Lesson in interval robotics

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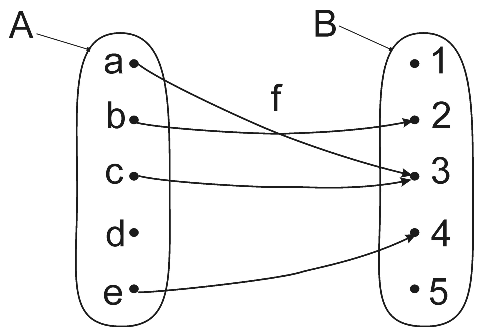
1 Interval computation

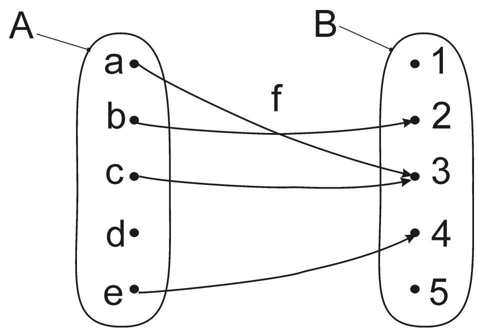
1.1 Notions on set theory

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= ?. \\ f^{-1}(B) &= ?. \\ f^{-1}(f(A)) &= ? \\ f^{-1}(f(\{b,c\})) &= ?. \end{aligned}$$

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= \{2,3,4\} = \mathsf{Im}(f). \\ f^{-1}(B) &= \{a,b,c,e\} = \mathsf{dom}(f). \\ f^{-1}(f(A)) &= \{a,b,c,e\} \subset A \\ f^{-1}(f(\{b,c\})) &= \{a,b,c\}. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= ? \\ f^{-1}([4,9]) &= ?. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= [4,9] \\ f^{-1}([4,9]) &= [-3,-2] \cup [2,3]. \end{aligned}$$

This is consistent with the property

$$f^{-1}\left(f\left(\mathbb{Y}\right)\right) \supset \mathbb{Y}.$$

1.2 Interval arithmetic

Exercise.

$$\begin{aligned} [-1,3]+[2,5] &= [?,?], \\ [-1,3]\cdot[2,5] &= [?,?], \\ [-2,6]/[2,5] &= [?,?]. \end{aligned}$$

Solution.

$$\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3].[2,5] &= [-5,15], \\
[-2,6]/[2,5] &= [-1,3].
\end{aligned}$$

Exercise. Compute

$$[-2,2]/[-1,1] \quad = [?,?].$$

Solution.

$$[-2, 2]/[-1, 1] \quad = [-\infty, \infty].$$

$$
\begin{aligned}
[x^-, x^+] + [y^-, y^+] = \quad & [x^- + y^-, x^+ + y^+], \\
[x^-, x^+] \cdot [y^-, y^+] = \quad & [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+, \\
& x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+],
\end{aligned}
$$

Exercise.

$$\begin{aligned}
\mathsf{sin}\,([0,\pi]) &= ?,\\
\mathsf{sqr}\,([-1,3]) &= [-1,3]^2 = ?,\\
\mathsf{abs}\,([-7,1]) &= ?,\\
\mathsf{sqrt}\,([-10,4]) &= \sqrt{[-10,4]} = ?,\\
\mathsf{log}\,([-2,-1]) &= ?.
\end{aligned}$$

Solution.

$$
\begin{aligned}
\mathsf{sin}\,([0,\pi]) &= [0,1],\\
\mathsf{sqr}\,([-1,3]) &= [-1,3]^2 = [0,9],\\
\mathsf{abs}\,([-7,1]) &= [0,7],\\
\mathsf{sqrt}\,([-10,4]) &= \sqrt{[-10,4]} = [0,2],\\
\mathsf{log}\,([-2,-1]) &= \emptyset.
\end{aligned}
$$

1.3 Inclusion function

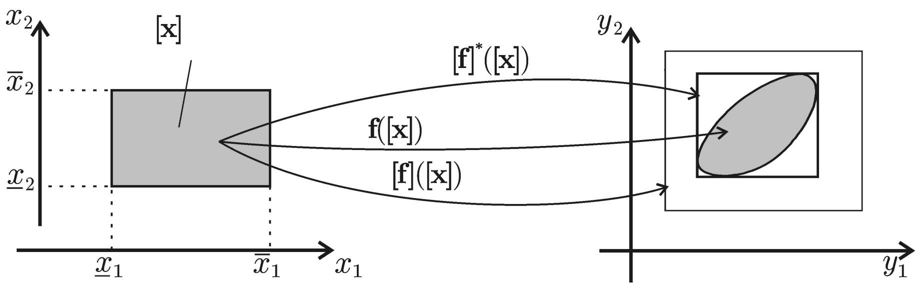
A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* of $\mathbf{f}$ if

$$\forall\, [\mathbf{x}] \in \mathbb{IR}^n, \quad \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3, 4]$, we have

$$\begin{aligned} [f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\ &= [0,16] + [-6,8] + 4 \\ &= [-2,28]. \end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{ccc} \mathbb{IR}^2 & \rightarrow & \mathbb{IR}^3 \\ ([x_1], [x_2]) & \rightarrow & \left([x_1] * [x_2], [x_1]^2, [x_1] - [x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by

Algorithm $\mathbf{f}$(in: $\mathbf{x} = (x_1, x_2, x_3)$, out: $\mathbf{y} = (y_1, y_2)$)

$z := x_1$;
for $k := 0$ to 100
    $z := x_2(z + k \cdot x_3)$;
next;
$y_1 := z$;
$y_2 := \sin(z x_1)$;

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in: $[\mathbf{x}]$, out: $[\mathbf{y}]$)
$[z] := [x_1]$;
for $k := 0$ to 100
$[z] := [x_2] * ([z] + k \cdot [x_3])$;
next;
$[y_1] := [z]$;
$[y_2] := \sin([z] * [x_1])$;

Is $[\mathbf{f}]$ convergent? thin? monotonic?

1.4 Boolean intervals

A *Boolean number* is an element of

$$\mathbb{B} \triangleq \{\mathit{false}, \mathit{true}\} = \{0, 1\} .$$

If we define the relation $\leq$ as

$$0 \leq 0, \quad 0 \leq 1, \quad 1 \leq 1,$$

then, the set $(\mathbb{B}, \leq)$ is a lattice for which intervals can be defined.

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{?, ?, ?, ?\},$$

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{\emptyset, 0, 1, [0, 1]\},$$

Boolean interval arithmetic

$$
\begin{aligned}
[a] \vee [b] &= \{a \vee b \mid a \in [a], b \in [b]\}, \\
[a] \wedge [b] &= \{a \wedge b \mid a \in [a], b \in [b]\}, \\
\neg [a] &= \{\neg a \mid a \in [a]\}.
\end{aligned}
$$

Exercise: Compute

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = \text{?}$$

Solution: We have

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = 1 \wedge [0,1] = [0,1].$$

2 Subpavings

2.1 Definition

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.

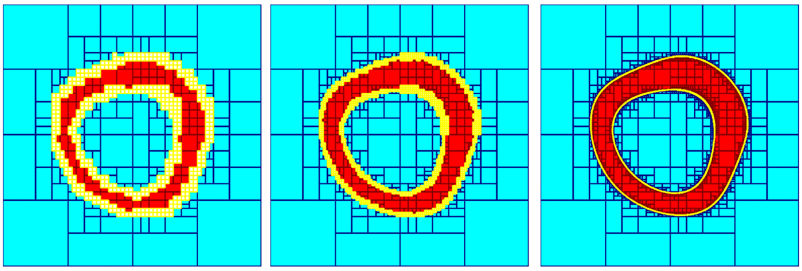
Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Exercise. The set

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}$$

are approximated by $\mathbb{X}^-$ and $\mathbb{X}^+$ for different accuracies.



2.2 Set inversion

If $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$.

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

Exercise. Define the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 x_2 + \sin x_2 \leq 0 \text{ and } x_1 - x_2 = 1\}.$$

Show that it is a set inversion problem.

Solution. We have

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

with

$$\mathbf{f}\left(\mathbf{x}\right) = \left(\begin{array}{c} x_1 x_2 + \sin x_2 \\ x_1 - x_2 \end{array} \right) \text{ and } \mathbb{Y} = [-\infty, 0] \times \{1\}\,.$$

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

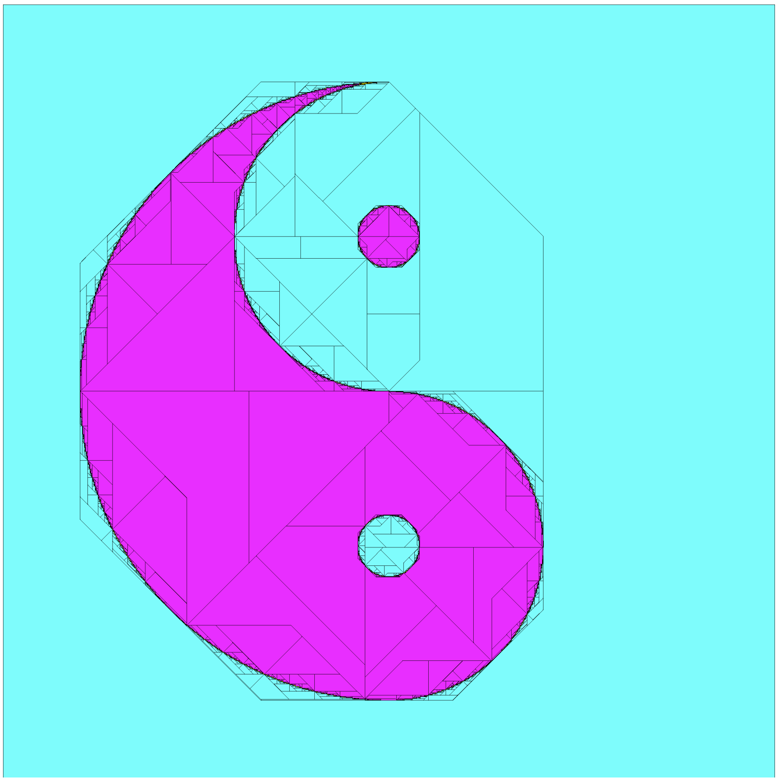
Algorithm Sivia(in: $[\mathbf{x}](0), \mathbf{f}, \mathbb{Y}$)

1 $\mathcal{L} := \{[\mathbf{x}](0)\}$;
2 pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$, 'red');
4 elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$, 'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$, 'yellow')};
6 else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$;
7 if $\mathcal{L} \neq \emptyset$, go to 2

If $\Delta\mathbb{X}$ denotes the union of yellow boxes and if $\mathbb{X}^-$ is the union of red boxes then:

$$\mathbb{X}^- \subset \mathbb{X} \subset \underbrace{\mathbb{X}^- \cup \Delta\mathbb{X}}_{\mathbb{X}^+}.$$

Sivia works with other abstract domains (or wrappers).



Sivia with octogones (made by D. Massé)

2.3 Bounded-error estimation

Exercise. Consider a parabola of the form

$$\phi(\mathbf{p}, t) = p_1 t^2 + p_2 t + p_3.$$

where $\mathbf{p} = (p_1, p_2, p_3)^\mathsf{T}$ is an unkown parameter vector. Assume that

$$\phi(\mathbf{p}, 1) \in [2, 3], \ \phi(\mathbf{p}, 4) \in [5, 6], \ \phi(\mathbf{p}, 7) \in [8, 9].$$

Show that the set $\mathbb{P}$ of all feasible $\mathbf{p}$ can be defined as a set inversion problem.

Solution. We have

$$\mathbb{P} = \mathbf{f}^{-1}(\mathbb{Y}),$$

where

$$\mathbf{f}(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p}, 1) \\ \phi(\mathbf{p}, 4) \\ \phi(\mathbf{p}, 7) \end{pmatrix} = \begin{pmatrix} p_1 + p_2 + p_3 \\ 16p_1 + 4p_2 t + p_3 \\ 49p_1 + 7p_2 + p_3 \end{pmatrix}$$

and

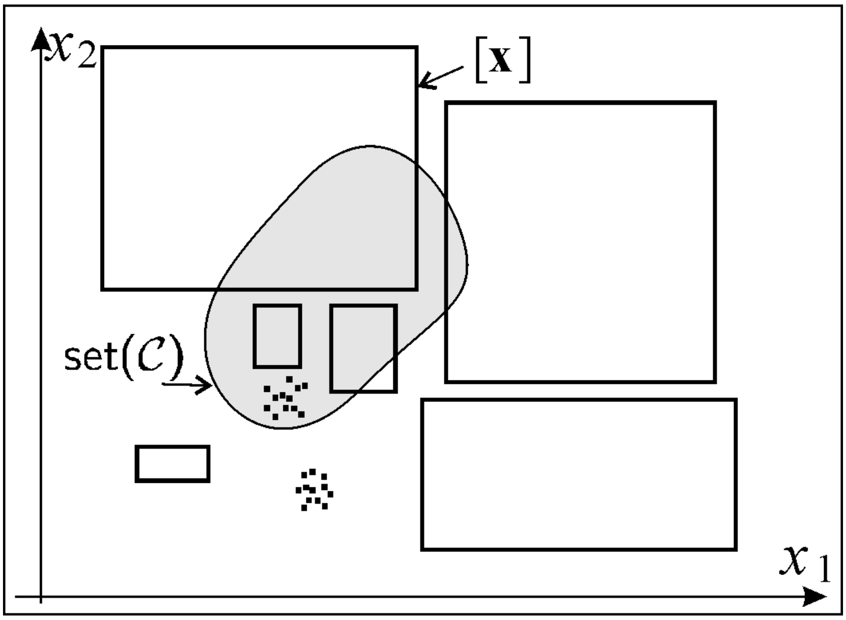
$$\mathbb{Y} = [2, 3] \times [5, 6] \times [8, 9].$$

3 Contractors

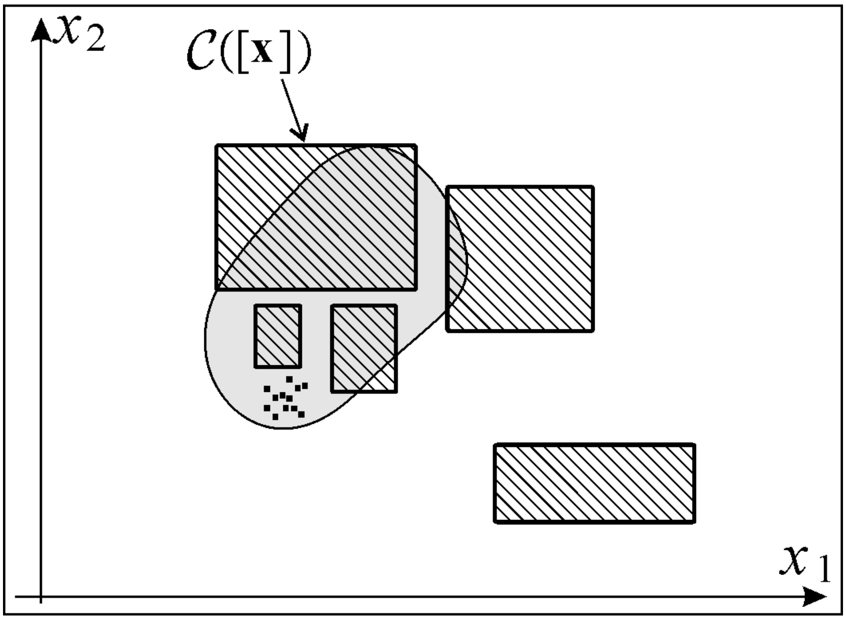
3.1 Definition

The operator $\mathcal{C}_{\mathbb{X}} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \left\{ \begin{array}{ll} \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)}, \\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness)}. \end{array} \right.$$



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
x_1

The operator $\mathcal{C} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall[\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) \end{cases}$$

Exercice. Let x, y, z be 3 variables such that

$$\begin{array}{rcl} x & \in & [-\infty, 5], \\ y & \in & [-\infty, 4], \\ z & \in & [6, \infty], \\ z & = & x + y. \end{array}$$

Contract the intervals for x, y, z.

Solution.

$$
\begin{aligned}
[x] &= [2, 5] \\
[y] &= [1, 4] \\
[z] &= [6, 9]
\end{aligned}
$$

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$\begin{aligned}
z = x + y \Rightarrow \quad z \in \quad & [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) \\
& = [6, \infty] \cap [-\infty, 9] = [6, 9]. \\
x = z - y \Rightarrow \quad x \in \quad & [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) \\
& = [-\infty, 5] \cap [2, \infty] = [2, 5]. \\
y = z - x \Rightarrow \quad y \in \quad & [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) \\
& = [-\infty, 4] \cap [1, \infty] = [1, 4].
\end{aligned}$$

The contractor associated with $z = x + y$ is:

Algorithm Cadd(inout: $[z], [x], [y]$)
1 $[z] := [z] \cap ([x] + [y])$;
2 $[x] := [x] \cap ([z] - [y])$;
3 $[y] := [y] \cap ([z] - [x])$.

The contractor associated with $z = x \cdot y$ is

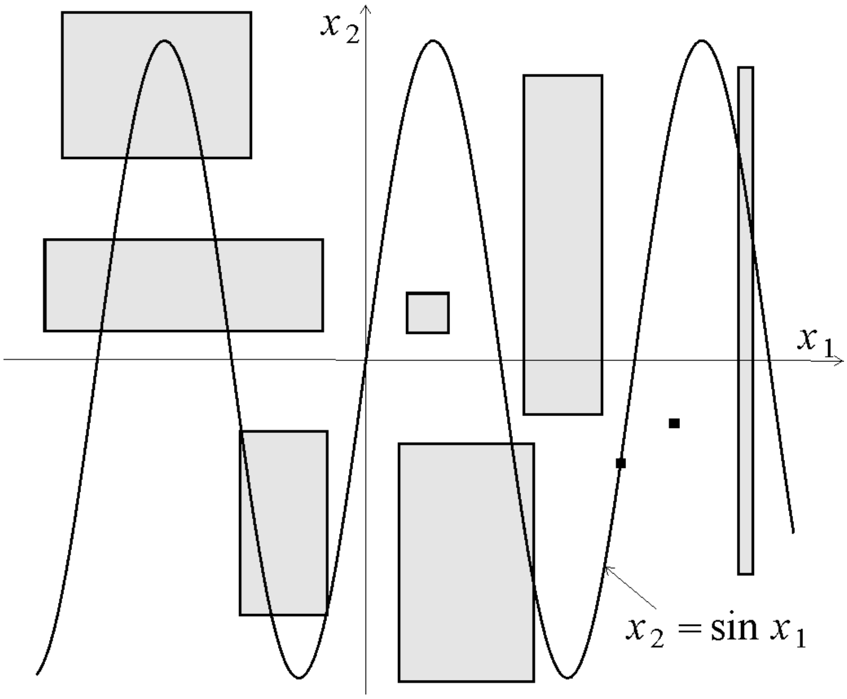
Algorithm pmul(inout: $[z], [x], [y]$)
1 $[z] := [z] \cap ([x] \cdot [y])$;
2 $[x] := [x] \cap ([z] \cdot 1/[y])$;
3 $[y] := [y] \cap ([z] \cdot 1/[x])$.

