Méthodes Intervalles et Applications

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1 Interval computation

1.1 Set theory

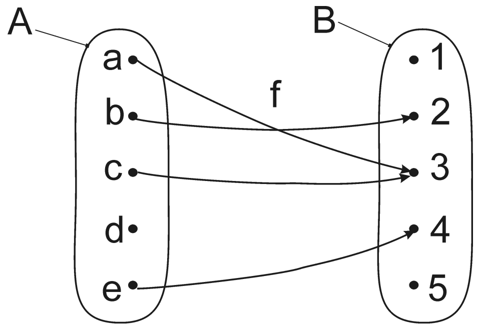
The *direct image* of $\mathbb{X}$ by f is

$$f(\mathbb{X}) \triangleq \{f(x) \mid x \in \mathbb{X}\}.$$

The *reciprocal image* of $\mathbb{Y}$ by f is

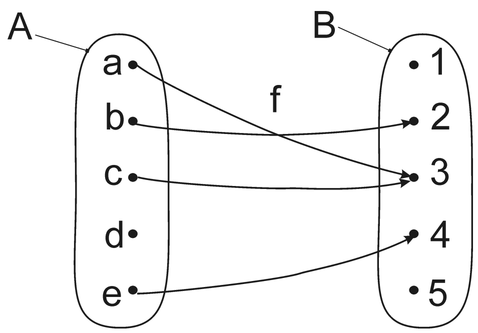
$$f^{-1}(\mathbb{Y}) \triangleq \{x \in \mathbb{X} \mid f(x) \in \mathbb{Y}\}.$$

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= ?. \\ f^{-1}(B) &= ?. \\ f^{-1}(f(A)) &= ? \\ f^{-1}(f(\{b,c\})) &= ?. \end{aligned}$$

Exercise: If f is defined as follows



$$
\begin{aligned}
f(A) &= \{2, 3, 4\} = \mathsf{Im}(f). \\
f^{-1}(B) &= \{a, b, c, e\} = \mathsf{dom}(f). \\
f^{-1}(f(A)) &= \{a, b, c, e\} \subset A \\
f^{-1}(f(\{b, c\})) &= \{a, b, c\}.
\end{aligned}
$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= ? \\ f^{-1}([4,9]) &= ?. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= [4,9] \\ f^{-1}([4,9]) &= [-3,-2] \cup [2,3]. \end{aligned}$$

This is consistent with the property

$$f^{-1}\left(f\left(\mathbb{Y}\right)\right) \supset \mathbb{Y}.$$

1.2 Interval arithmetic

If $\diamond \in \{+, -, \cdot, /, \mathsf{max}, \mathsf{min}\}$

$$[x] \diamond [y] \quad = \quad [\ \{x \diamond y \mid x \in [x], y \in [y]\}\ ]\,.$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A} \subset \mathbb{R}$.

Exercise.

$$\begin{aligned}
[-1,3]+[2,5] &= [?,?], \\
[-1,3]\cdot[2,5] &= [?,?], \\
[-2,6]/[2,5] &= [?,?].
\end{aligned}$$

Solution.

$$\begin{aligned} [-1,3]+[2,5] &= [1,8],\\ [-1,3].[2,5] &= [-5,15],\\ [-2,6]/[2,5] &= [-1,3]. \end{aligned}$$

$$
\begin{aligned}
[x^-, x^+] + [y^-, y^+] = \quad & [x^- + y^-, x^+ + y^+], \\
[x^-, x^+] \cdot [y^-, y^+] = \quad & [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+, \\
& x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+],
\end{aligned}
$$

If $f \in \{$cos, sin,sqr, sqrt, log, exp, $\ldots\}$

$$f\left([x]\right) = \left[\{f(x) \mid x \in [x]\}\right].$$

Exercise.

$$\begin{aligned}
\sin([0,\pi]) &= ?,\\
\mathrm{sqr}([-1,3]) &= [-1,3]^2 = ?,\\
\mathrm{abs}([-7,1]) &= ?,\\
\mathrm{sqrt}([-10,4]) &= \sqrt{[-10,4]} = ?,\\
\log([-2,-1]) &= ?.
\end{aligned}$$

Solution.

$$
\begin{aligned}
\operatorname{sin}([0,\pi]) &= [0,1], \\
\operatorname{sqr}([-1,3]) &= [-1,3]^2 = [0,9], \\
\operatorname{abs}([-7,1]) &= [0,7], \\
\operatorname{sqrt}([-10,4]) &= \sqrt{[-10,4]} = [0,2], \\
\log([-2,-1]) &= \emptyset.
\end{aligned}
$$

1.3 Boxes

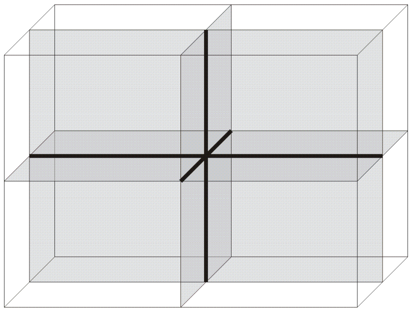
A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

The *width* $w([\mathbf{x}])$ of a box $[\mathbf{x}]$ is the length of its largest side. For instance

