, \mathcal{G}^{+}\right].
\end{aligned}
$$

Initialization

$[\mathbf{x}](t) = [-\infty, \infty]$ if $t > 0$ and $[\mathbf{x}](0) = \mathbf{0}$.

$\mathbb{A}(i) = \mathbb{R}^2$.

$\mathbb{H}_i \in \left[\emptyset, \mathbb{R}^2\right]$.

$\mathbb{F} \in \left[\emptyset, \mathbb{R}^2\right]$.

$\mathcal{G} \in [\emptyset, \top]$

Constraints

(i) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$

(ii) $\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$

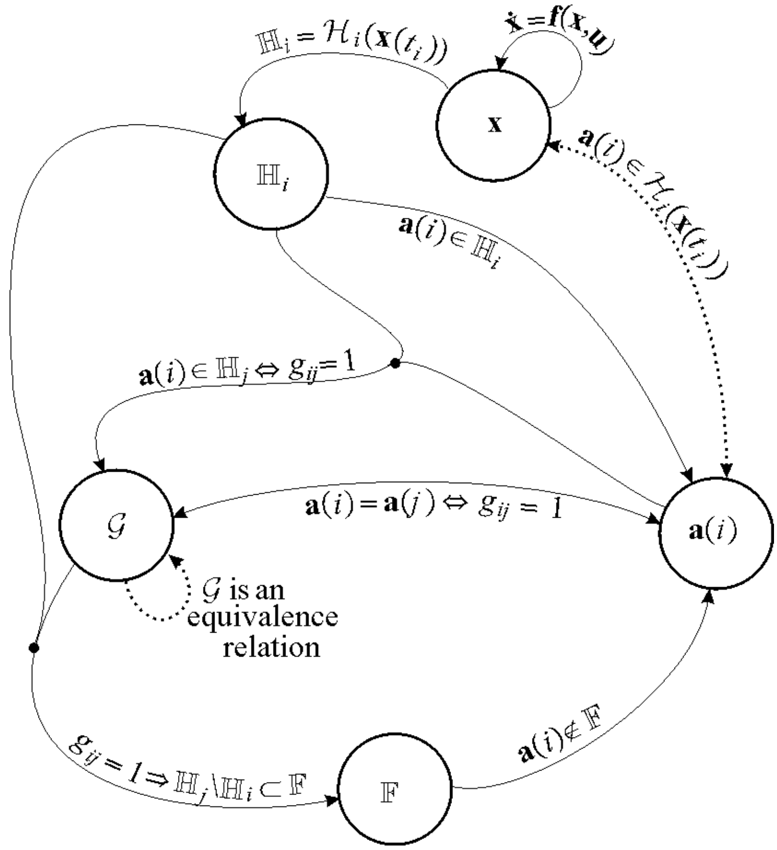
(iii) $\mathbf{a}(i) \in \mathbb{H}_i$

(iv) $\mathbf{a}(i) = \mathbf{a}(j) \Leftrightarrow g_{ij} = \mathbf{1}$

(v) $\mathbf{a}(i) \in \mathbb{H}_j \Leftrightarrow g_{ij} = \mathbf{1}$

(vi) $g_{ij} = \mathbf{1} \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}$

(vii) $\mathbf{a}(i) \notin \mathbb{F}$



$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})$
$\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$
$\mathbf{x}$
$\mathbb{H}_i$
$\mathbf{a}(i) \in \mathcal{H}_i(\mathbf{x}(t_i))$
$\mathbf{a}(i) \in \mathbb{H}_i$
$\mathbf{a}(i) \in \mathbb{H}_j \Leftrightarrow g_{ij} = 1$
$\mathcal{G}$
$\mathbf{a}(i) = \mathbf{a}(j) \Leftrightarrow g_{ij} = 1$
$\mathbf{a}(i)$
$\mathcal{G}$ is an equivalence relation
$\mathbf{a}(i) \notin \mathbb{F}$
$g_{ij} = 1 \Rightarrow \mathbb{H}_j \setminus \mathbb{H}_i \subset \mathbb{F}$
$\mathbb{F}$

9 Test-case

Generation of the data.

A simulated robot follows a cycloid for 100sec.

10 marks inside $[-8, 8] \times [-8, 8]$.

A rangefinder collects the distance $\tilde{d}$ to the nearest mark.

Resolution. The robot is

$$\begin{cases} \dot{x}_1 &= u_1 \cos u_2 \\ \dot{x}_2 &= u_1 \sin u_2. \end{cases}$$

The set-valued sector functions are

$$\begin{array}{lcl} \mathcal{H}_i\left(\mathbf{x}\left(t_i\right)\right) &=& \left\{\mathbf{a} \mid \ \left\|\mathbf{a}-\mathbf{x}\left(t_i\right)\right\| \ \in \ \left[d_i\right]\right\} \\ \mathcal{H}_{i+1}\left(\mathbf{x}\left(t_{i+1}\right)\right) &=& \left\{\mathbf{a} \mid \ \left\|\mathbf{a}-\mathbf{x}\left(t_{i+1}\right)\right\| \ < \ \delta_{i+1}\right\}. \end{array}$$