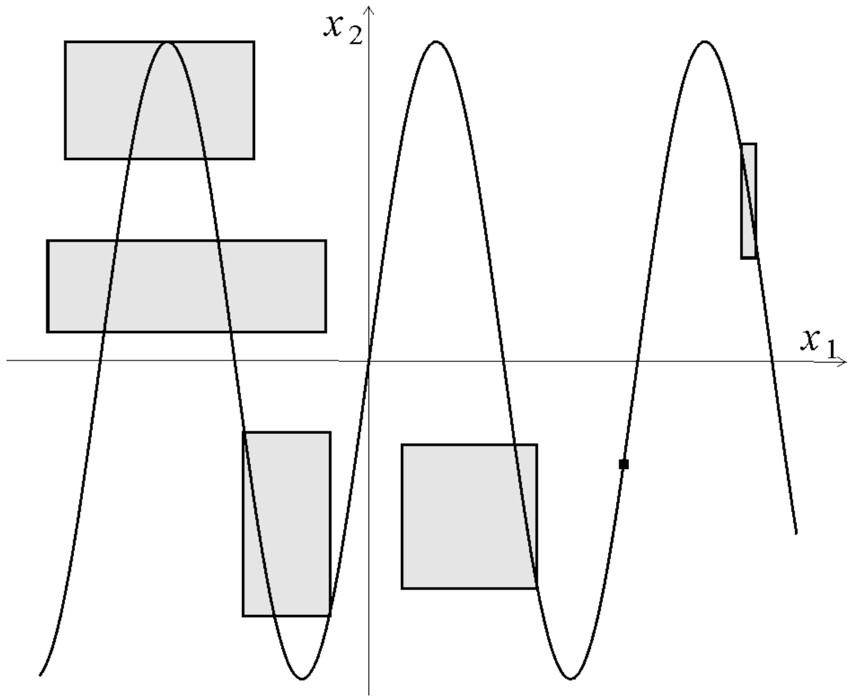
The contractor associated with $y = \exp(y)$ is

Algorithm Cexp(inout: $[y], [x]$)
1 $[y] := [y] \cap \exp([x])$;
2 $[x] := [x] \cap \log([y])$.

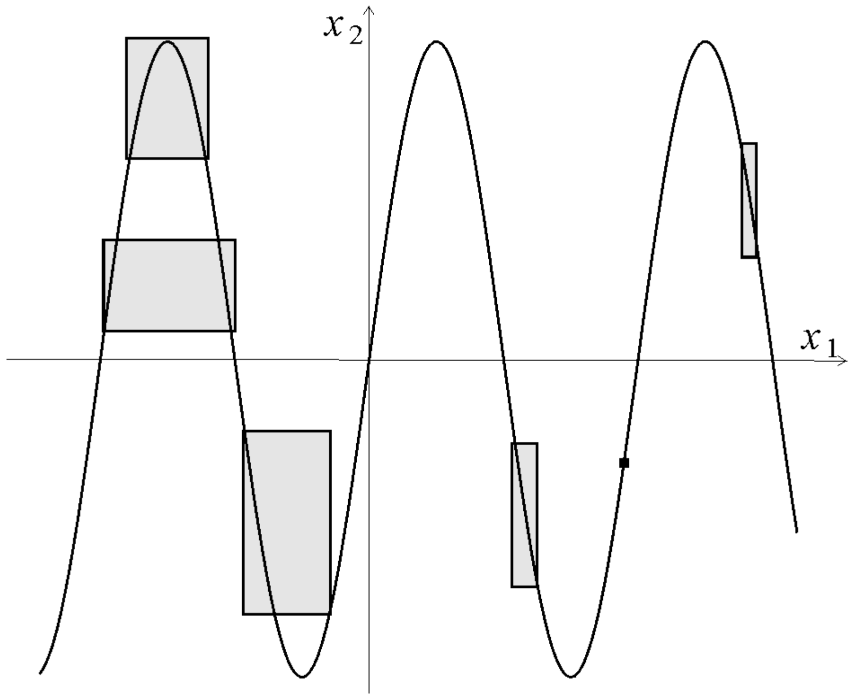
Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$


x_2
x_1
$x_2 = \sin x_1$



Forward contraction



Backward contraction

Decomposition

$$x + \mathsf{sin}(xy) \leq 0,$$
$$x \in [-1, 1], y \in [-1, 1]$$

Decomposition

$$\begin{aligned} & x + \mathsf{sin}(xy) \le 0, \\ & x \in [-1, 1], y \in [-1, 1] \end{aligned}$$

can be decomposed into

$$\left\{ \begin{array}{l} a = xy \\ b = \mathsf{sin}(a) \\ c = x + b \end{array} \right. , \qquad \begin{array}{ll} x \in [-1, 1] & a \in [-\infty, \infty] \\ y \in [-1, 1] & b \in [-\infty, \infty] \\ & c \in [-\infty, 0] \end{array}$$

Forward-backward contractor (HC4 revise)

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2]\,,$$

we have the following contractor:

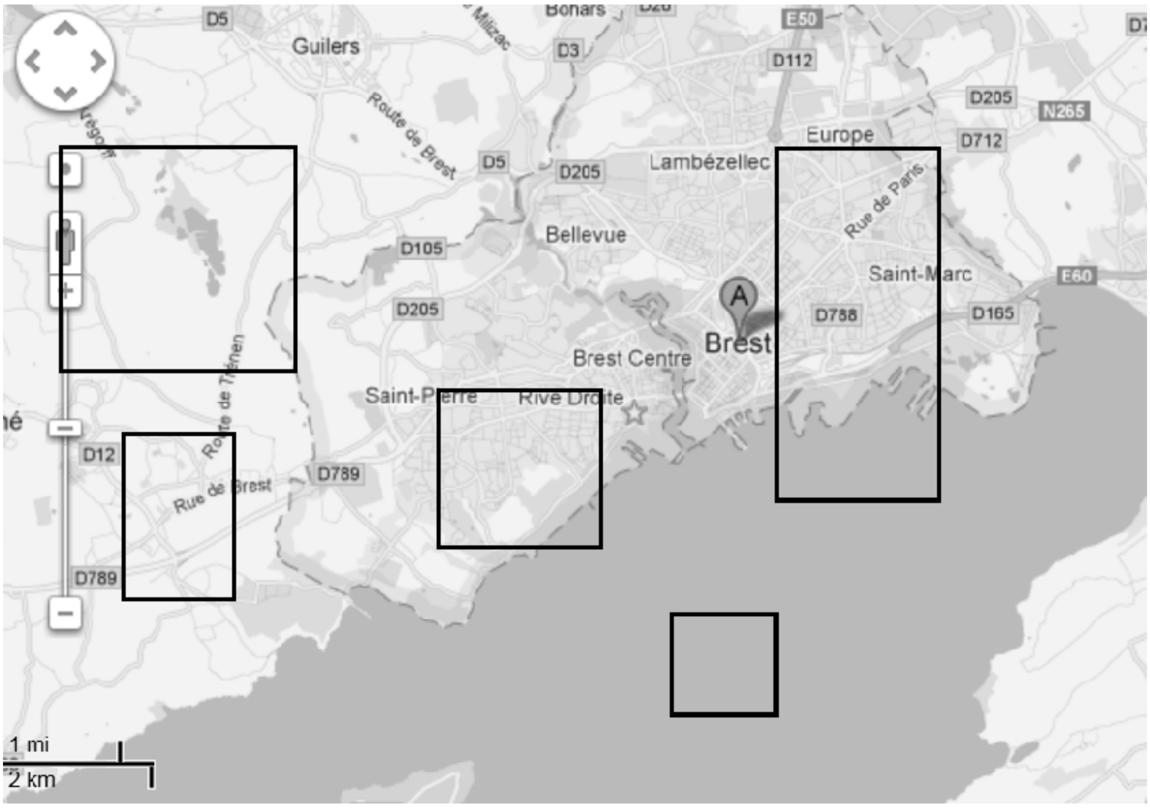
algorithm $\mathcal{C}$ (inout $[x_1]\,,[x_2]\,,[x_3]$)	
$[a] = [x_1] + [x_2]$	$//\ a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	$//\ b = a \cdot x_3$
$[b] = [b] \cap [1, 2]$	$//\ b \in [1, 2]$
$[x_3] = [x_3] \ \cap \ \frac{[b]}{[a]}$	$//\ x_3 = \frac{b}{a}$
$[a] = [a] \ \cap \ \frac{[b]}{[x_3]}$	$//\ a = \frac{b}{x_3}$
$[x_1] = [x_1] \ \cap \ \ [a] - [x_2]$	$//\ x_1 = a - x_2$
$[x_2] = [x_2] \ \cap \ \ [a] - [x_1]$	$//\ x_2 = a - x_1$

Properties

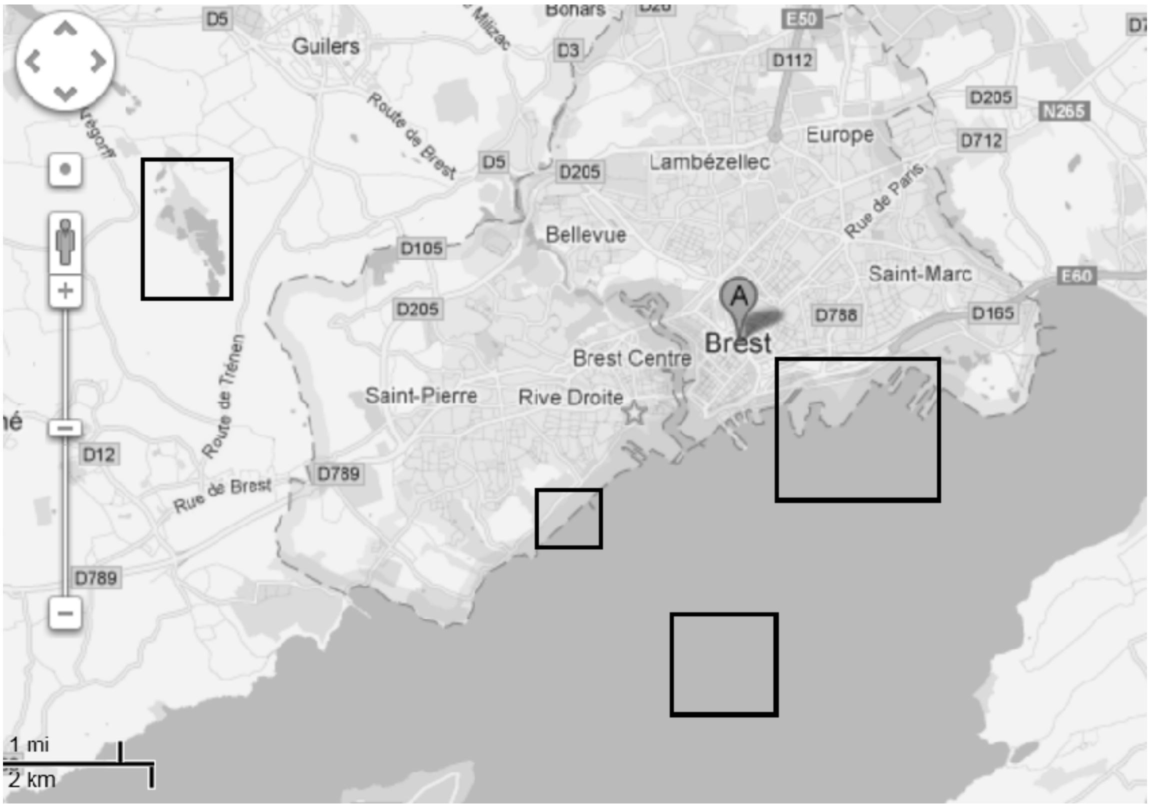
$$
\begin{aligned}
(\mathcal{C}_1^{\infty} \cap \mathcal{C}_2^{\infty})^{\infty} &= (\mathcal{C}_1 \cap \mathcal{C}_2)^{\infty} \\
(\mathcal{C}_1 \cap (\mathcal{C}_2 \cup \mathcal{C}_3)) &\supset (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_1 \cap \mathcal{C}_3) \\
\left\{ \begin{array}{l} \mathcal{C}_1 \text{ minimal} \\ \mathcal{C}_2 \text{ minimal} \end{array} \right. &\Rightarrow \mathcal{C}_1 \cup \mathcal{C}_2 \text{ minimal}
\end{aligned}
$$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.



Guilers
Bohars
Milizac
D5
D3
D26
E50
D112
D205
N265
D712
Europe
Lambézellec
Route de Brest
D5
D205
Bellevue
D105
D205
Rue de Paris
Saint-Marc
E60
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Tréinen
Rue de Brest
D12
D789
D789
1 mi
2 km



Guilers
Bohars
Route de Brest
Lambézellec
Europe
Rue de Paris
Bellevue
Saint-Marc
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
Rue de Brest
D5
D3
D112
E50
D205
N265
D712
D105
E60
D788
D165
D12
D789
1 mi
2 km

3.2 Propagation

A CN (Constraint Network) is composed of
1) a set of variables $\mathcal{V} = \{x_1, \ldots, x_n\}$,
2) a set of constraints $\mathcal{C} = \{c_1, \ldots, c_m\}$ and
3) a set of interval domains $\{[x_1], \ldots, [x_n]\}$.

Principle of propagation: contract $[\mathbf{x}] = [x_1] \times \cdots \times [x_n]$ as follows:

$$((((((([\mathbf{x}] \sqcap c_1) \sqcap c_2) \sqcap \ldots) \sqcap c_m) \sqcap c_1) \sqcap c_2) \ldots,$$

until a steady box is reached.

3.3 Example 1

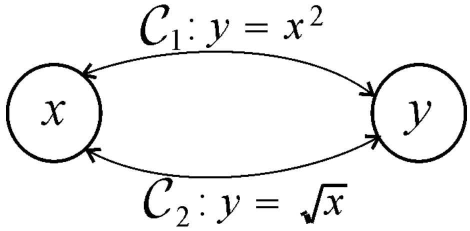
Consider the system of two equations.