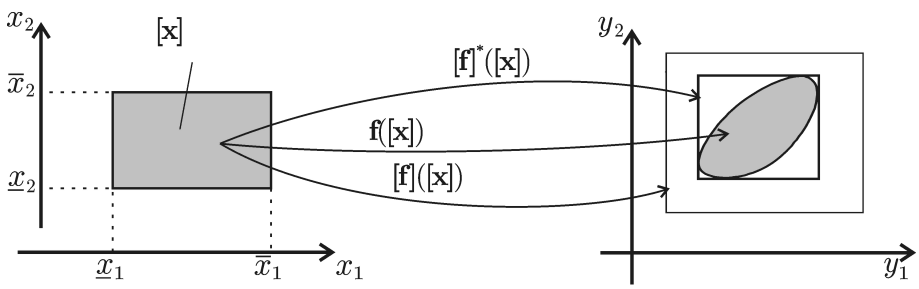
$$w([1,2]\times[-1,3])=4$$



1.4 Inclusion function

The interval function $[\mathbf{f}]$ from $\mathbb{IR}^n$ to $\mathbb{IR}^m$, is an *inclusion function* of $\mathbf{f}$ if

$$\forall\,[\mathbf{x}] \in \mathbb{IR}^n, \quad \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

The inclusion function $[\mathbf{f}]$ is

monotonic	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([\mathbf{f}]([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{y}]))$
minimal	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [\mathbf{f}]([\mathbf{x}]) = [\mathbf{f}([\mathbf{x}])]$
thin	if	$w([\mathbf{x}]) = 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) = 0$
convergent	if	$w([\mathbf{x}]) \to 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) \to 0.$

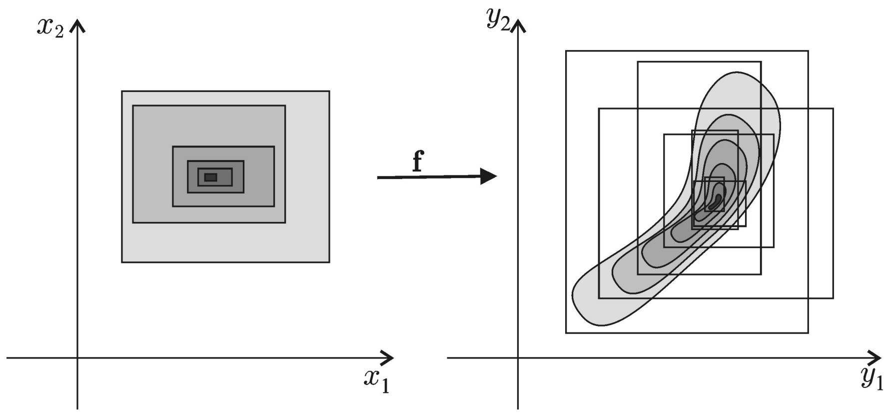
Exercise

The figure provides a nested sequence of boxes $[\mathbf{x}](k)$, their image $\mathbf{f}([\mathbf{x}])$ by a function $\mathbf{f}$ and the image by an inclusion function $[\mathbf{f}]$.

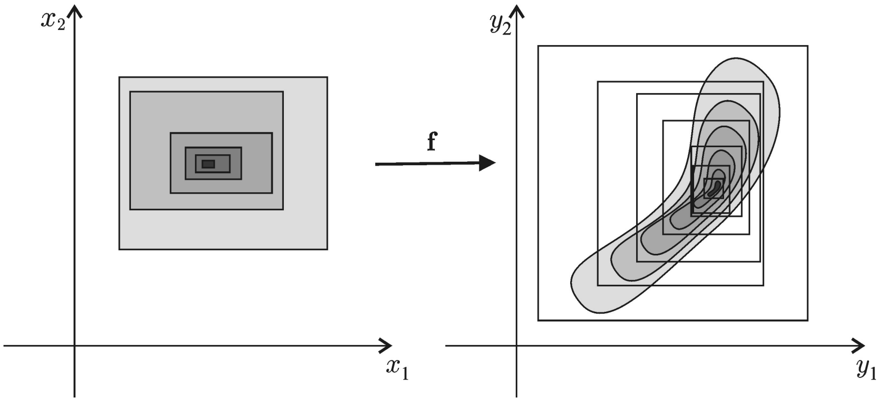
a) $[\mathbf{f}]$ is convergent.

b) $[\mathbf{f}]$ is monotonic

c) $[\mathbf{f}]$ is minimal.



Solution. $[\mathrm{f}]$ is convergent, non-monotonic, non-minimal.



Convergent and monotonic

Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3, 4]$, we have

$$\begin{aligned}
[f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\
&= [0,16] + [-6,8] + 4 \\
&= [-2,28].
\end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{ccc} \mathbb{IR}^2 & \to & \mathbb{IR}^3 \\ ([x_1], [x_2]) & \to & \left([x_1] * [x_2], [x_1]^2, [x_1] - [x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by the algorithm

Algorithm $\mathbf{f}$(in: $\mathbf{x} = (x_1, x_2, x_3)$, out: $\mathbf{y} = (y_1, y_2)$)

$z := x_1$;
for $k := 0$ to 100
    $z := x_2 \cdot (z + k \cdot x_3)$;
next;
$y_1 := z$;
$y_2 := \sin(z x_1)$;

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in: $[\mathbf{x}]$, out: $[\mathbf{y}]$)

$[z] := [x_1]$;
for $k := 0$ to 100
    $[z] := [x_2] \cdot ([z] + k \cdot [x_3])$;
next;
$[y_1] := [z]$;
$[y_2] :=$ sin$([z] * [x_1])$;

Is $[\mathbf{f}]$ convergent? thin? monotonic?

2 Subpavings

2.1 Definition

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.