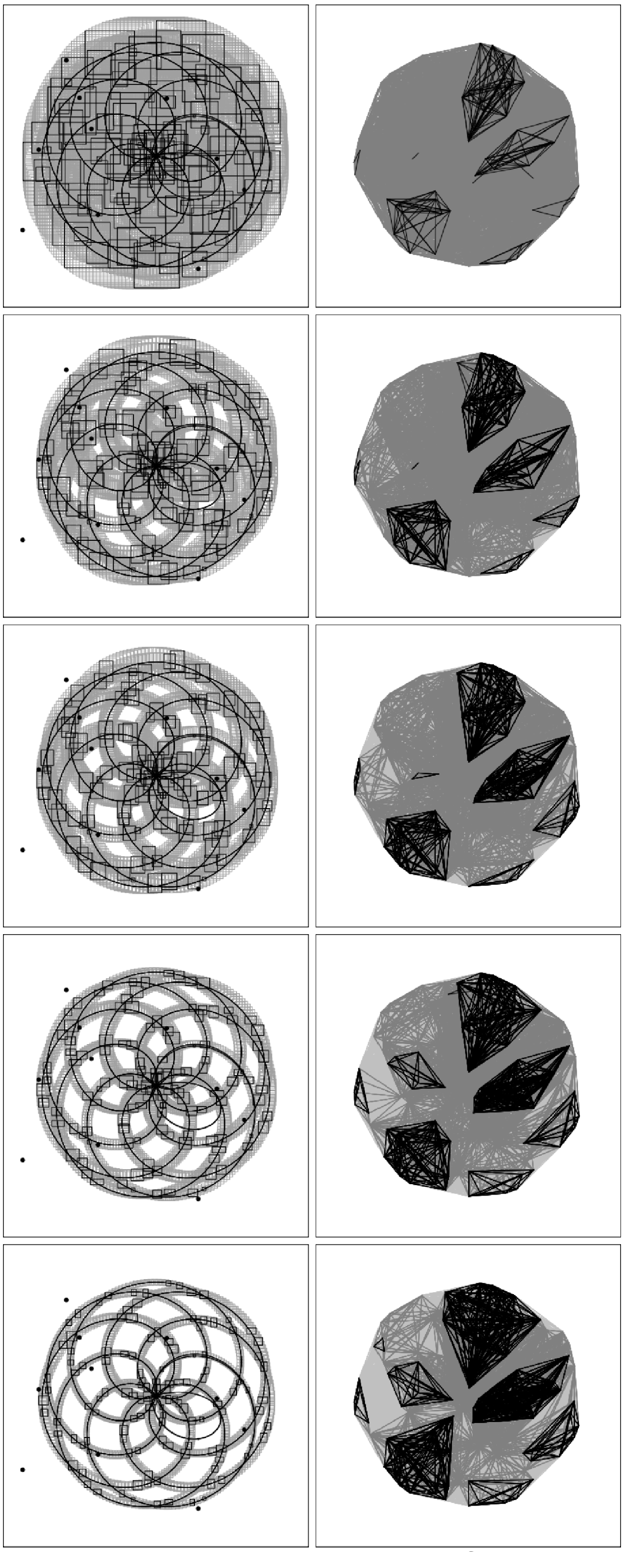
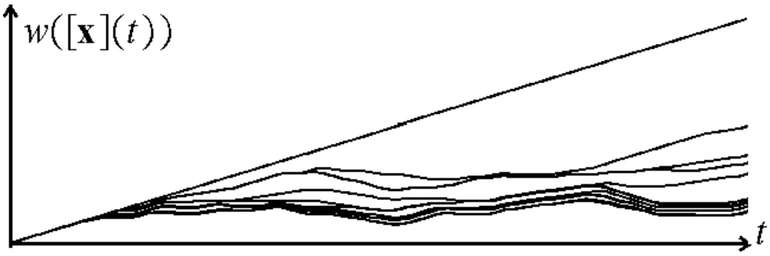
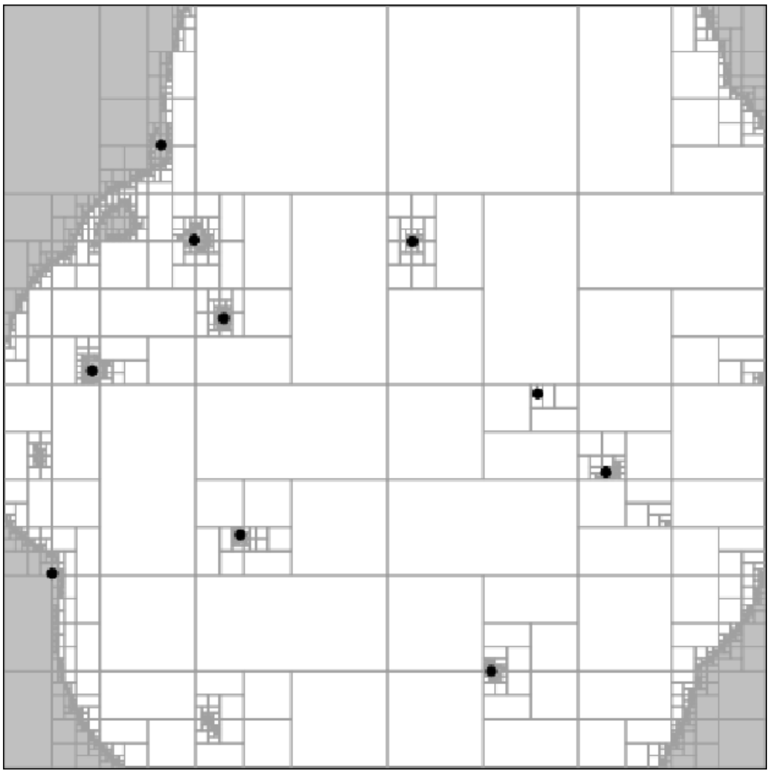


Illustration of the propagation. Left: the tube becomes more and more accurate. Right: The association graph has more and more arcs.



Superposition of the width of the tube $[\mathbf{x}](t)$

Associations. At the fixed point, 3888 associations have been found, 29128 pairs $(\mathbf{a}(i), \mathbf{a}(j))$ have been proven disjoint and 5400 pairs $(\mathbf{a}(i), \mathbf{a}(j))$ have not been classified.



Free space $\mathbb{F}$.

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