$$
\begin{aligned}
y &= x^2 \\
y &= \sqrt{x}.
\end{aligned}
$$

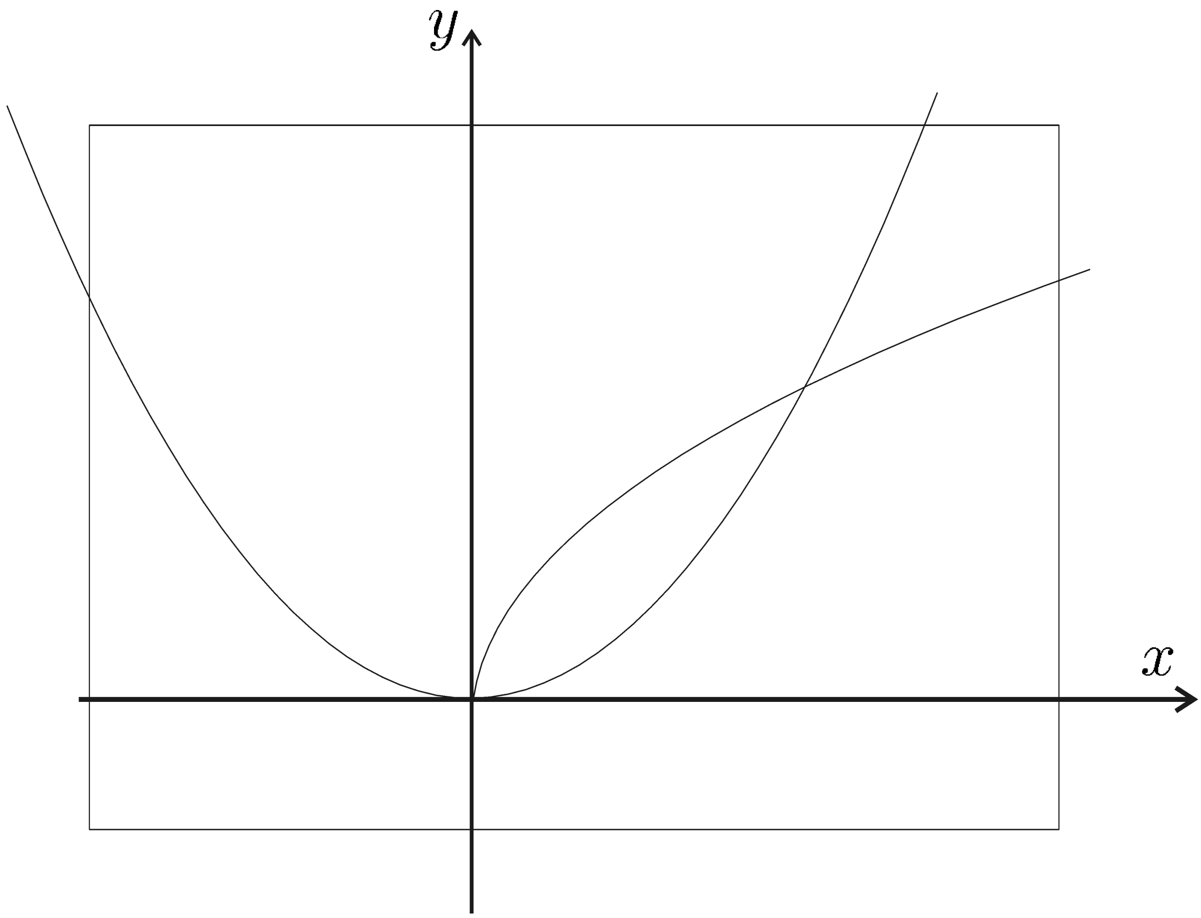
We can build two contractors

$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

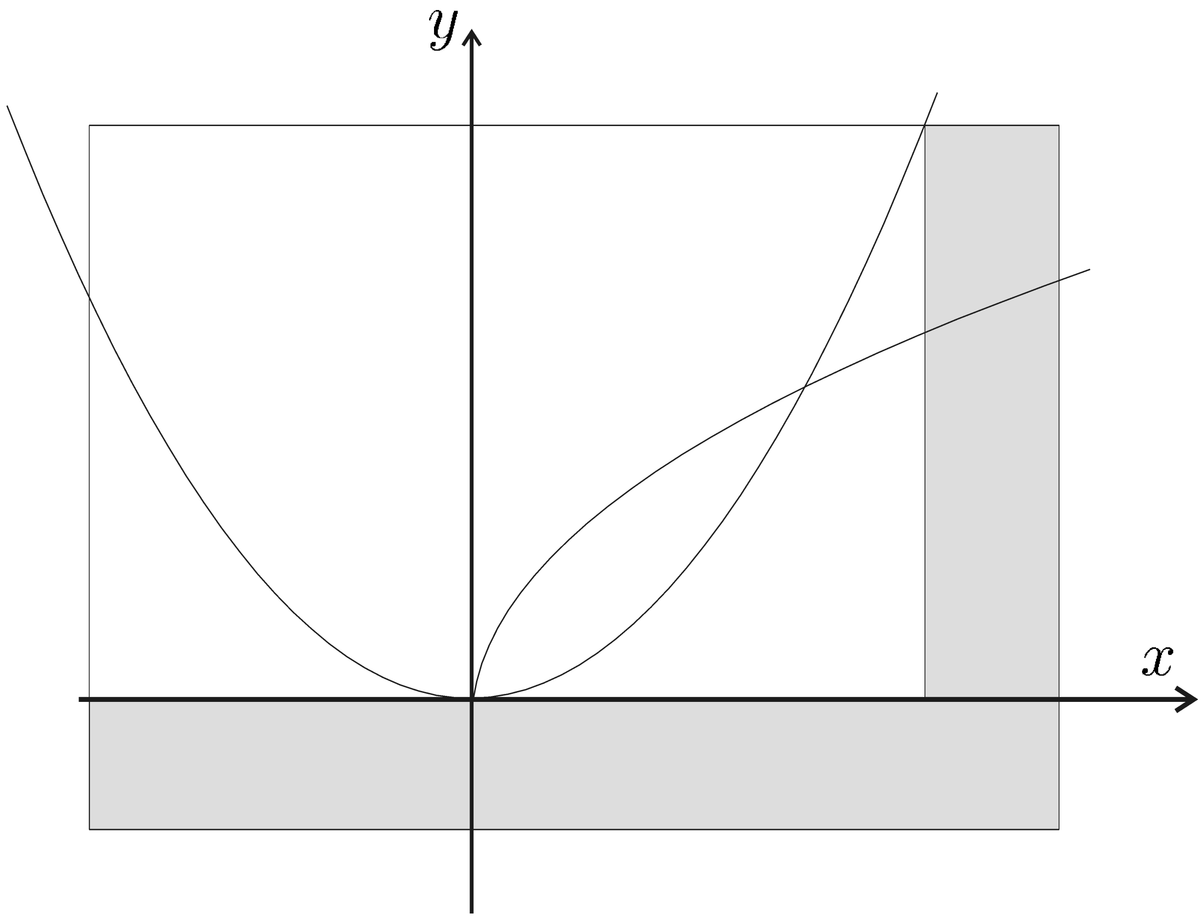
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$