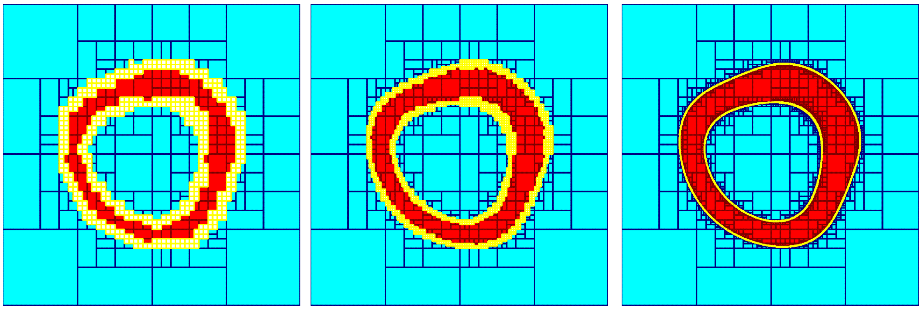
Compact sets $\mathbb{X} \subset \mathbb{R}^n$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example. The set

$$\mathbb{X} = \{(x_1, x_2) \,\big|\, x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}.$$

can be approximated by subpavings.



Set operations such as $\mathbb{Z} := \mathbb{X} + \mathbb{Y},\ \mathbb{X} := \mathbf{f}^{-1}(\mathbb{Y}), \mathbb{Z} := \mathbb{X} \cap \mathbb{Y} \ldots$ can be approximated by subpaving operations.

2.2 Set inversion

If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$.

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

$$\begin{array}{lll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

Algorithm Sivia(in: $[\mathbf{x}](0), \mathbf{f}, \mathbb{Y}$)

1 $\mathcal{L} := \{[\mathbf{x}](0)\}$;
2 pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$, 'red');
4 elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$, 'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$, 'yellow')};
6 else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$;
7 if $\mathcal{L} \neq \emptyset$, go to 2

If $\mathbf{\Delta}\mathbb{X}$ denotes the union of yellow boxes and if $\mathbb{X}^{-}$ is the union of red boxes then :

$$\mathbb{X}^{-} \subset \mathbb{X} \subset \mathbb{X}^{-} \cup \mathbf{\Delta}\mathbb{X}.$$

2.3 Bounded-error estimation

Model : $\phi(\mathbf{p}, t) = p_1 e^{-p_2 t}$.

Prior feasible box for the parameters : $[\mathbf{p}] \subset \mathbb{R}^2$

Measurement times : $t_1, t_2, \ldots, t_m$

Data bars : $[y_1^-, y_1^+], [y_2^-, y_2^+], \ldots, [y_m^-, y_m^+]$

$\mathbb{S} = \{\mathbf{p} \in [\mathbf{p}], \phi(\mathbf{p}, t_1) \in [y_1^-, y_1^+], \ldots, \phi(\mathbf{p}, t_m) \in [y_m^-, y_m^+]\}$.

If

$$\boldsymbol{\phi}\left(\mathbf{p}\right)=\left(\begin{array}{c}\phi\left(\mathbf{p},t_1\right)\\ \\ \phi\left(\mathbf{p},t_m\right)\end{array}\right)$$

and

$$[\mathbf{y}]=[y_1^-,y_1^+]\times\cdots\times[y_m^-,y_m^+]$$

then

$$\mathbb{S}=[\mathbf{p}]\cap\boldsymbol{\phi}^{-1}\left([\mathbf{y}]\right).$$

3 Contractors

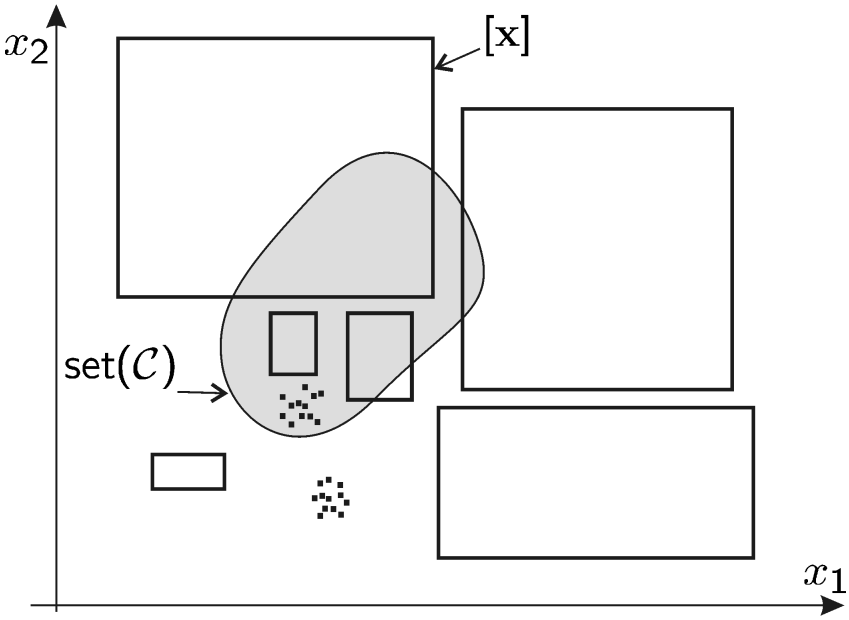
To characterize $\mathbb{X} \subset \mathbb{R}^n$, bisection algorithms bisect all boxes in all directions and become inefficient. Interval methods can still be useful if

- the solution set $\mathbb{X}$ is small (optimization problem, solving equations),
- contraction procedures are used as much as possible,
- bisections are used only as a last resort.