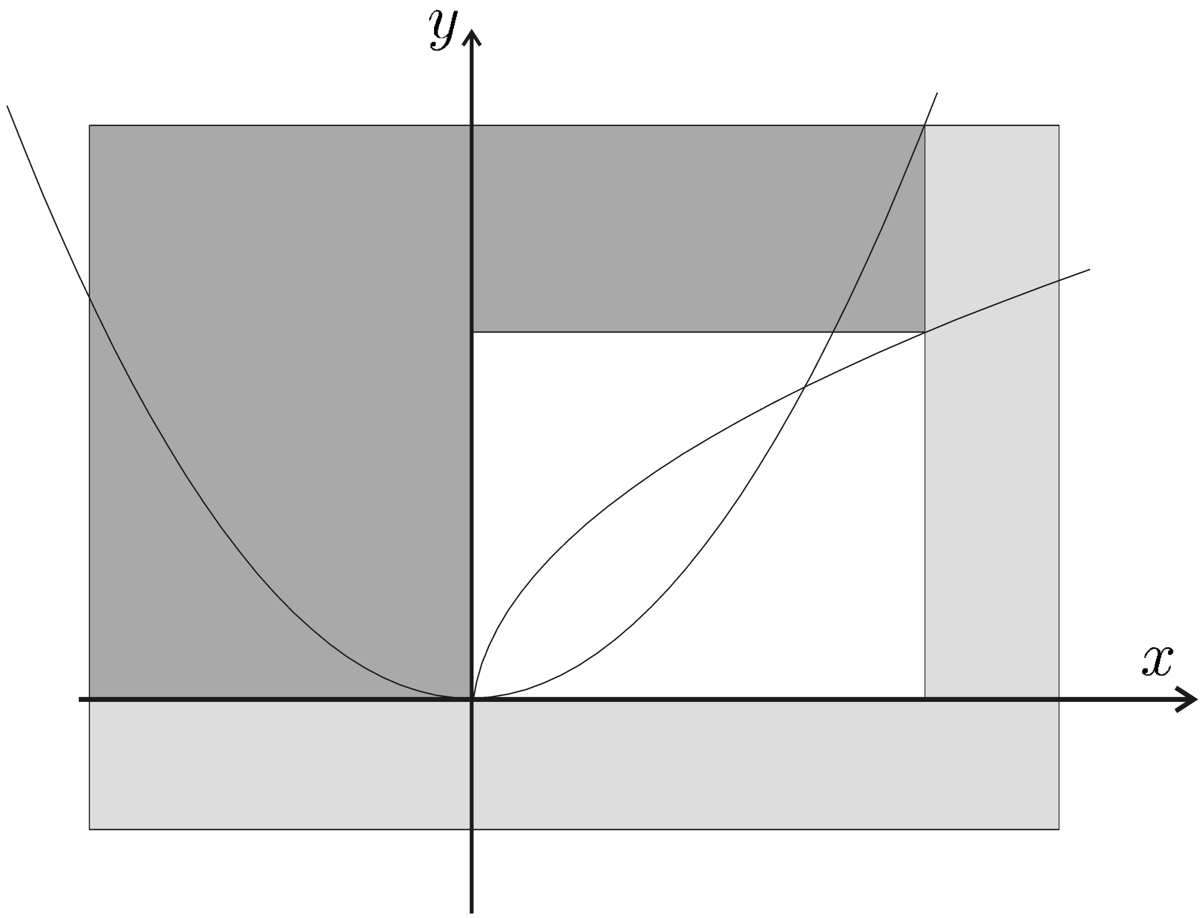
Contractor graph



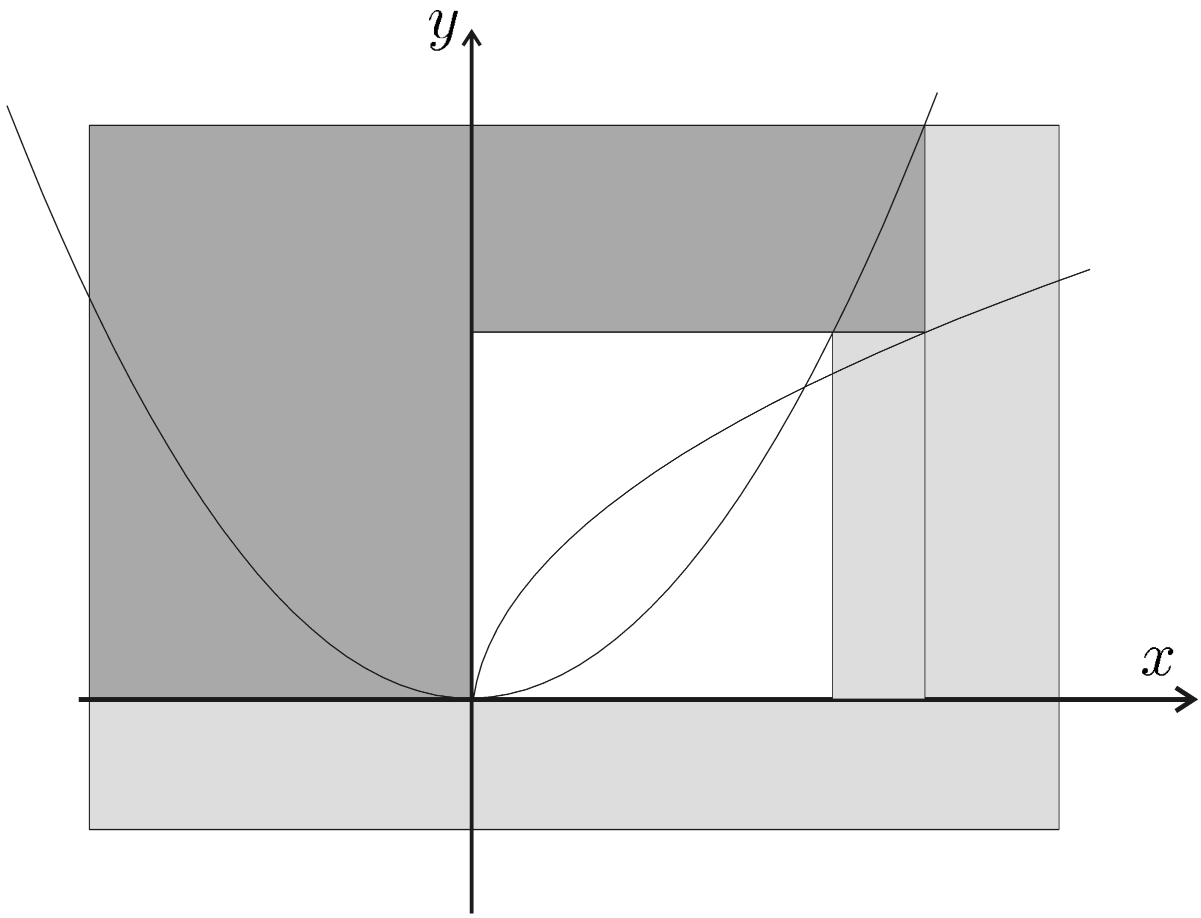
y
x



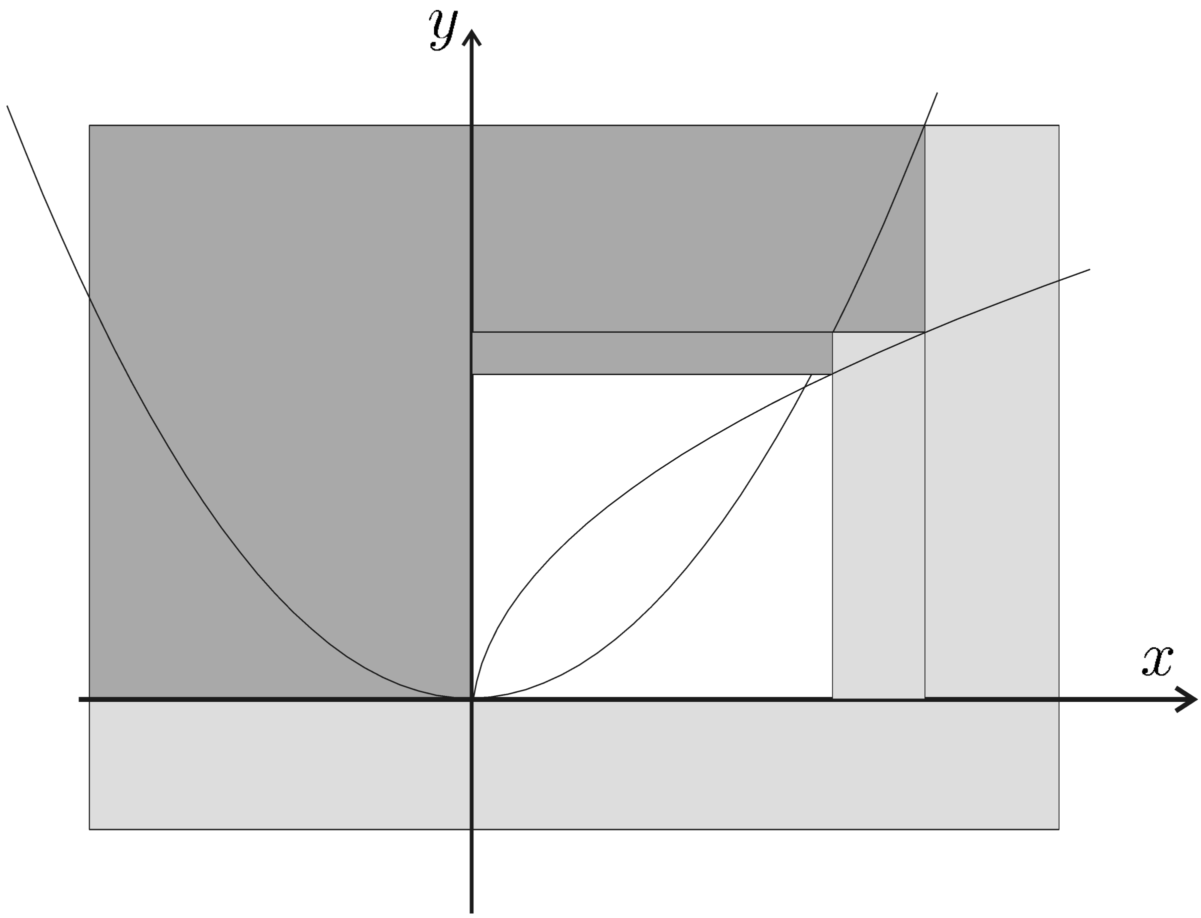
y
x



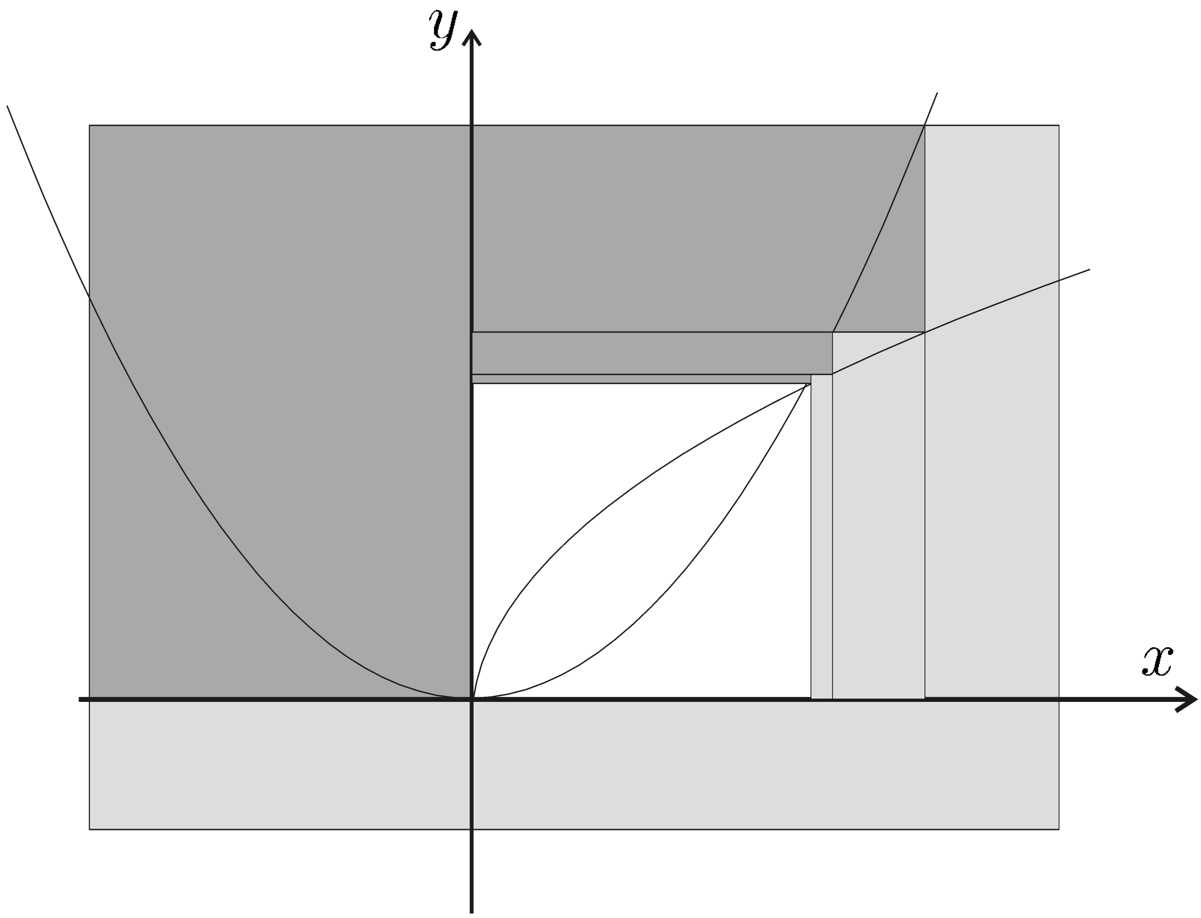
y
x



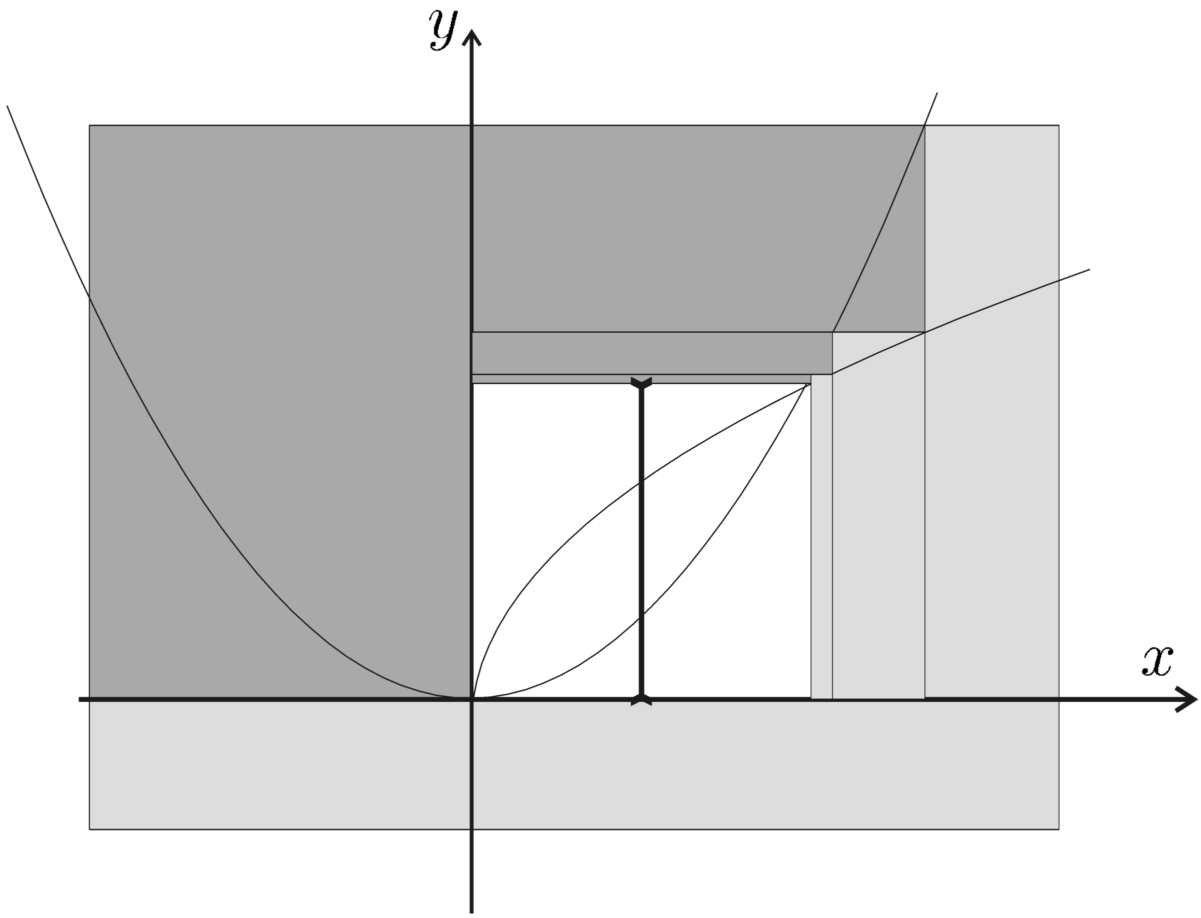
y
x

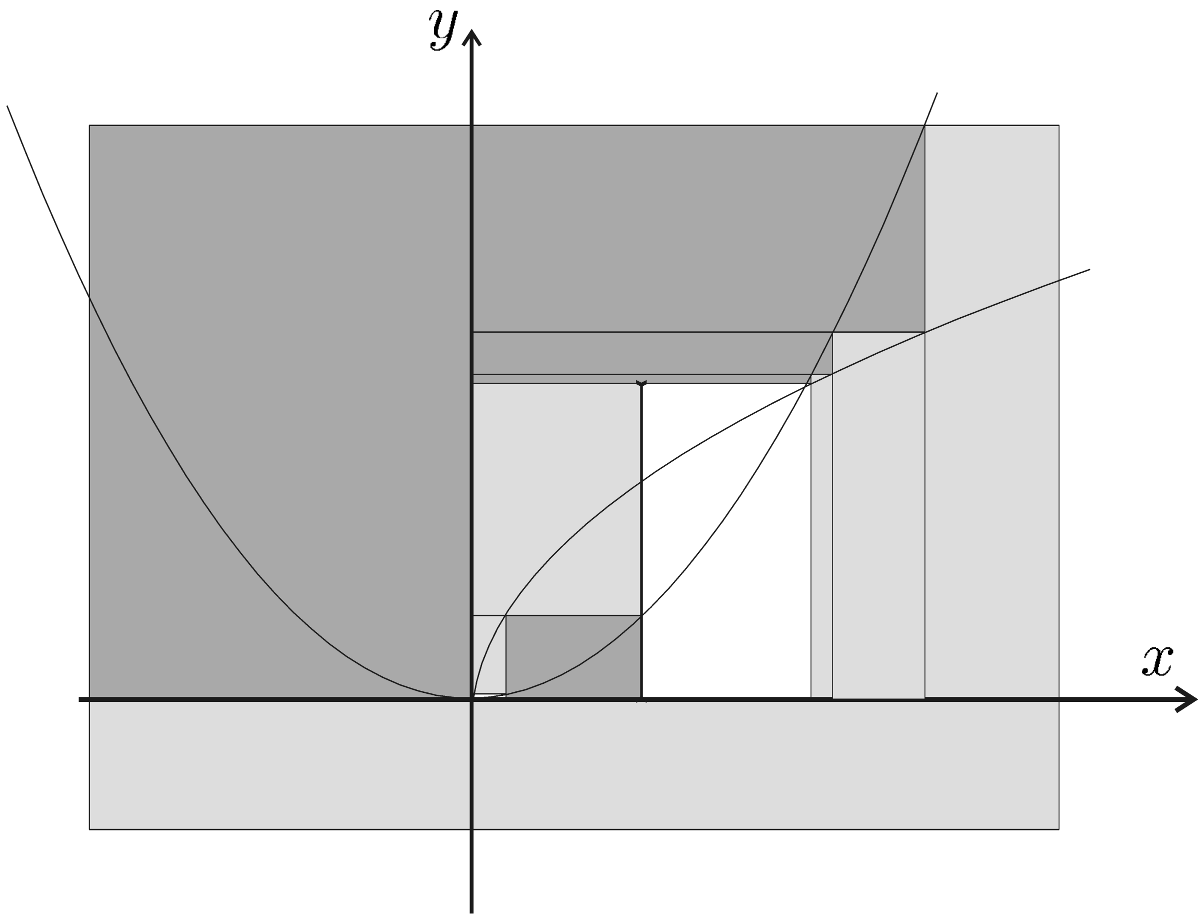


y
x

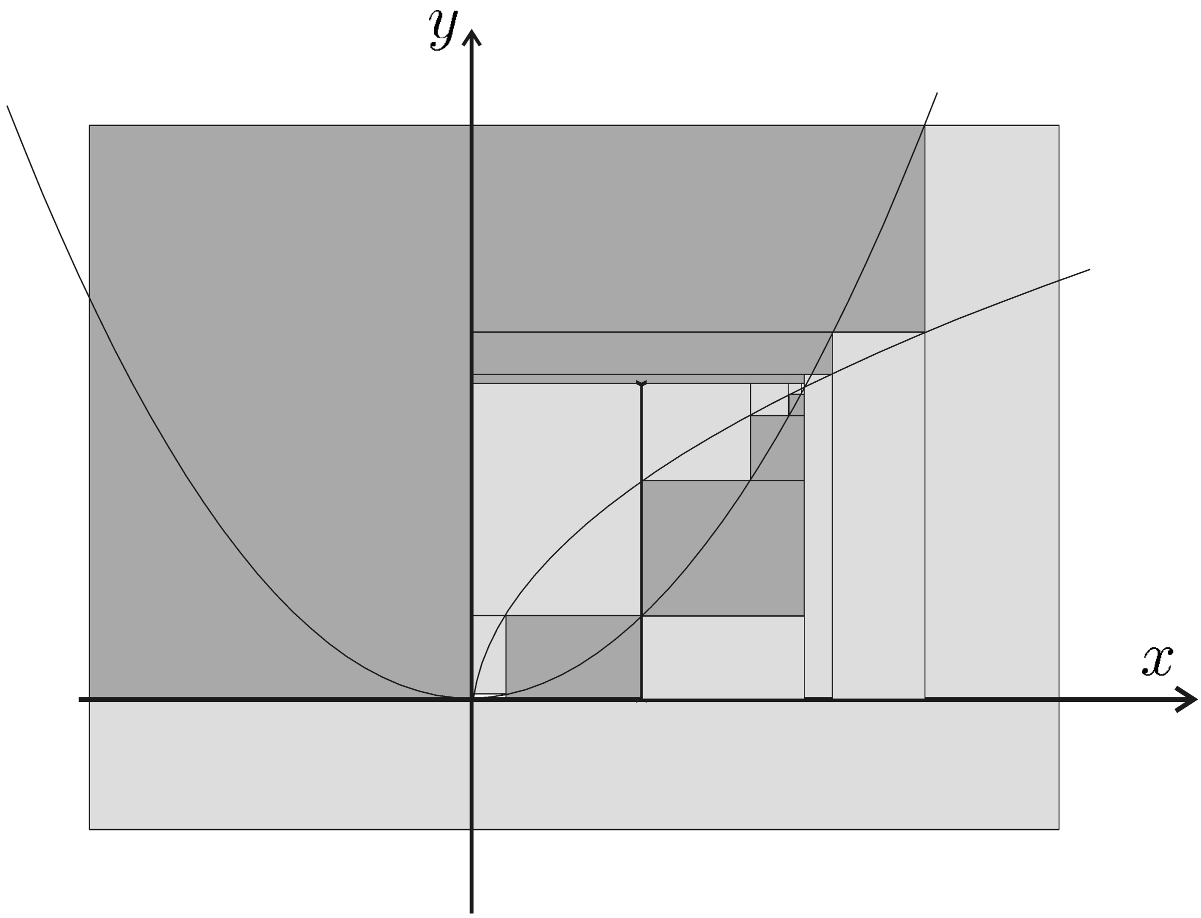


y
x


y
x



y
x

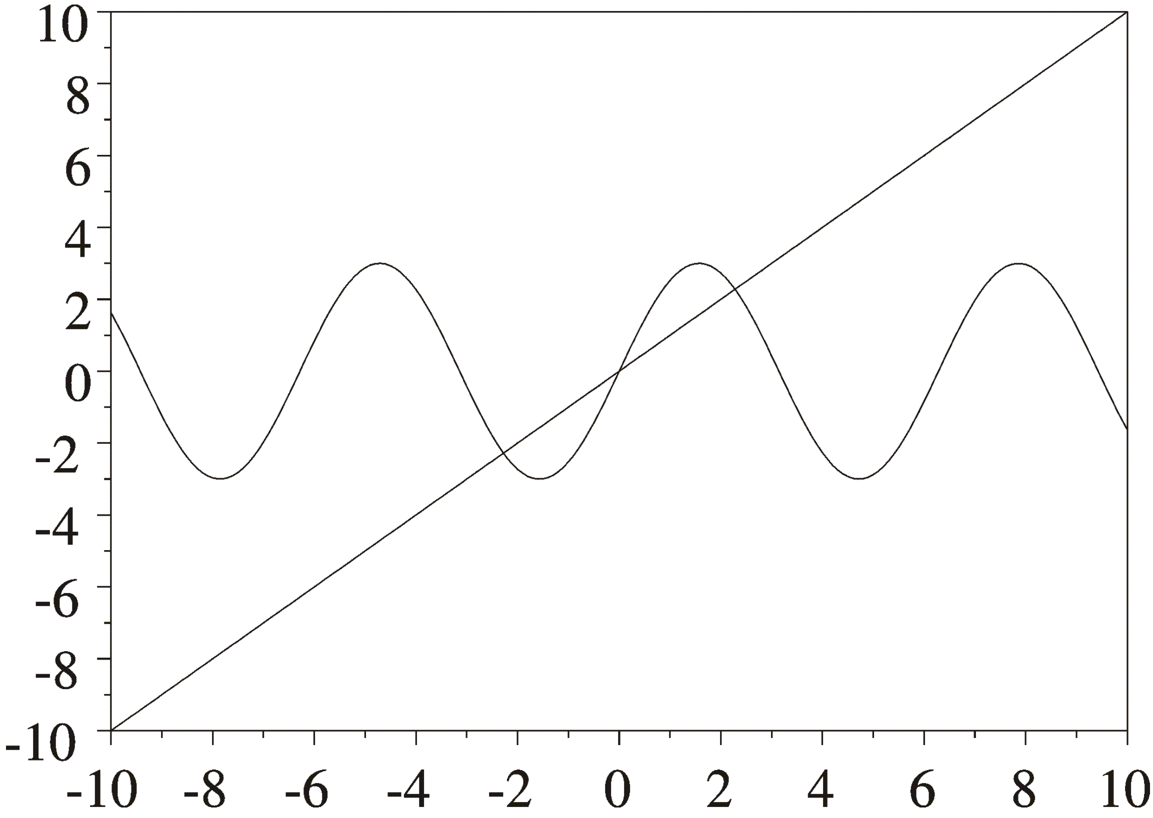


y
x

3.4 Example 2

Exemple. Consider the system

$$\left\{ \begin{array}{lcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$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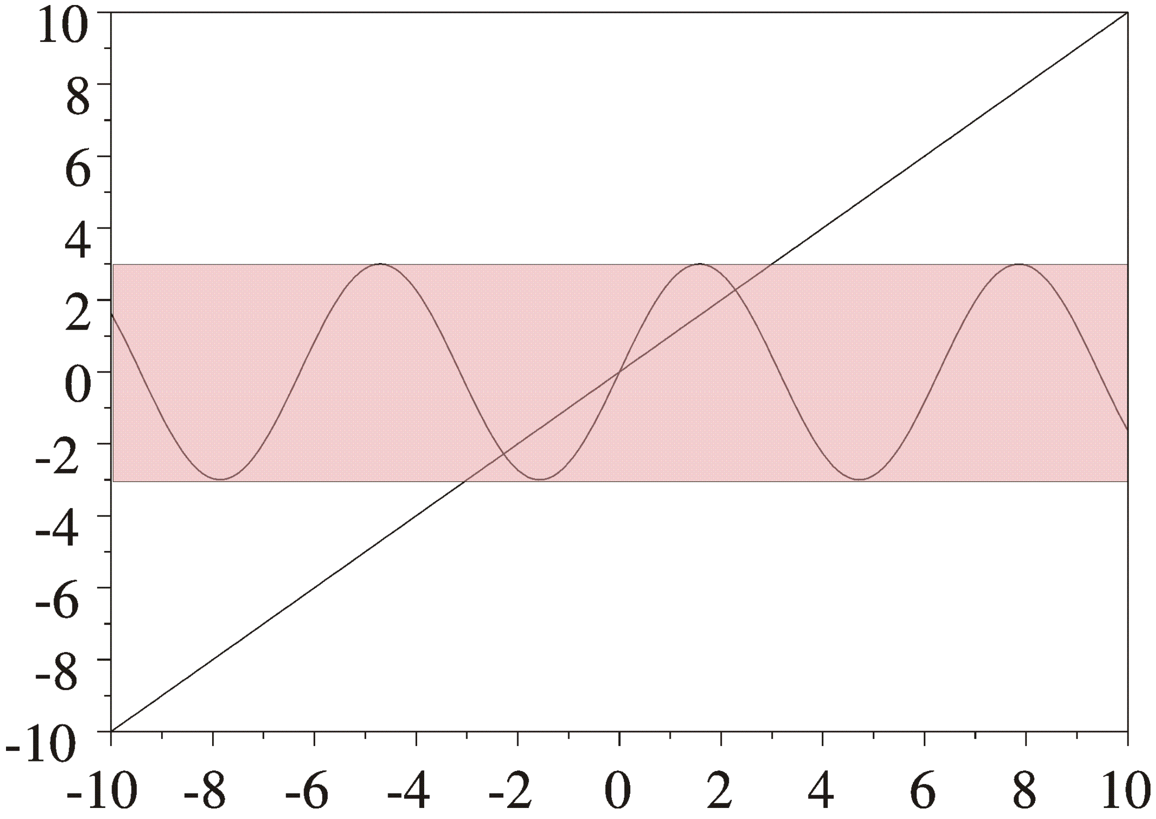
-10 -8 -6 -4 -2 0 2 4 6 8 10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8
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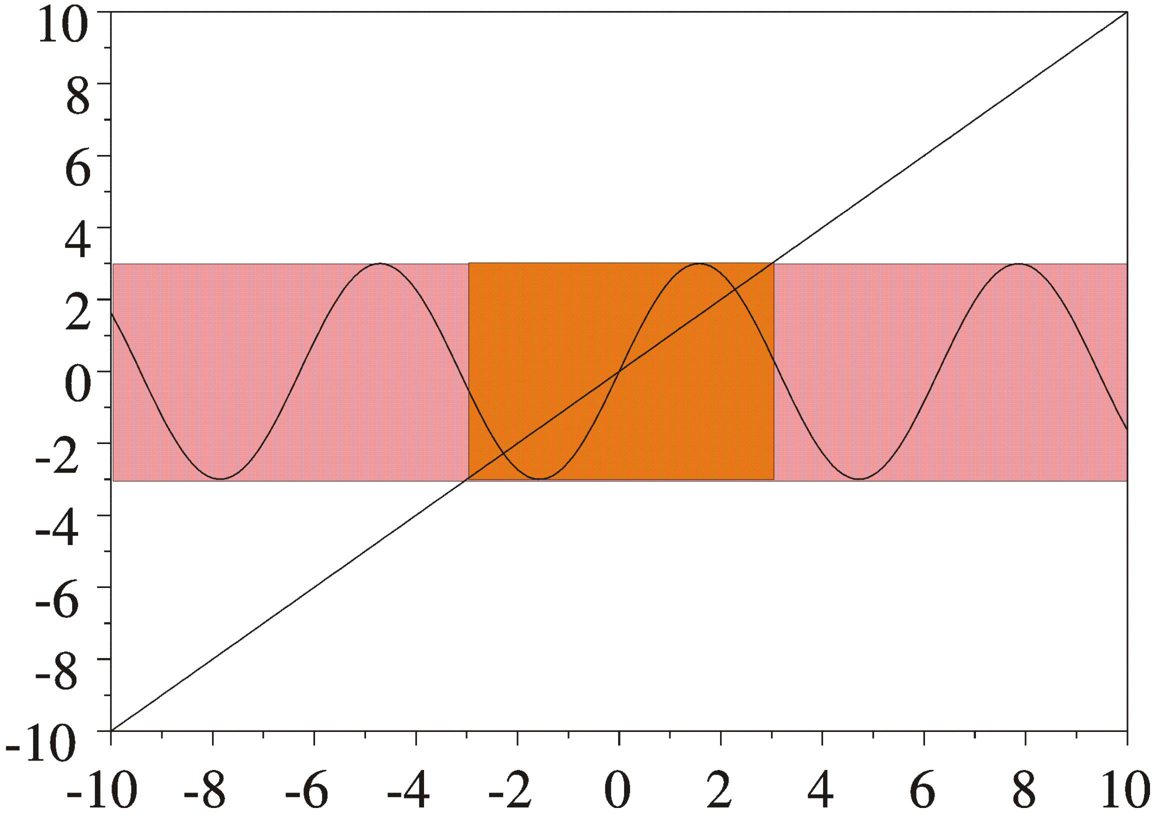