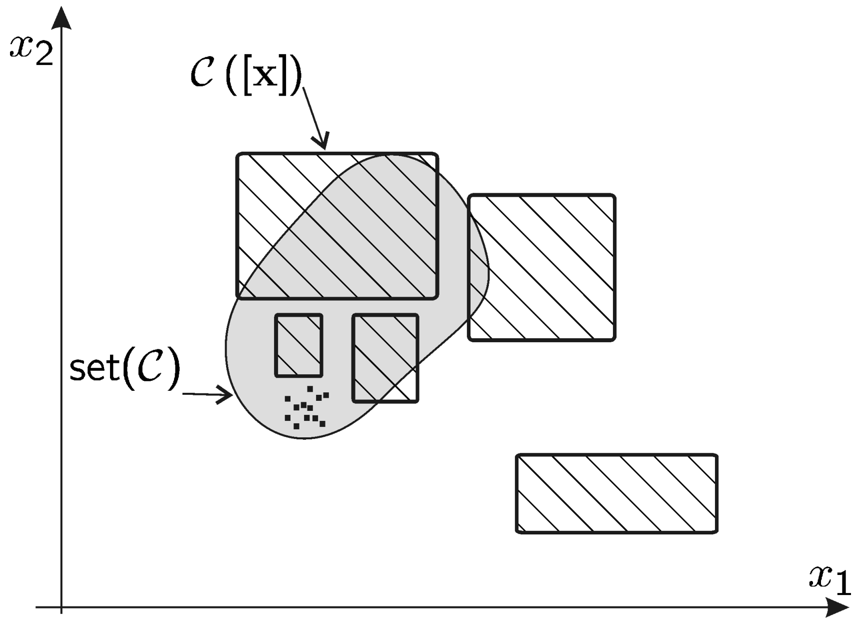
3.1 Definition

The operator $\mathcal{C}_{\mathbb{X}} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \left\{ \begin{array}{ll} \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)}, \\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness)}. \end{array} \right.$$



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
set($\mathcal{C}$)
x_1

The operator $\mathcal{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) \end{cases}$$

$\mathcal{C}_{\mathbb{X}}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset \mathcal{C}_{\mathbb{X}}([\mathbf{y}])$
$\mathcal{C}_{\mathbb{X}}$ is *minimal* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) = [[\mathbf{x}] \cap \mathbb{X}]$
$\mathcal{C}_{\mathbb{X}}$ is *thin* if	$\forall \mathbf{x} \in \mathbb{R}^n,\ \mathcal{C}_{\mathbb{X}}(\{\mathbf{x}\}) = \{\mathbf{x}\} \cap \mathbb{X}$
$\mathcal{C}_{\mathbb{X}}$ is *idempotent* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}_{\mathbb{X}}\left(\mathcal{C}_{\mathbb{X}}([\mathbf{x}])\right) = \mathcal{C}_{\mathbb{X}}([\mathbf{x}])$.

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
répétition	$\mathcal{C}^\infty \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

$\mathcal{C}_{\mathbb{X}}$ is said to be *convergent* if

$$[\mathbf{x}](k) \to \mathbf{x} \quad \Rightarrow \quad \mathcal{C}_{\mathbb{X}}([\mathbf{x}]\,(k)) \to \{\mathbf{x}\} \cap \mathbb{X}.$$

3.2 Projection of constraints

Let x, y, z be 3 variables such that

$$\begin{array}{rcl} x & \in & [-\infty, 5], \\ y & \in & [-\infty, 4], \\ z & \in & [6, \infty], \\ z & = & x + y. \end{array}$$

The values < 2 for x, < 1 for y and > 9 for z are inconsistent.

To *project* a constraint (here, $z = x + y$), is to compute the smallest intervals which contains all consistent values.

For our example, this amounts to project onto x, y and z the set

$$\mathbb{S} = \{(x, y, z) \in [-\infty, 5] \times [-\infty, 4] \times [6, \infty] \mid z = x + y\}\,.$$

3.3 Numerical method for projection

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$\begin{aligned}
z = x + y \Rightarrow \quad z \in \quad & [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) \\
& = [6, \infty] \cap [-\infty, 9] = [6, 9]. \\
x = z - y \Rightarrow \quad x \in \quad & [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) \\
& = [-\infty, 5] \cap [2, \infty] = [2, 5]. \\
y = z - x \Rightarrow \quad y \in \quad & [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) \\
& = [-\infty, 4] \cap [1, \infty] = [1, 4].
\end{aligned}$$

The contractor associated with $z = x + y$ is.

Algorithm pplus(inout: $[z], [x], [y]$)
1 $[z] := [z] \cap ([x] + [y])$;
2 $[x] := [x] \cap ([z] - [y])$;
3 $[y] := [y] \cap ([z] - [x])$.

The projection procedure developed for plus can be extended to other ternary constraints such as mult: $z = x * y$, or equivalently

$$\text{mult} \triangleq \left\{(x, y, z) \in \mathbb{R}^3 \mid z = x * y\right\}.$$

The resulting projection procedure becomes

Algorithm pmult(inout: $[z], [x], [y]$)
1 $[z] := [z] \cap ([x] * [y])$;
2 $[x] := [x] \cap ([z] * 1/[y])$;
3 $[y] := [y] \cap ([z] * 1/[x])$.

Consider the binary constraint

$$\mathsf{exp} \triangleq \{(x, y) \in \mathbb{R}^n | y = \mathsf{exp}\,(x)\}\,.$$

The associated contractor is

Algorithm pexp(inout: $[y], [x]$)

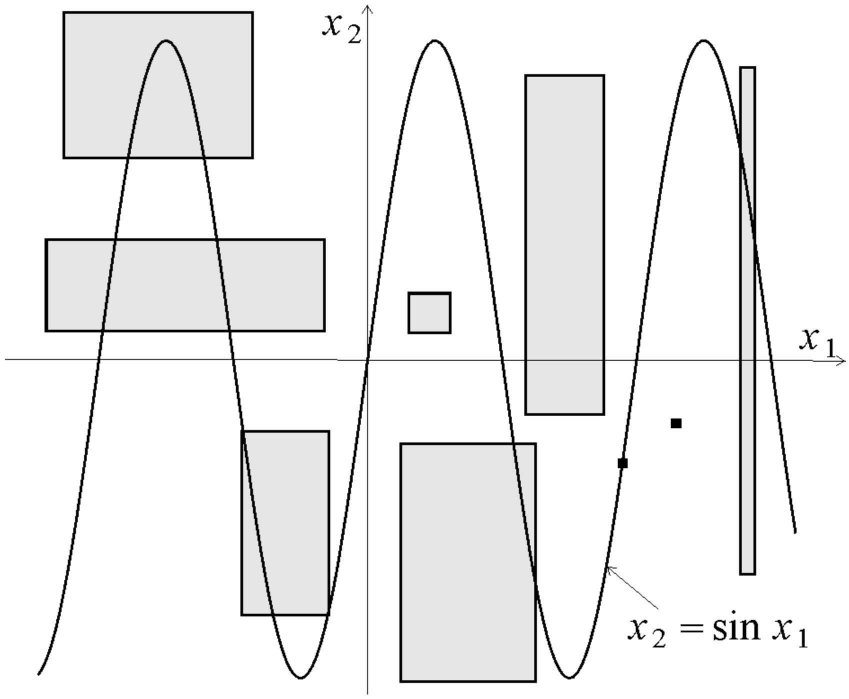
$[y] := [y] \cap \mathsf{exp}\,([x])\,;$

$[x] := [x] \cap \mathsf{log}\,([y])\,.$

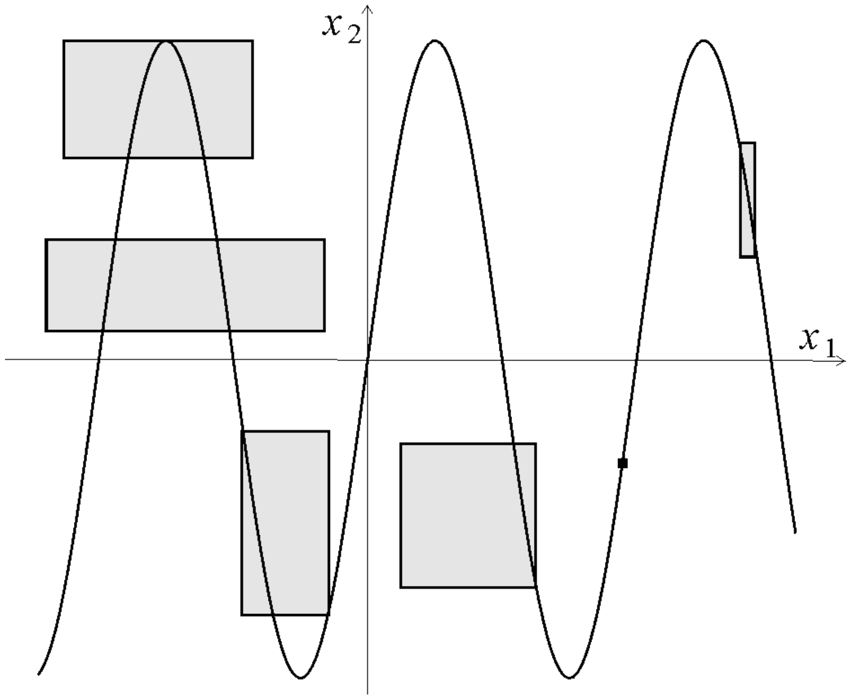
Any constraint for which such a projection procedure is available will be called a *primitive constraint*.

Example. Consider the primitive equation:

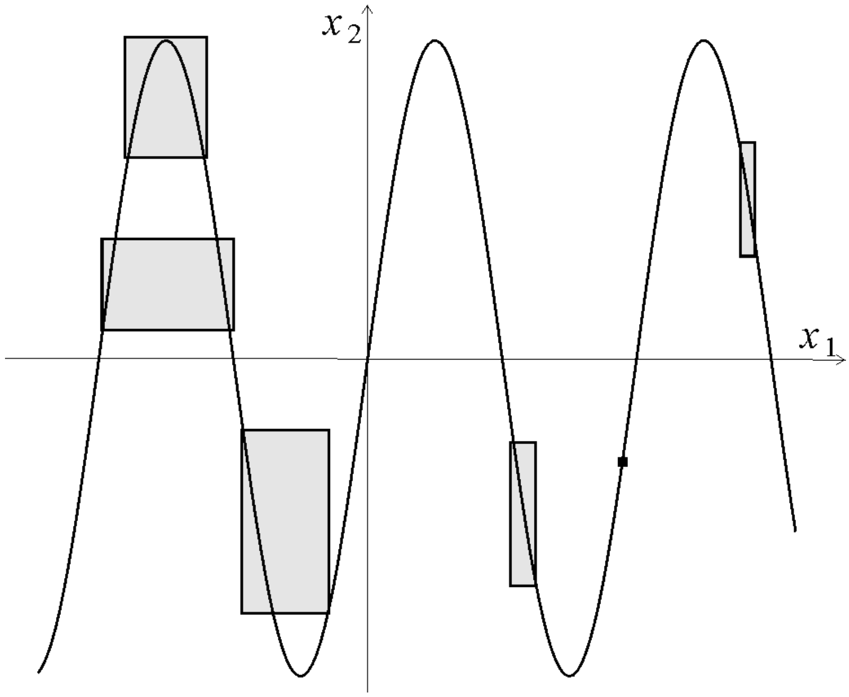
$$x_2 = \sin x_1.$$



x_2
x_1
$x_2 = \sin x_1$



Forward contraction



Backward contraction

Decomposition

$$x + \mathsf{sin}(xy) \leq 0,$$
$$x \in [-1, 1], y \in [-1, 1]$$

Decomposition

$$x + \mathsf{sin}(xy) \leq 0,$$
$$x \in [-1, 1], y \in [-1, 1]$$

can be decomposed into

$$\left\{ \begin{array}{l} a = xy \\ b = \mathsf{sin}(a) \\ c = x + b \end{array} \right. , \qquad \begin{array}{l} x \in [-1, 1] \\ y \in [-1, 1] \\ \end{array} \quad \begin{array}{l} a \in [-\infty, \infty] \\ b \in [-\infty, \infty] \\ c \in [-\infty, 0] \end{array}$$

Forward-backward contractor (HC4 revise)

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2]\,,$$

we have the following contractor:

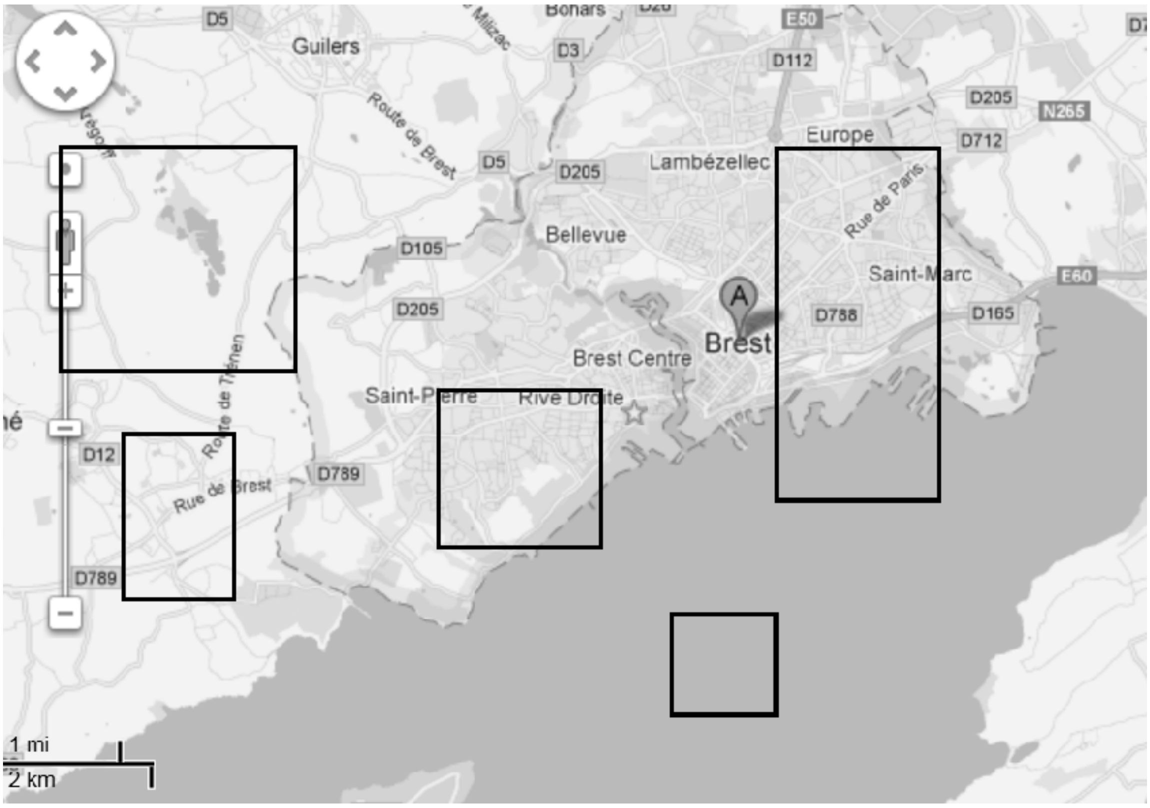
algorithm $\mathcal{C}$ (inout $[x_1]\,, [x_2]\,, [x_3]$)	
$[a] = [x_1] + [x_2]$	$//\ a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	$//\ b = a \cdot x_3$
$[b] = [b] \cap [1, 2]$	$//\ b \in [1, 2]$
$[x_3] = [x_3] \ \cap \ \frac{[b]}{[a]}$	$//\ x_3 = \frac{b}{a}$
$[a] = [a] \ \cap \ \frac{[b]}{[x_3]}$	$//\ a = \frac{b}{x_3}$
$[x_1] = [x_1] \ \cap \ \ [a] - [x_2]$	$//\ x_1 = a - x_2$
$[x_2] = [x_2] \ \cap \ \ [a] - [x_1]$	$//\ x_2 = a - x_1$