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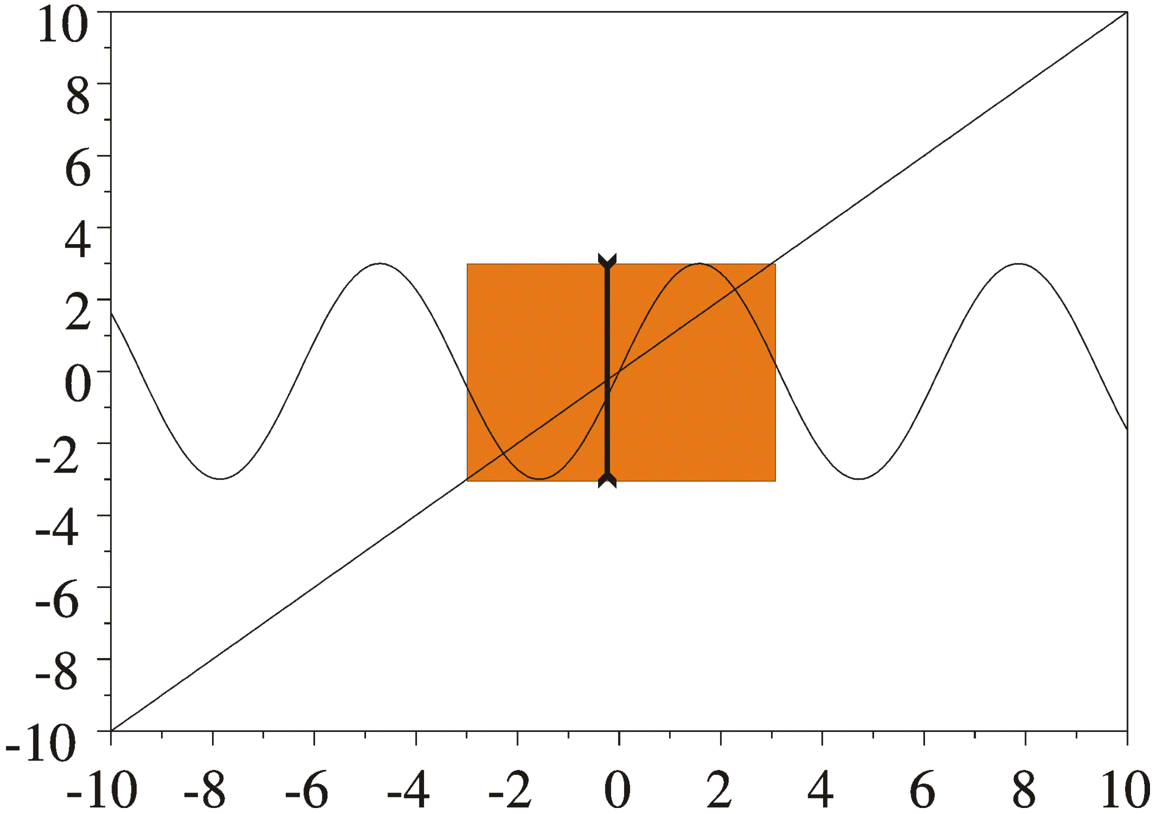
-10 -8 -6 -4 -2 0 2 4 6 8 10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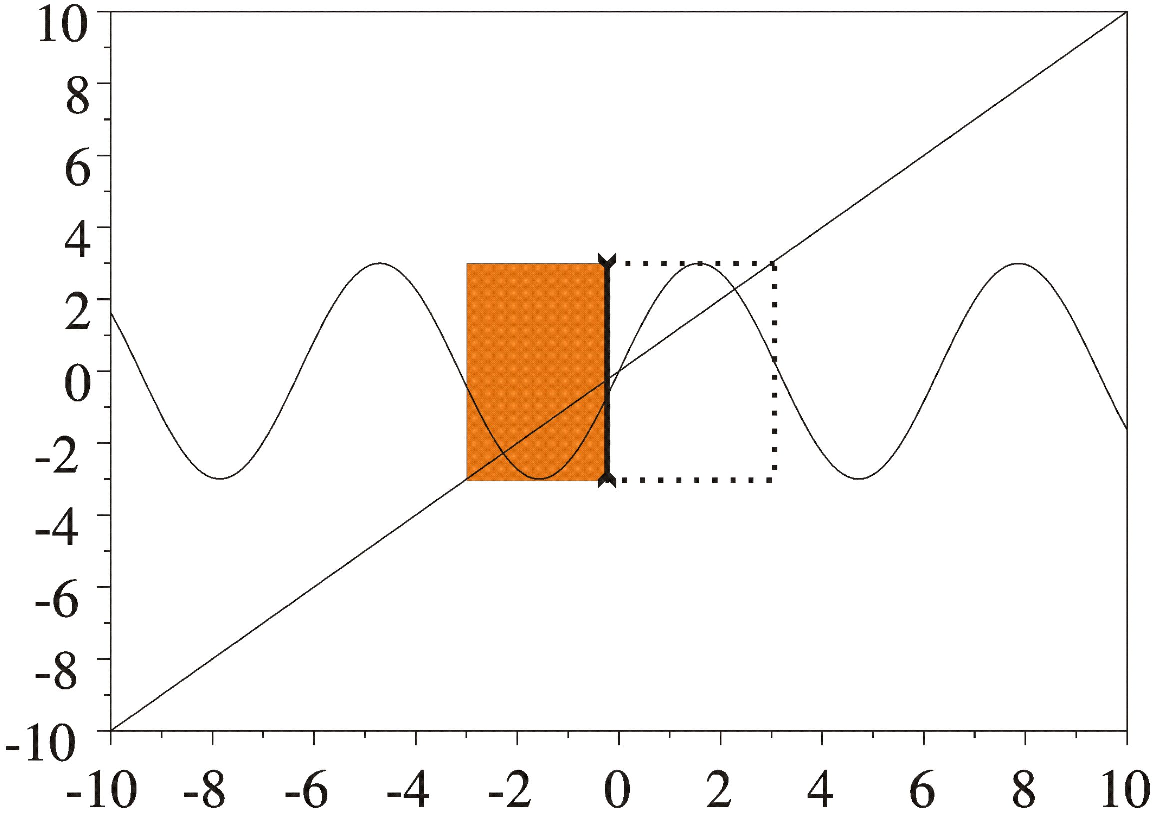

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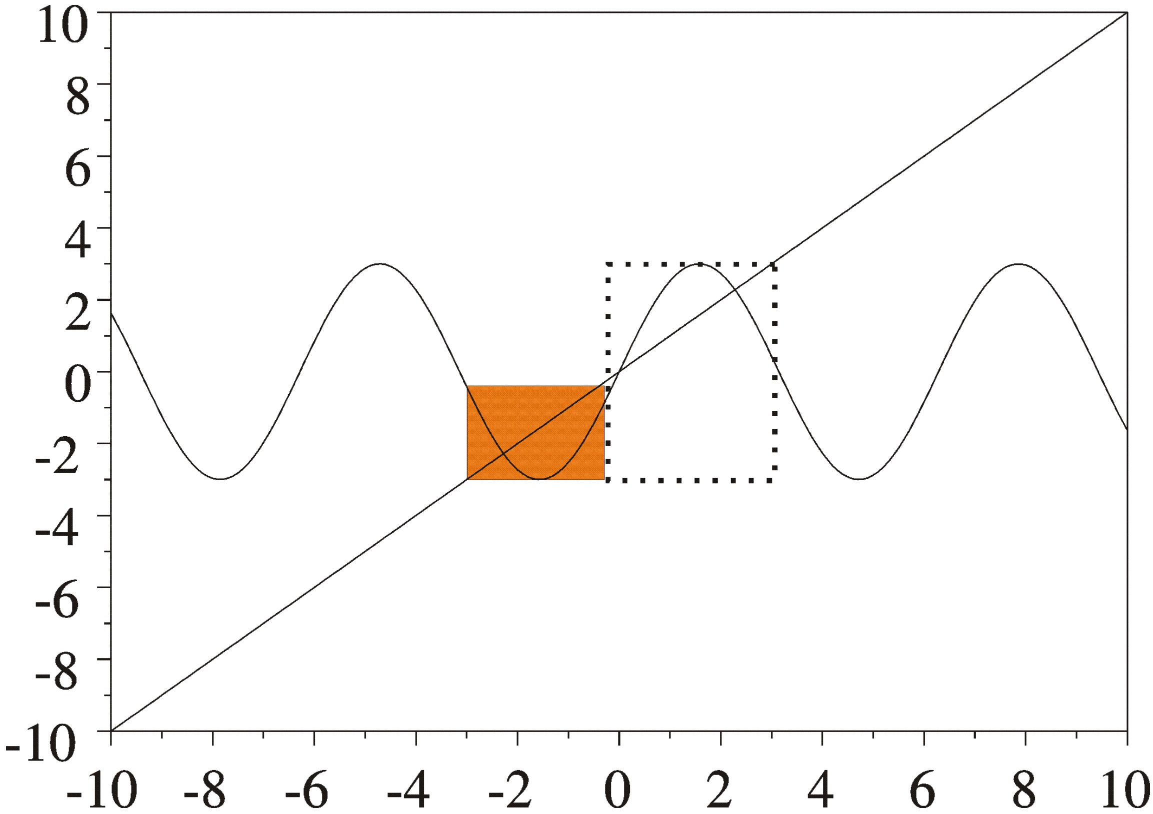

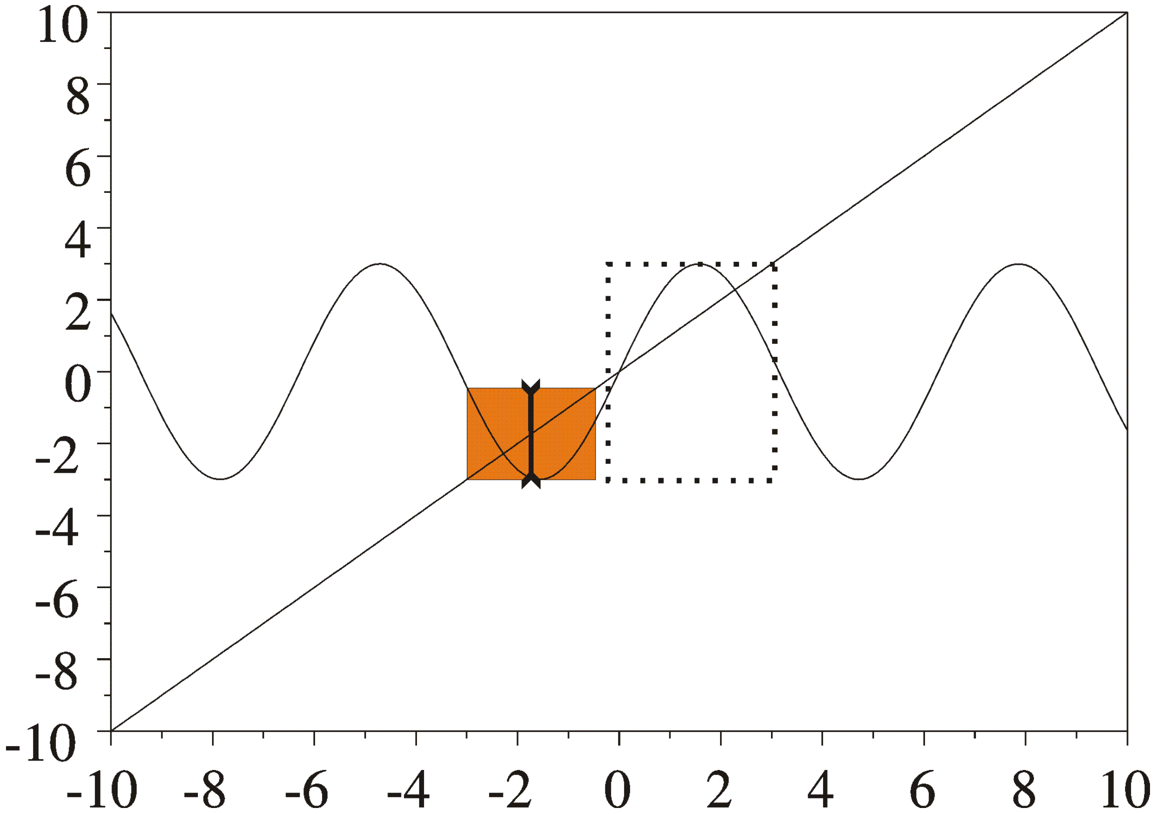

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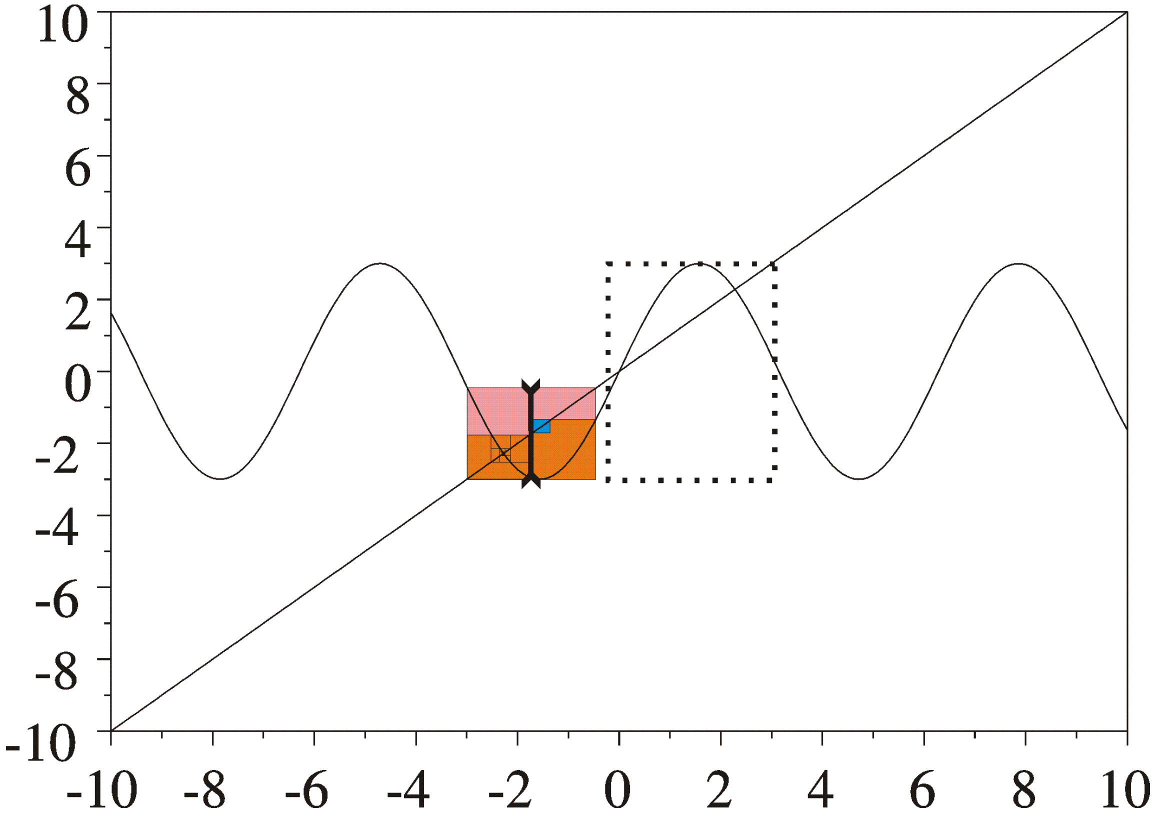
8
10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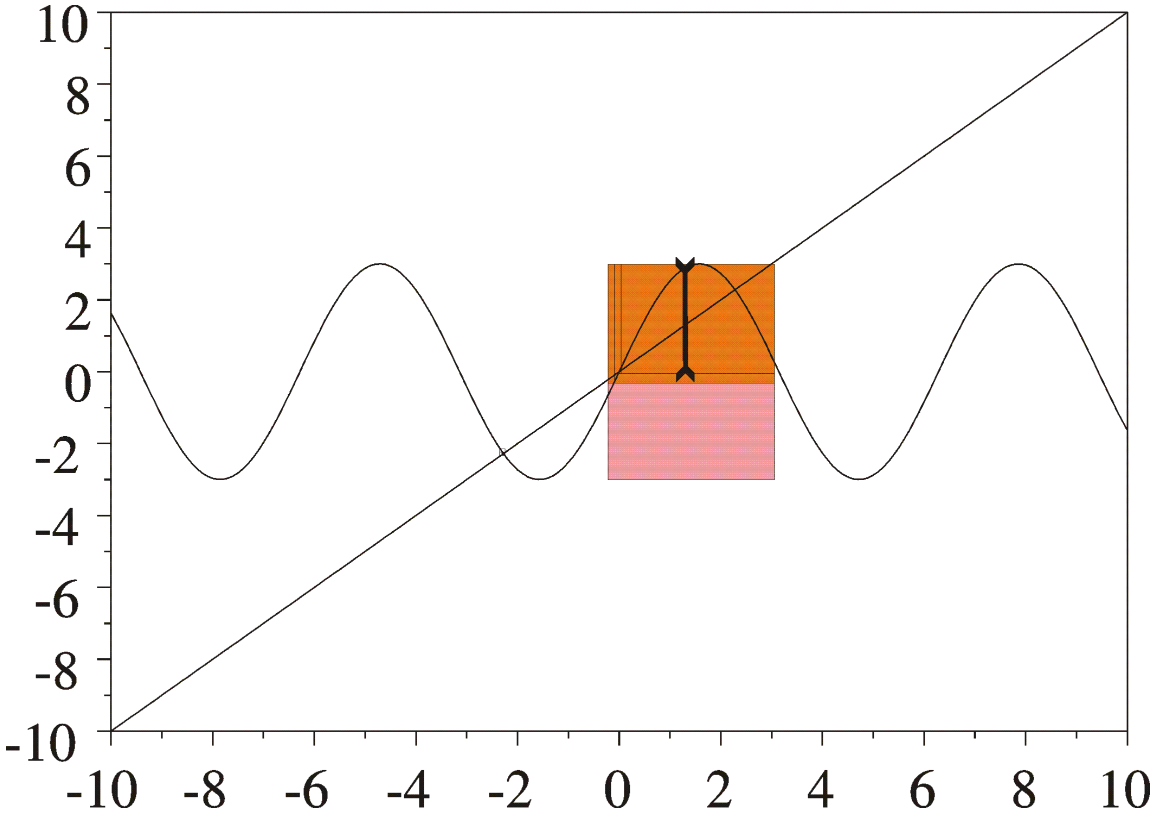
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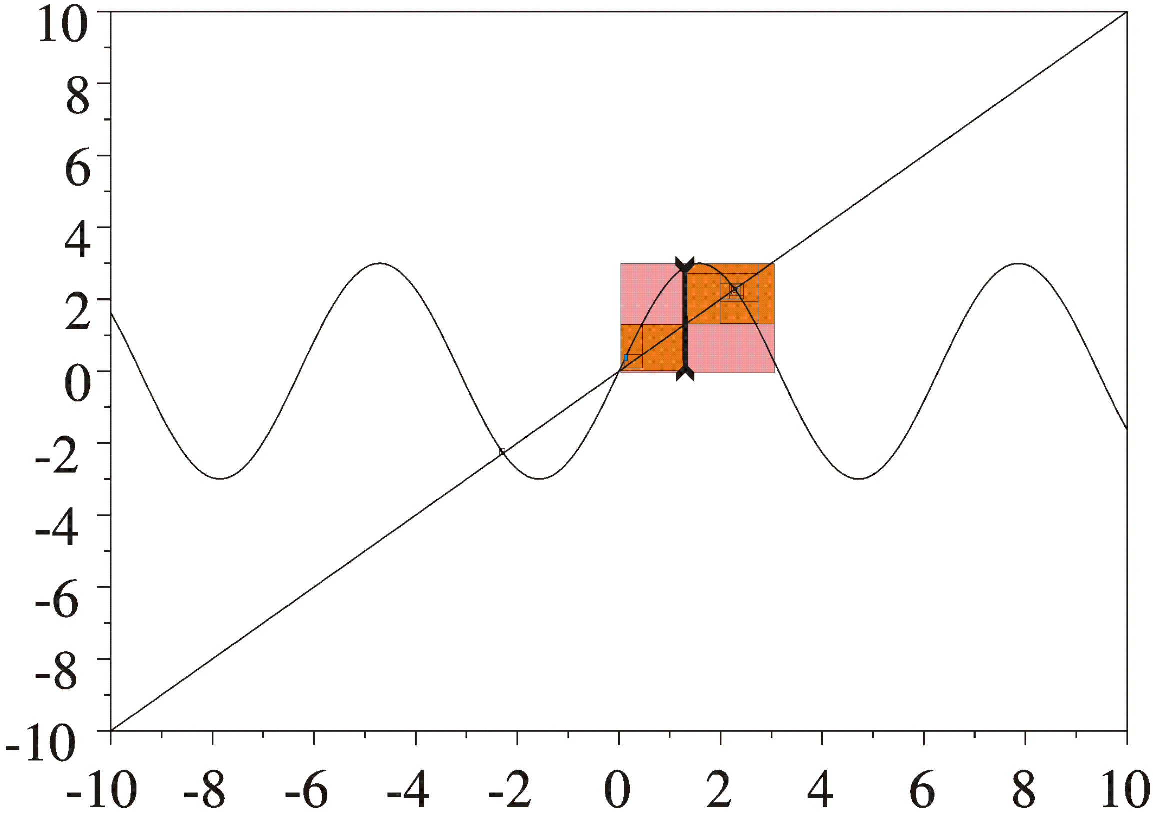
-10 -8 -6 -4 -2 0 2 4 6 8 10