Properties

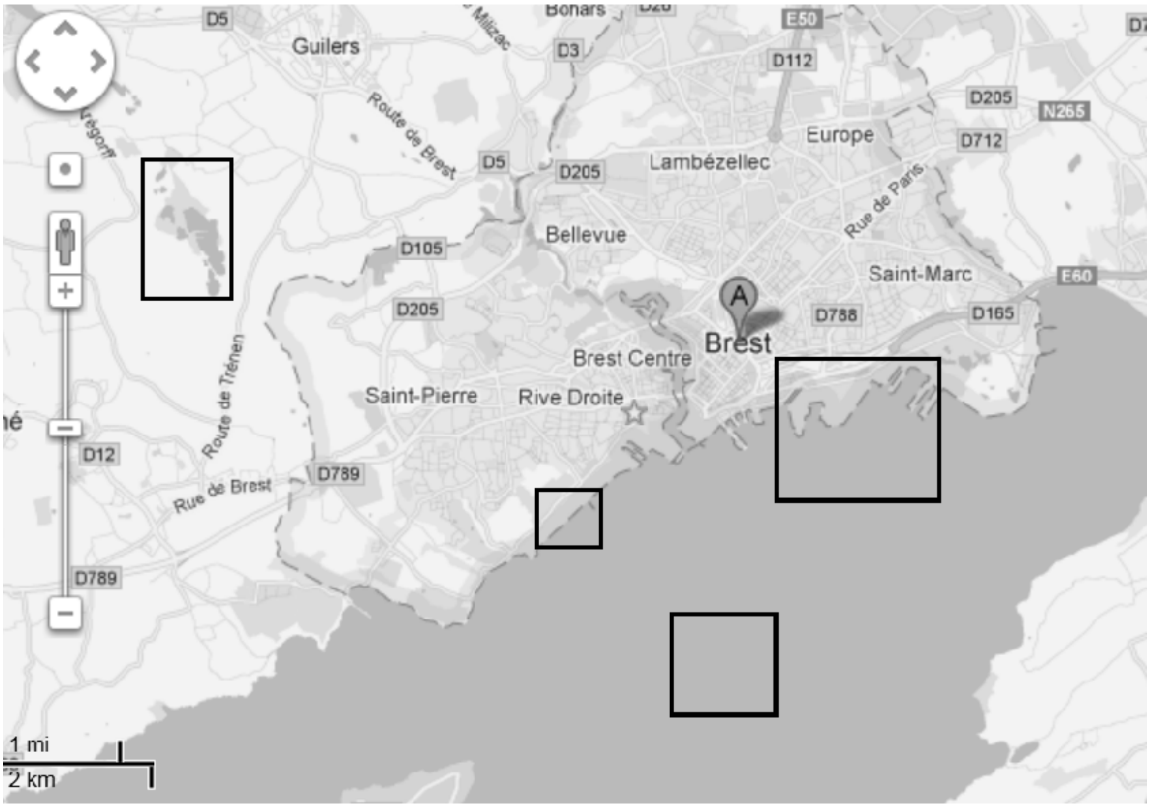
$$\begin{array}{rcl}
(\mathcal{C}_1^\infty \cap \mathcal{C}_2^\infty)^\infty & = & (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty \\
(\mathcal{C}_1 \cap (\mathcal{C}_2 \cup \mathcal{C}_3)) & \supset & (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_1 \cap \mathcal{C}_3) \\
\left\{ \begin{array}{l} \mathcal{C}_1 \text{ minimal} \\ \mathcal{C}_2 \text{ minimal} \end{array} \right. & \Rightarrow & \mathcal{C}_1 \cup \mathcal{C}_2 \text{ minimal}
\end{array}$$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Route de Brest
D3
E50
D112
D205
N265
Europe
D712
D5
D205
Lambézellec
Rue de Paris
Bellevue
D105
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
D12
Rue de Brest
D789
D789
1 mi
2 km



Guilers
Route de Brest
D5
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E50
D112
D205
N265
Europe
D712
Lambézellec
D205
Bellevue
Rue de Paris
D105
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
D12
D789
Rue de Brest
D789
1 mi
2 km

3.4 Propagation

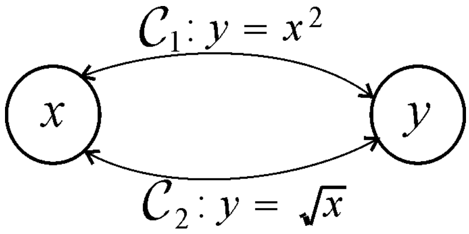
Example 1. Consider the system of two equations.