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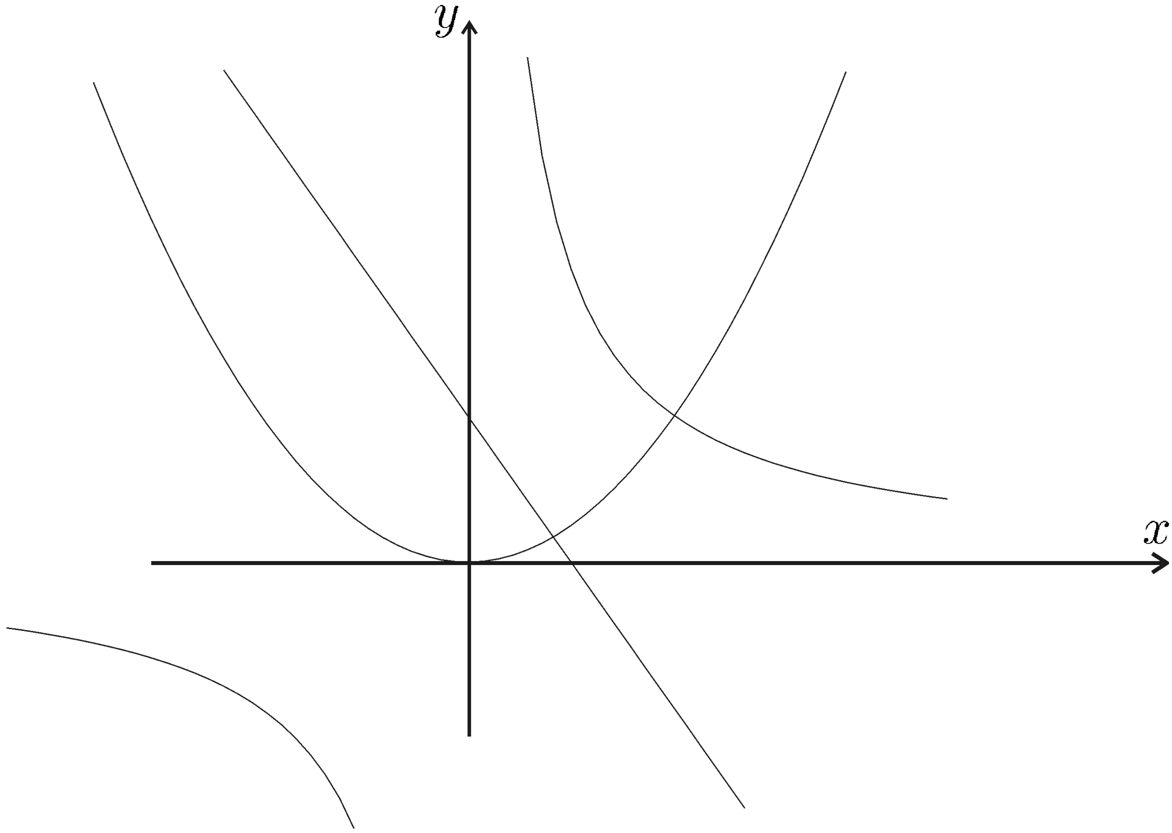
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10

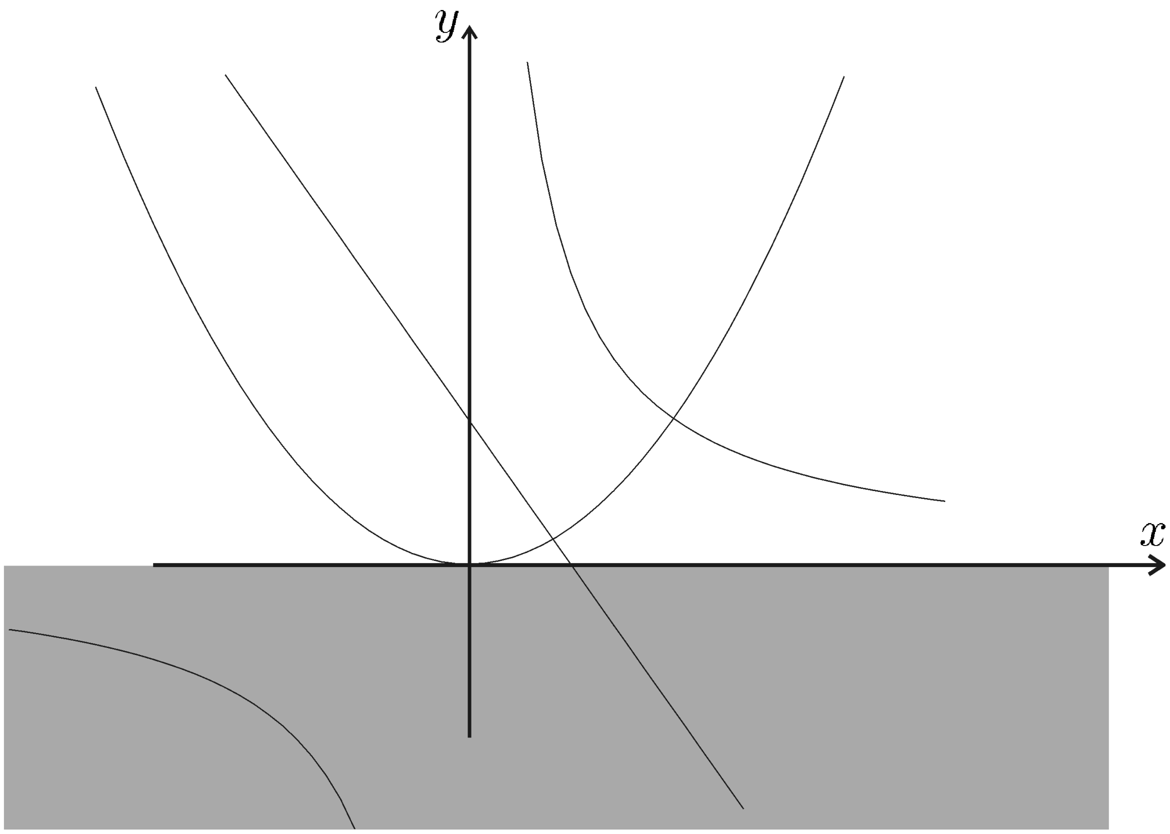
3.5 Example 3

Consider the problem:

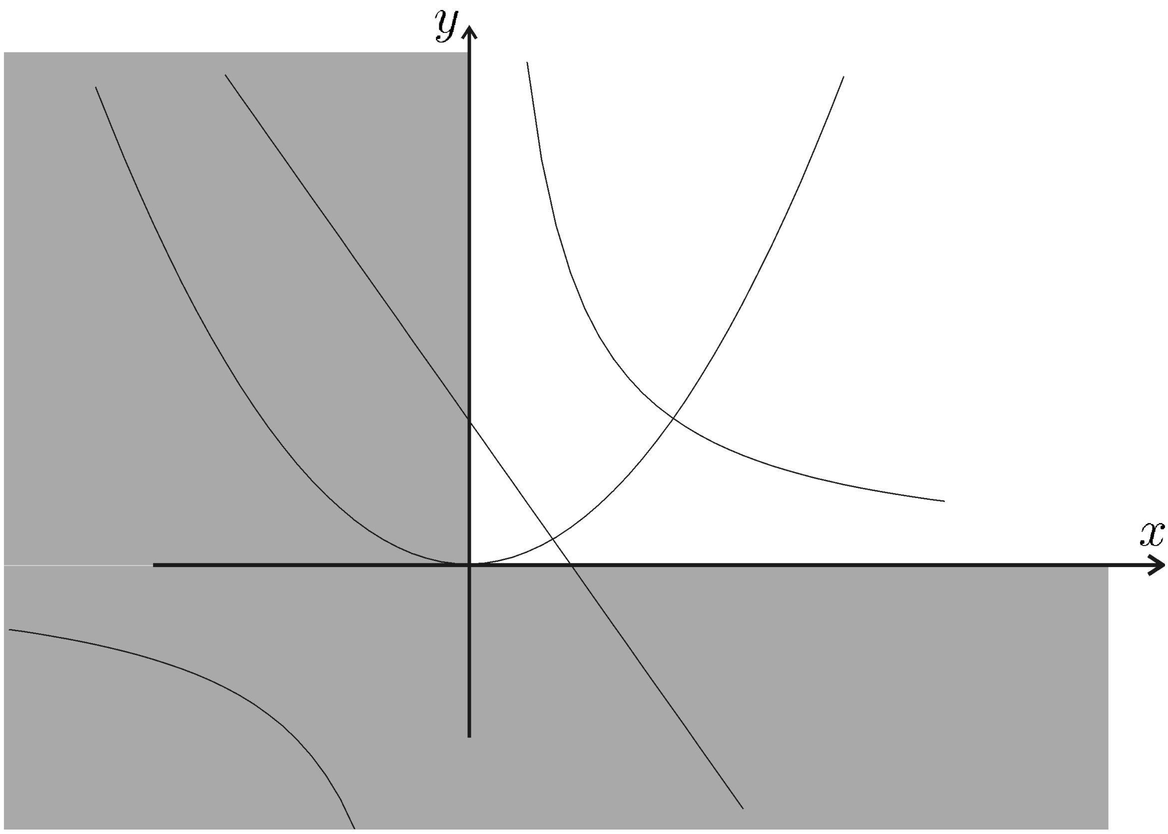
$$\begin{cases} (C_1): & y = x^2 \\ (C_2): & xy = 1 \\ (C_3): & y = -2x + 1 \end{cases}$$



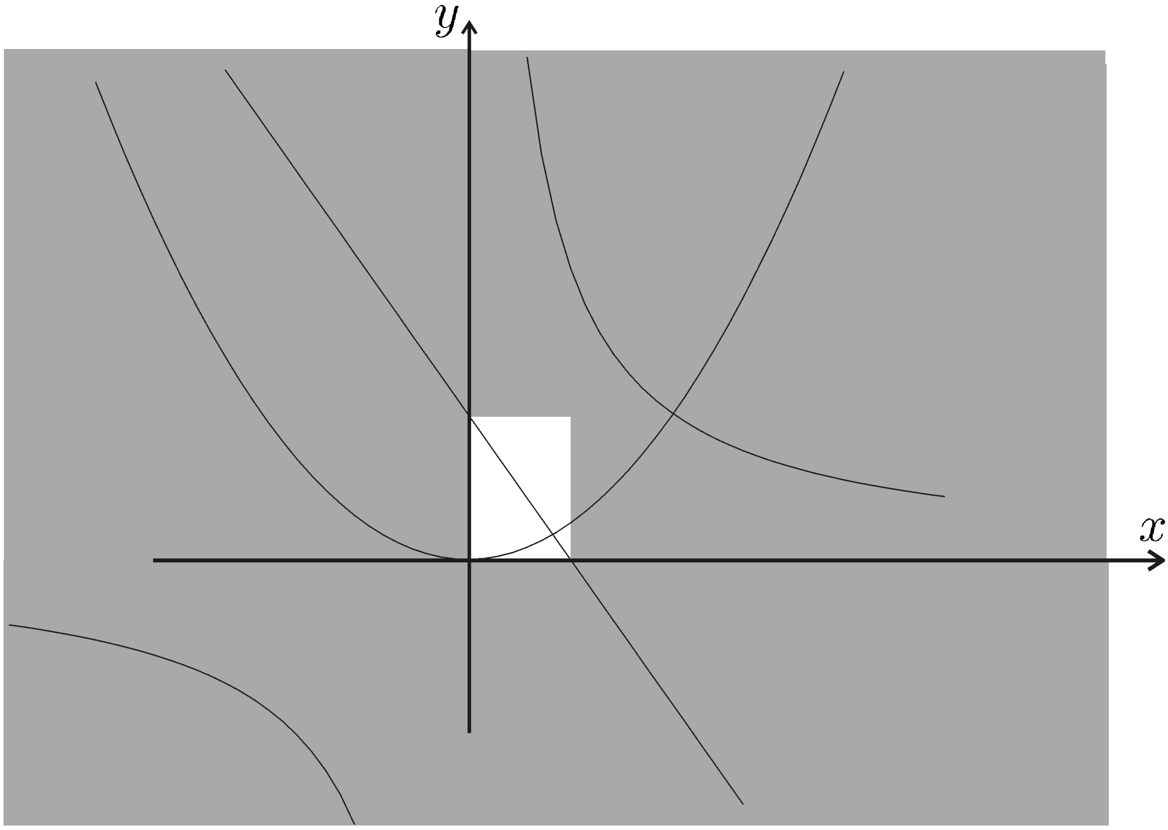
y
x



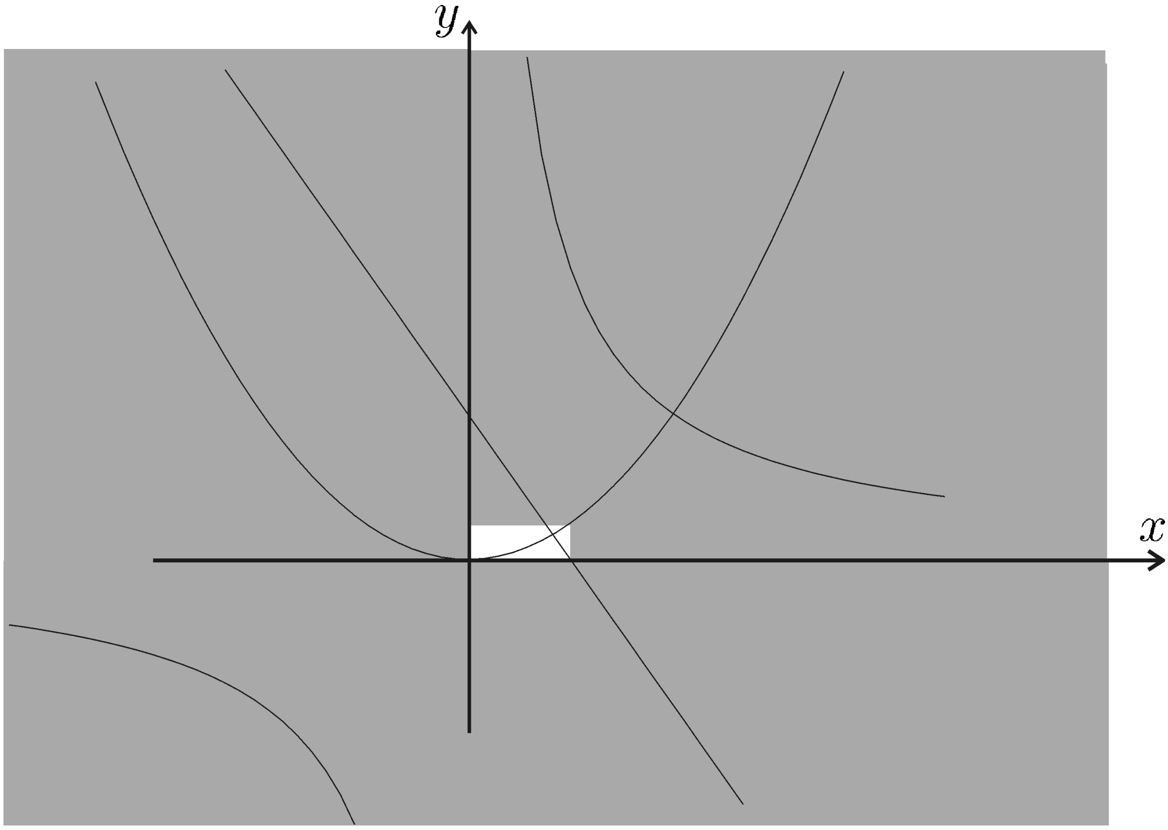
y
x



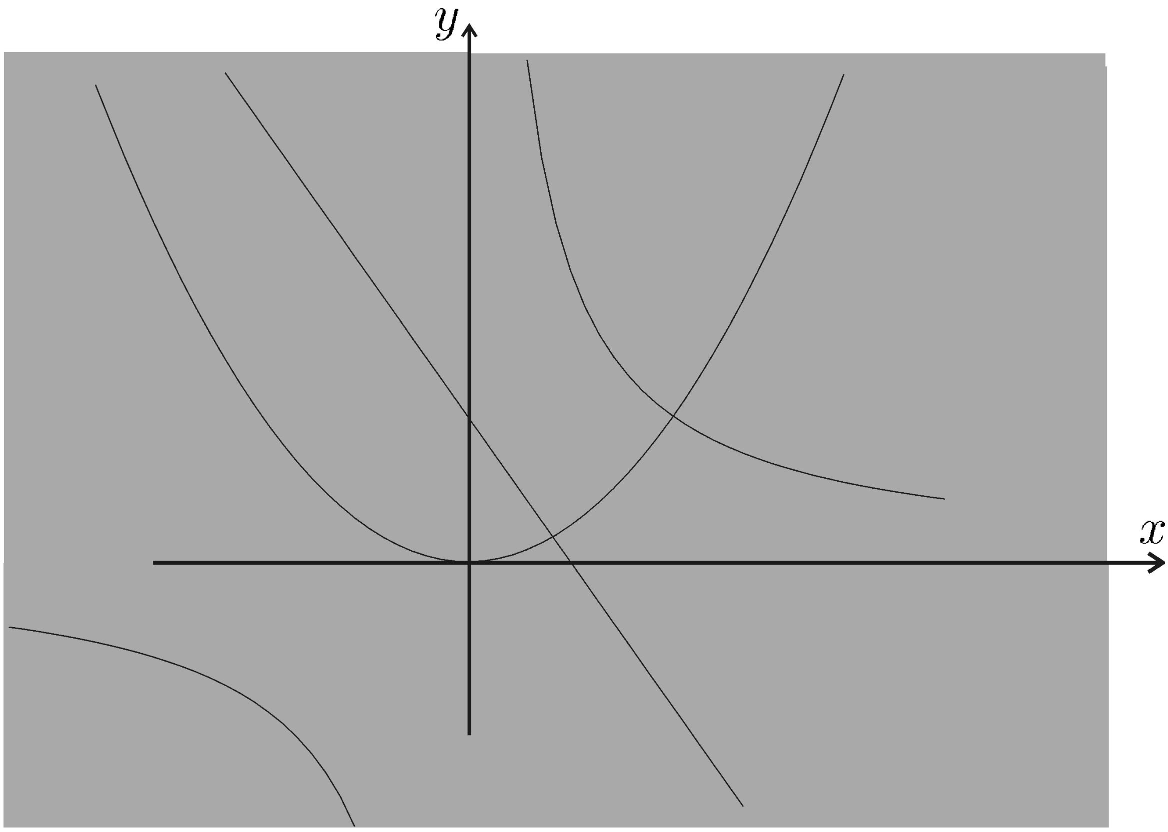
y
x



y
x



y
x



y
x

$$
\begin{array}{lll}
(C_1) \Rightarrow & y \in & [-\infty, \infty]^2 \ = \ [0, \infty] \\
(C_2) \Rightarrow & x \in & 1/[0, \infty] \ = [0, \infty] \\
(C_3) \Rightarrow & y \in & [0, \infty] \ \cap \ ((-2)\,.[0, \infty] + 1) \\
 & & = [0, \infty] \ \cap \ ([-\infty, 1]) = [0, 1] \\
 & x \in & [0, \infty] \ \cap \ (-[0, 1]/2 + 1/2) = [0, \frac{1}{2}] \\
(C_1) \Rightarrow & y \in & [0, 1] \cap [0, 1/2]^2 = [0, 1/4] \\
(C_2) \Rightarrow & x \in & [0, 1/2] \ \cap \ 1/[0, 1/4] = \emptyset \\
 & y \in & [0, 1/4] \cap 1/\emptyset = \emptyset
\end{array}
$$

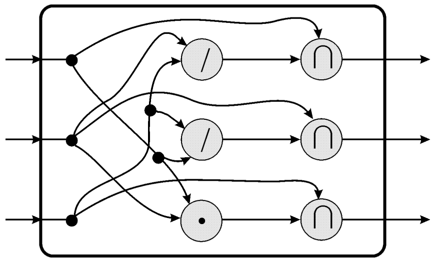
3.6 Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
repetition	$\mathcal{C}^\infty \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$
repeat intersection	$\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$
repeat union	$\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^\infty$

3.7 A link between matrices and contractors

$$\begin{array}{ccc} \text{linear application} & \longrightarrow & \text{matrices} \\ \mathcal{L}:\left\{\begin{array}{lcl} \alpha & = & 2a+3h \\ \gamma & = & h-5a \end{array}\right. & \longrightarrow & \mathbf{A}=\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \end{array}$$

We have a matrix algebra and Matlab.
We have: var$(\mathcal{L}) = \{a, h\}$, covar$(\mathcal{L}) = \{\alpha, \gamma\}$.
But we cannot write: var$(\mathbf{A}) = \{a, h\}$, covar$(\mathbf{A}) = \{\alpha, \gamma\}$.

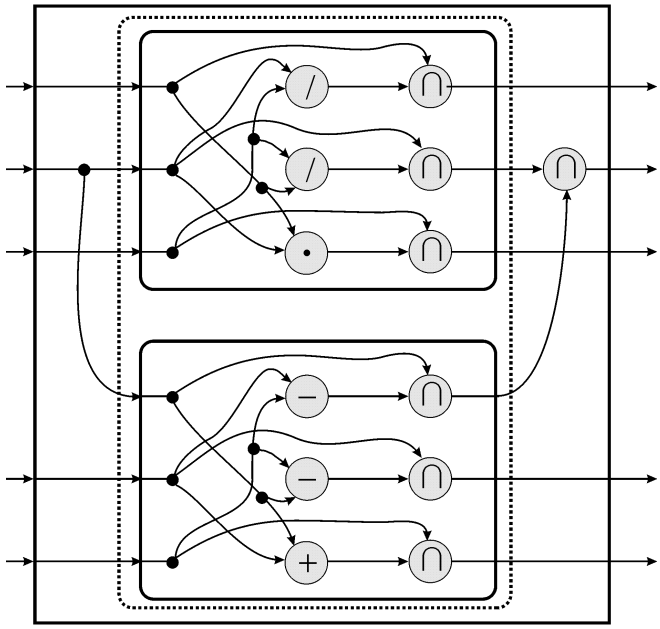
constraint	$\longrightarrow$	contractor
$a \cdot b = z$	$\longrightarrow$	

Contractor fusion

$$\begin{cases} a \cdot b = z & \longrightarrow \quad \mathcal{C}_1 \\ b + c = d & \longrightarrow \quad \mathcal{C}_2 \end{cases}$$

Since b occurs in both constraints, we fuse the two contractors as:

$$\begin{aligned} \mathcal{C} &= \mathcal{C}_1 \times \mathcal{C}_2 \rfloor_{(2,1)} \\ &= \mathcal{C}_1 | \mathcal{C}_2 \text{ (for short)} \end{aligned}$$



4 Robust parameter estimation

Exercise. A robot measures its own distance to three marks. The distances and the coordinates of the marks are

mark	x_i	y_i	d_i
1	0	0	$[22, 23]$
2	10	10	$[10, 11]$
3	30	-30	$[53, 54]$

1) Define the set $\mathbb{X}$ al all feasible positions.
2) Build the contractor associated with $\mathbb{X}$.
2) Build the contractor associated with $\overline{\mathbb{X}}$.

Solution.

$$\mathbb{X} = \bigcap_{i \in \{1,2,3\}} \underbrace{\left\{ (x,y) \;\mid\; (x-x_i)^2 + (y-y_i)^2 \in \left[d_i^-, d_i^+ \right] \right\}}_{\mathbb{X}_i}$$

$$
\begin{aligned}
\overline{\mathbb{X}} &= \overline{\bigcap_{i\in\{1,2,3\}} \mathbb{X}_i} = \bigcup_{i\in\{1,2,3\}} \overline{\mathbb{X}_i} \\
&= \bigcup_{i\in\{1,2,3\}} \Big\{(x,y) \;|\; (x-x_i)^2 + (y-y_i)^2 \in \Big[-\infty, d_i^-\Big] \\
&\qquad \cup \Big\{(x,y) \;|\; (x-x_i)^2 + (y-y_i)^2 \in \Big[d_i^+, \infty\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= \bigcap_{i \in \{1,2,3\}} \mathcal{D}_{\left[d_i^-, d_i^+\right]} \\
\overline{\mathcal{C}} &= \bigcup_{i \in \{1,2,3\}} \left(\mathcal{D}_{\left[-\infty, d_i^-\right]}\right) \cup \left(\mathcal{D}_{\left[d_i^+, \infty\right]}\right)
\end{aligned}
$$

4.1 Relaxed intersection

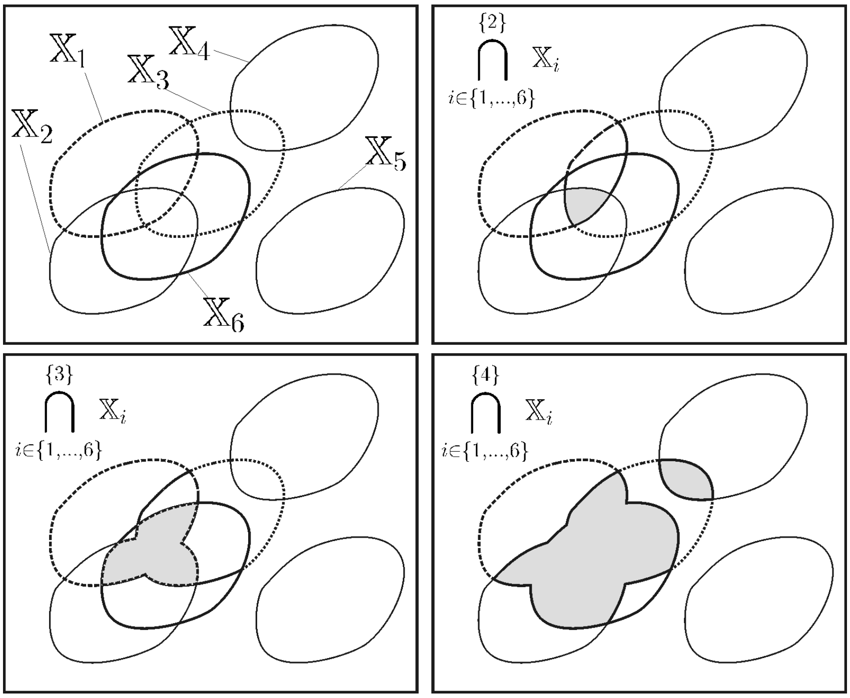
Dealing with outliers

$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Consider m sets $\mathbb{X}_{\mathbf{1}}, \ldots, \mathbb{X}_m$ of $\mathbb{R}^n$. The q-relaxed intersection $\bigcap^{\{q\}} \mathbb{X}_i$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most.

We have

$$\mathbf{x} \in \bigcap^{\{q\}} \mathbb{X}_i \Leftrightarrow \# \{i | \mathbf{x} \in \mathbb{X}_i\} \geq m - q$$


$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\bigcap_{i\in\{1,\ldots,6\}}^{\{2\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\ldots,6\}}^{\{3\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\ldots,6\}}^{\{4\}} \mathbb{X}_i$

Exercise. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{1\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{6\}} \mathbb{X}_i = ?$$

Solution. we have

$$\bigcap^{\{0\}} \mathbb{X}_i = \emptyset$$

$$\bigcap^{\{1\}} \mathbb{X}_i = \emptyset$$

$$\bigcap^{\{5\}} \mathbb{X}_i = \bigcup \mathbb{X}_i$$

$$\bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}^2$$

Exercise. Consider 8 intervals: $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?, \quad \bigcap^{\{1\}} \mathbb{X}_i = ?, \quad \bigcap^{\{2\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{3\}} \mathbb{X}_i = ?, \quad \bigcap^{\{4\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?, \quad \bigcap^{\{6\}} \mathbb{X}_i = ?.$$

Solution. For $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$, we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset, \ \bigcap^{\{1\}} \mathbb{X}_i = [3,4], \ \bigcap^{\{2\}} \mathbb{X}_i = [3,4], \\
\bigcap^{\{3\}} \mathbb{X}_i &= [2,4] \cup [6,7], \ \bigcap^{\{4\}} \mathbb{X}_i = [2,7], \\
\bigcap^{\{5\}} \mathbb{X}_i &= [1,9], \ \bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}.
\end{aligned}$$

If $\mathbb{X}_i$'s are intervals, the relaxed intersection can be computed with a complexity of $m \log m$.

Take all bounds of all intervals with their brackets.

Bounds	1	4	2	4	2	7	6	9	3	4	3	7
Brackets	[	]	[	]	[	]	[	]	[	]	[	]

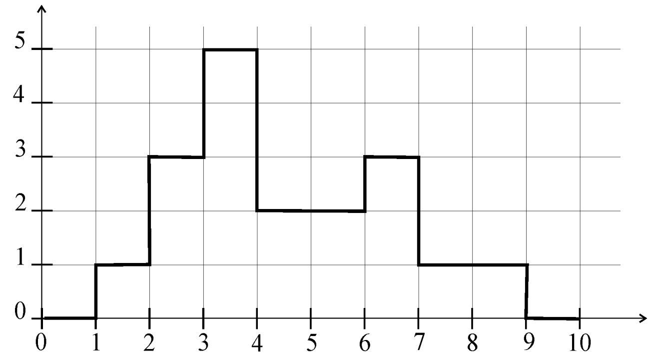
Sort the columns with respect the bounds:

Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]

Scan from left to right, counting $+1$ for '[' and -1 for ']':

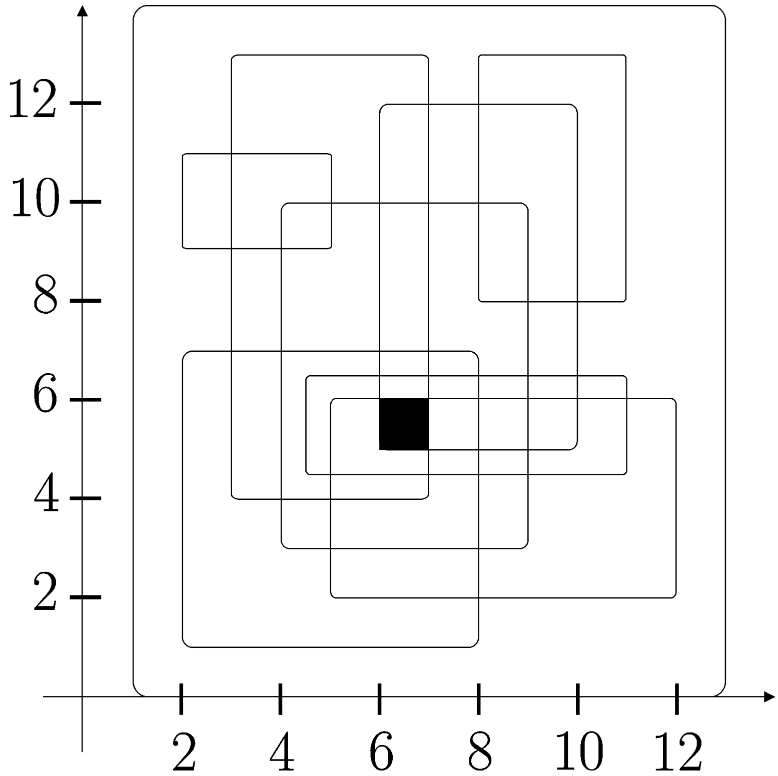
Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]
Sum	1	2	3	4	5	4	3	2	3	2	1	0

Read the q-intersections



Set-membership function associated with the 6 intervals

Computing the q relaxed intersection of m boxes is tractable.



The black box is the 2-intersection of 9 boxes

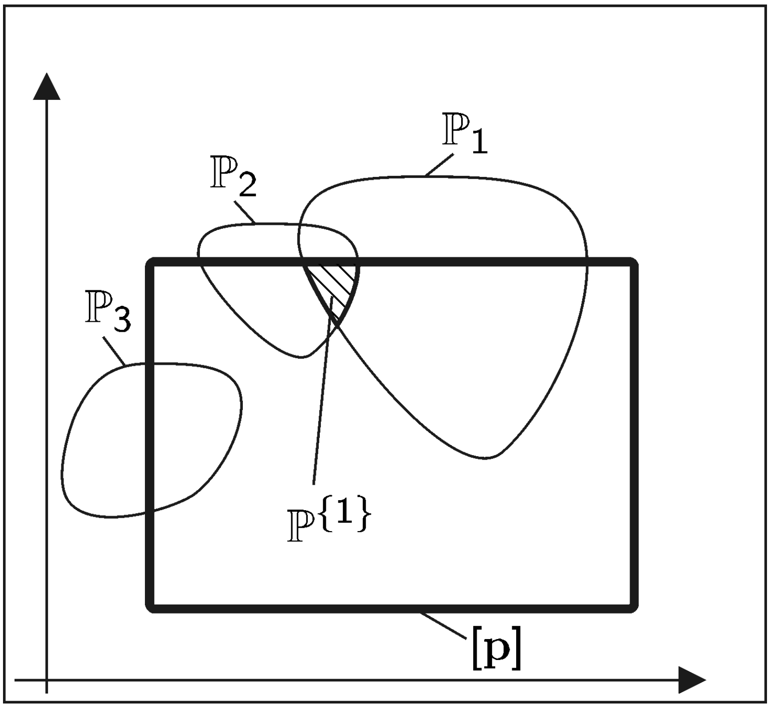
From the De Morgan's law, we get

$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} = \overline{\bigcup^{\{m-q-1\}} \mathbb{X}_i} = \bigcap^{\{m-q-1\}} \overline{\mathbb{X}_i}$$

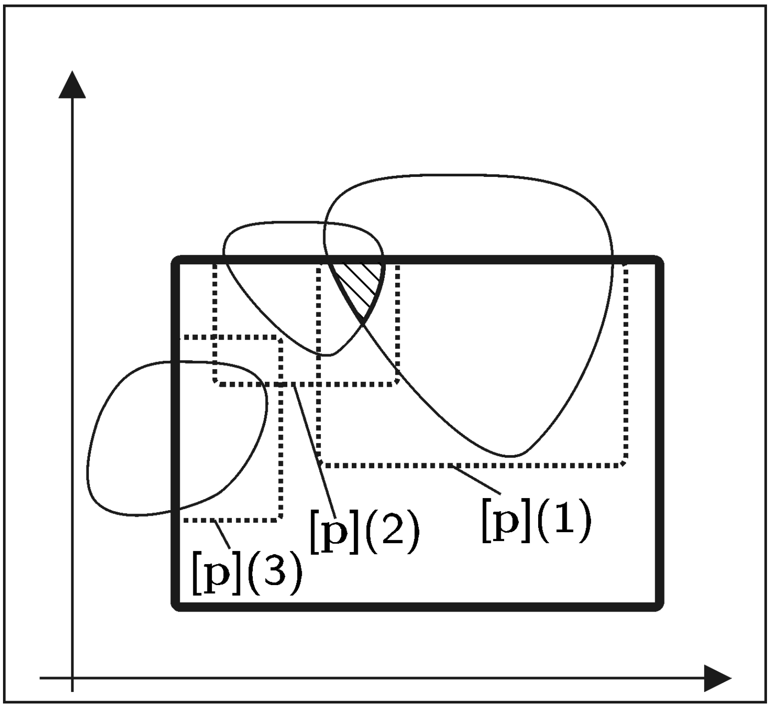
Relaxation of contractors

We define the q-relaxed intersection between m contractors

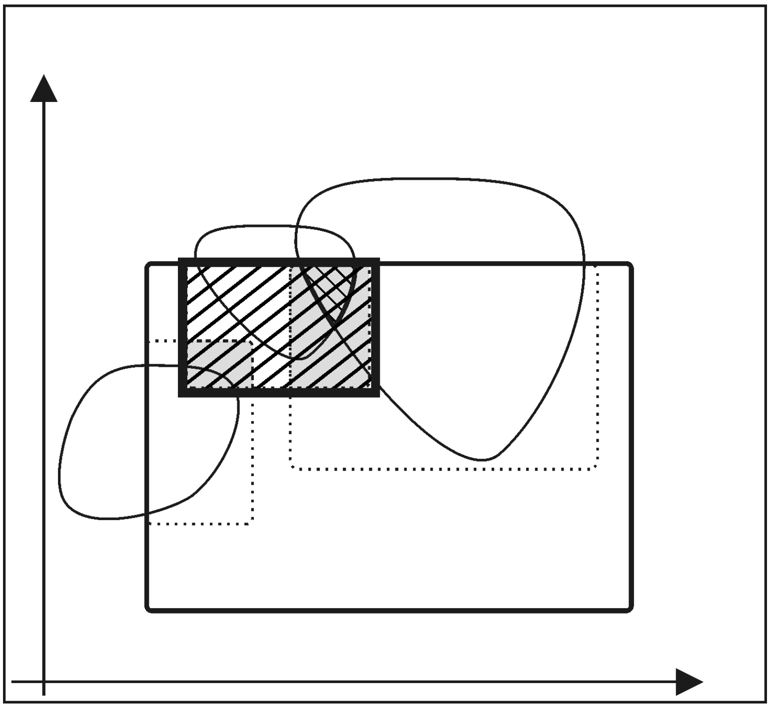
$$\mathcal{C} = \left(\bigcap_{i \in \{\mathbf{1},\dots,m\}}^{\{q\}} \mathcal{C}_i \right) \Leftrightarrow \forall \, [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\left([\mathbf{x}]\right) = \bigcap^{\{q\}} \mathcal{C}_i\left([\mathbf{x}]\right).$$

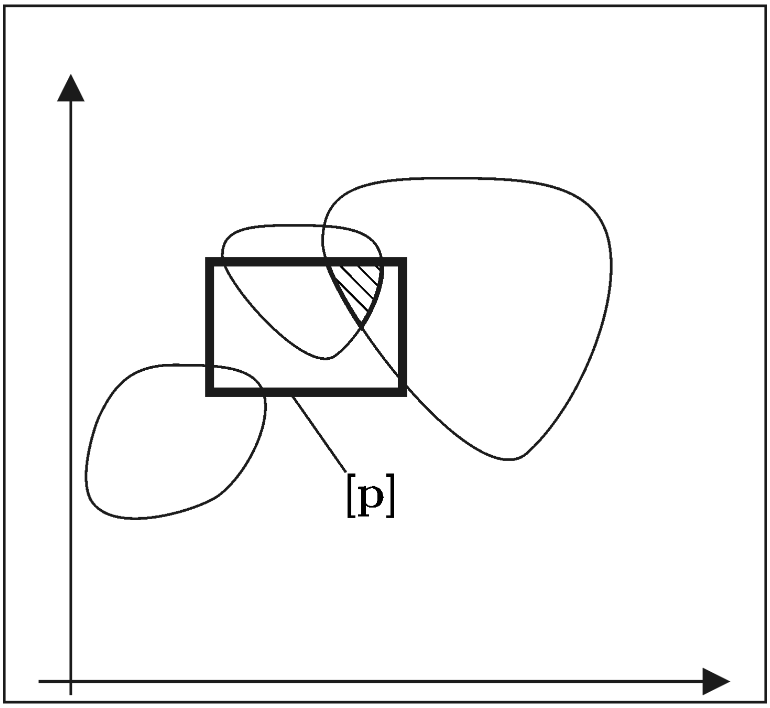


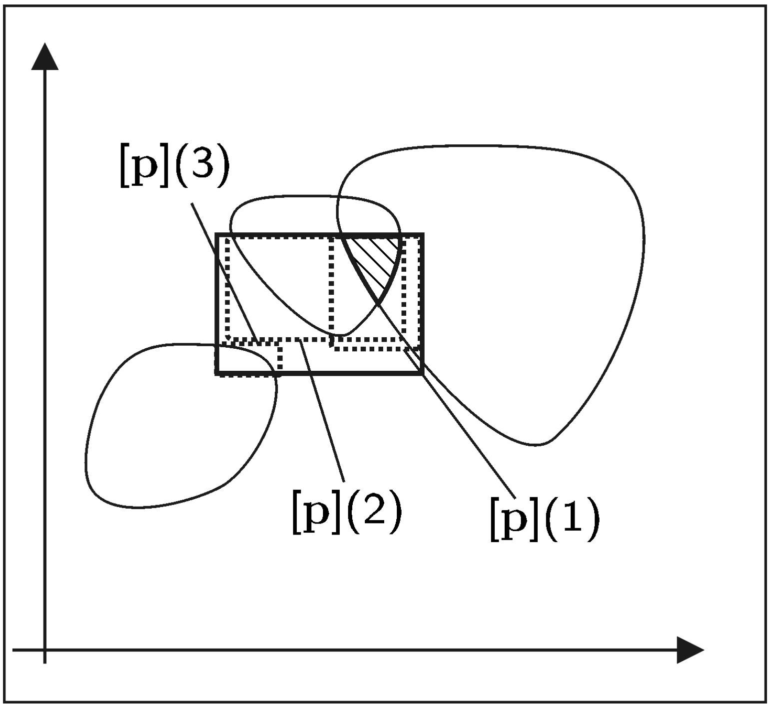
$\mathbb{P}_1$
$\mathbb{P}_2$
$\mathbb{P}_3$
$\mathbb{P}^{\{1\}}$
[p]

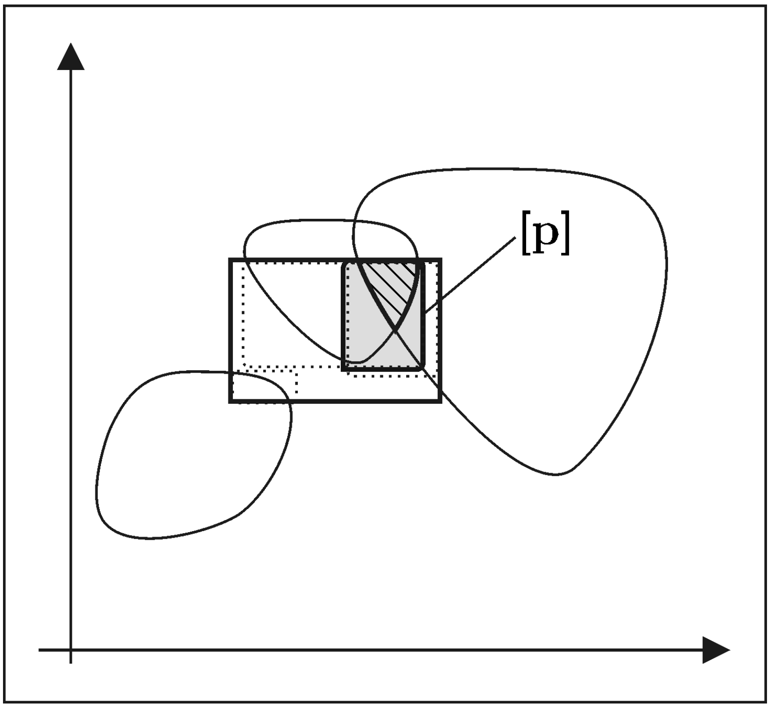


[p](1)
[p](2)
[p](3)



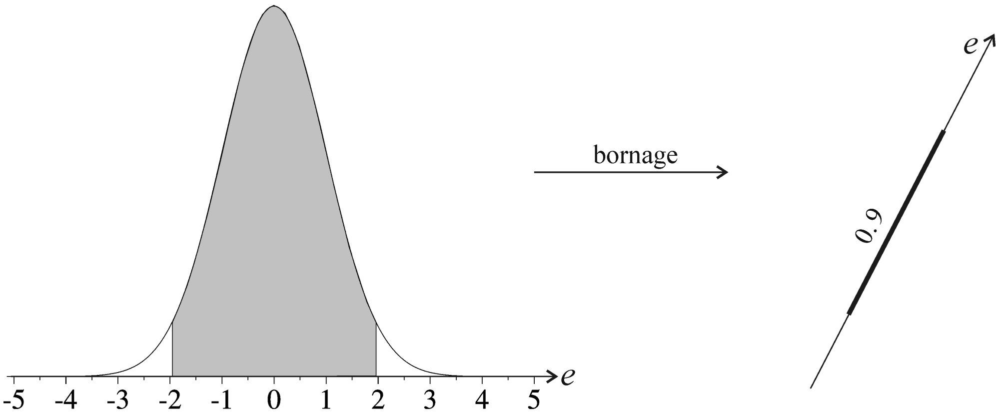

[p]


[p](3)
[p](2)
[p](1)

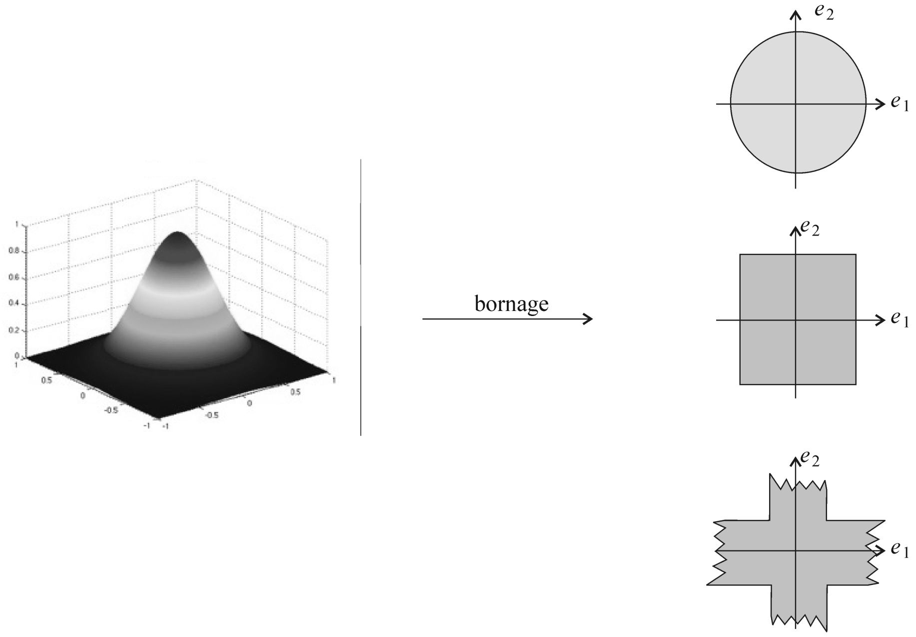


[p]

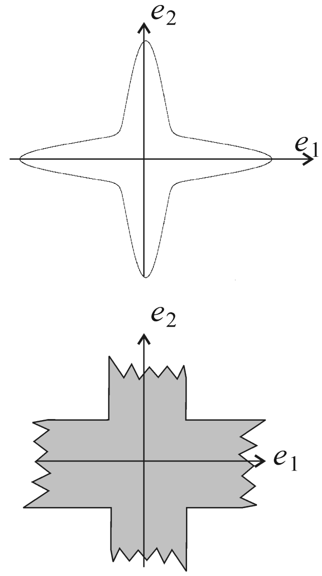
4.2 Probabilistic motivation



-5
-4
-3
-2
-1
0
1
2
3
4
5
e
bornage
0.9
e


bornage
e_2
e_1

$$\Pi(\mathbf{e}) \propto \left(\exp(-e_1^2) + \exp\left(-\frac{e_1^2}{10}\right)\right) * \left(\exp(-e_2^2) + \exp\left(-\frac{e_2^2}{10}\right)\right) \xrightarrow{\text{bornage}}$$



Consider the error model

$$\mathbf{e} = \underbrace{\mathbf{y} - \boldsymbol{\psi}(\mathbf{p})}_{\mathbf{f}(\mathbf{y},\mathbf{p})}.$$

y_i is an *inlier* if $e_i \in [e_i]$ and an *outlier* otherwise. We assume that

$$\forall i,\ \mathsf{Pr}\left(e_i \in [e_i]\right) = \pi$$

and that all e_i's are independent.

Equivalently,

$$\begin{cases} f_1(\mathbf{y}, \mathbf{p}) \in [e_1] & \text{with a probability } \pi \\ \vdots & \vdots \\ f_m(\mathbf{y}, \mathbf{p}) \in [e_m] & \text{with a probability } \pi \end{cases}$$

Having k inliers follows a binomial distribution

$$\frac{m!}{k!\,(m-k)!}\pi^k.\,(1-\pi)^{m-k}\,.$$

The probability of having more than q outliers is thus

$$\gamma\left(q,m,\pi\right) \stackrel{\text{def}}{=} \sum_{k=0}^{m-q-1} \frac{m!}{k!\left(m-k\right)!}\pi^k.\left(1-\pi\right)^{m-k}.$$

Example. If $m = 1000$, $q = 900, \pi = 0.2$, we get $\gamma\left(q,m,\pi\right) = 7.04 \times 10^{-16}$. Thus having more than 900 outliers can be seen as a rare event.

4.3 Robust bounded error estimation

$$\mathbb{S} = \bigcap_{i}^{\{q\}} \left\{\mathbf{p} \in \mathbb{R}^n \mid f_i\left(\mathbf{p}\right) \in [y_i]\right\}$$

We build the following contractors

$$
\begin{array}{rcl}
\mathcal{C}_i & : & f_i(\mathbf{p}) \in [y_i] \\
\overline{\mathcal{C}_i} & : & f_i(\mathbf{p}) \notin [y_i] \\
\mathcal{C} & = & \bigcap_{i}^{\{q\}} \mathcal{C}_i \\
\overline{\mathcal{C}} & = & \overline{\bigcap_{i}^{\{q\}} \mathcal{C}_i} = \bigcap^{\{n-q-1\}} \overline{\mathcal{C}_i}
\end{array}
$$

Then we call a paver with $\overline{\mathcal{C}}$ and $\mathcal{C}$.

4.4 Application to localization

A robot measures distances to three beacons.

beacon	x_i	y_i	$[d_i]$
1	1	3	$[1, 2]$
2	3	1	$[2, 3]$
3	-1	-1	$[3, 4]$

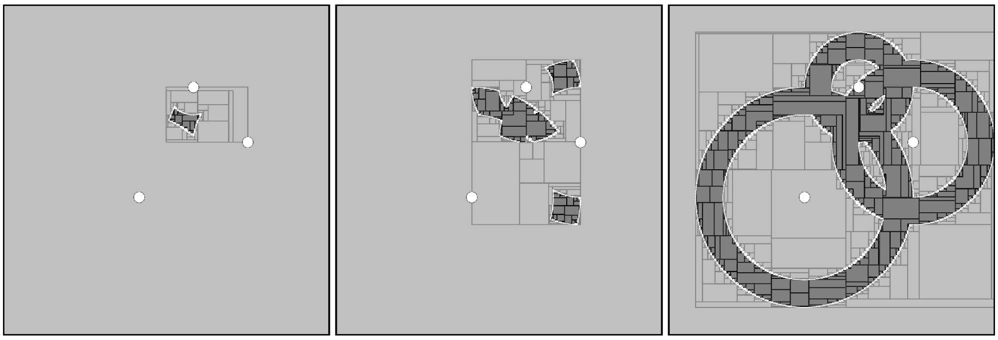
The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5]\right.$$

where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$.

$$
\begin{aligned}
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{0\}}\right) &= 0.729 \\
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{1\}}\right) &= 0.972 \\
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{2\}}\right) &= 0.999
\end{aligned}
$$



Probabilistic sets $\mathbb{P}^{\{0\}}, \mathbb{P}^{\{1\}}, \mathbb{P}^{\{2\}}$.