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

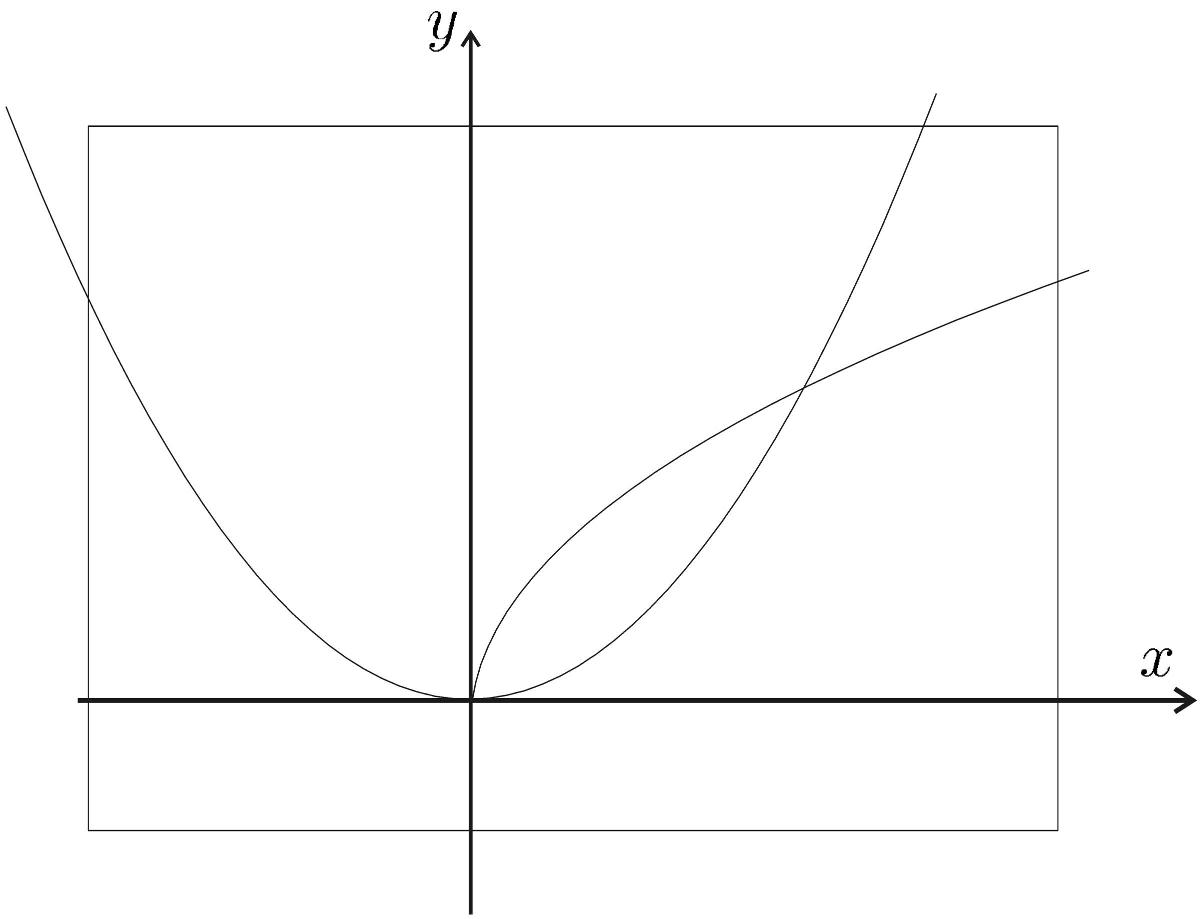
We can build two contractors

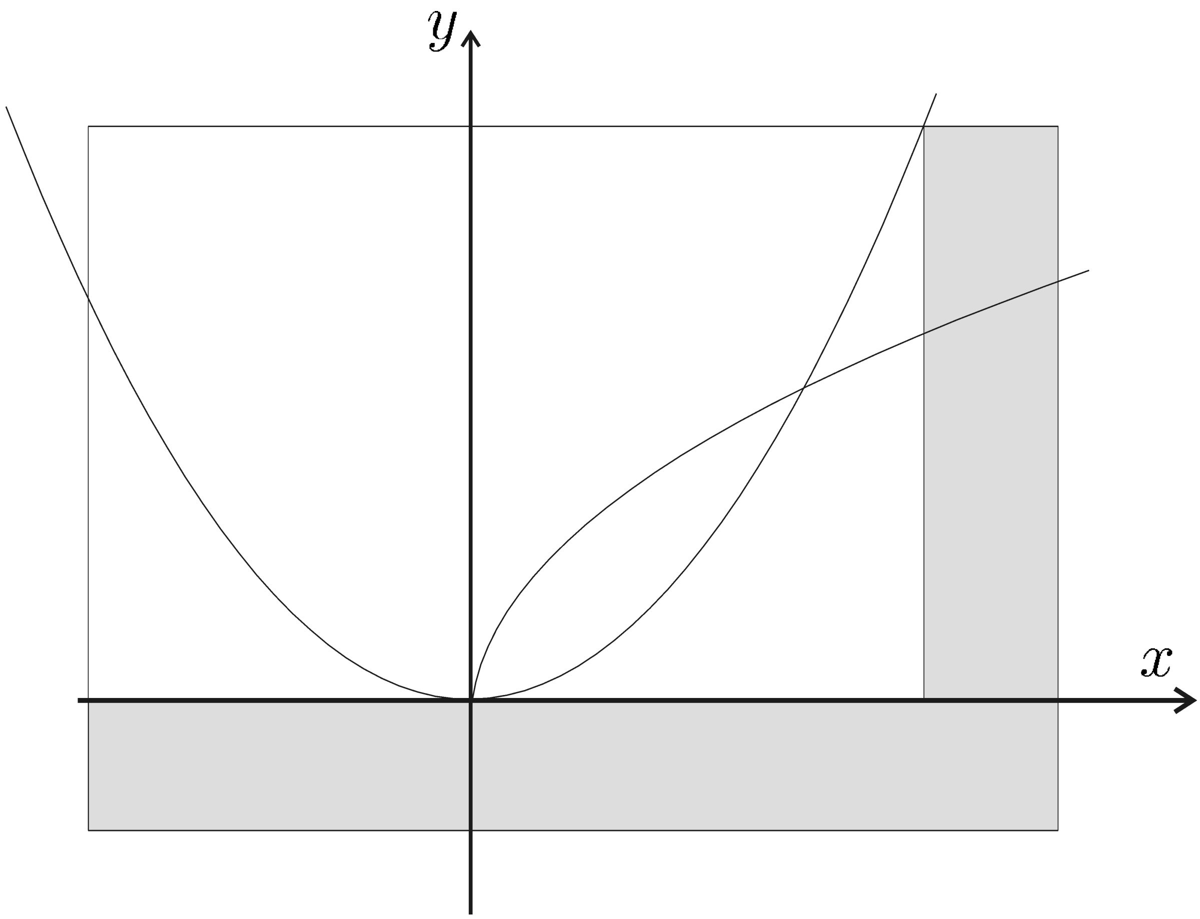
$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$

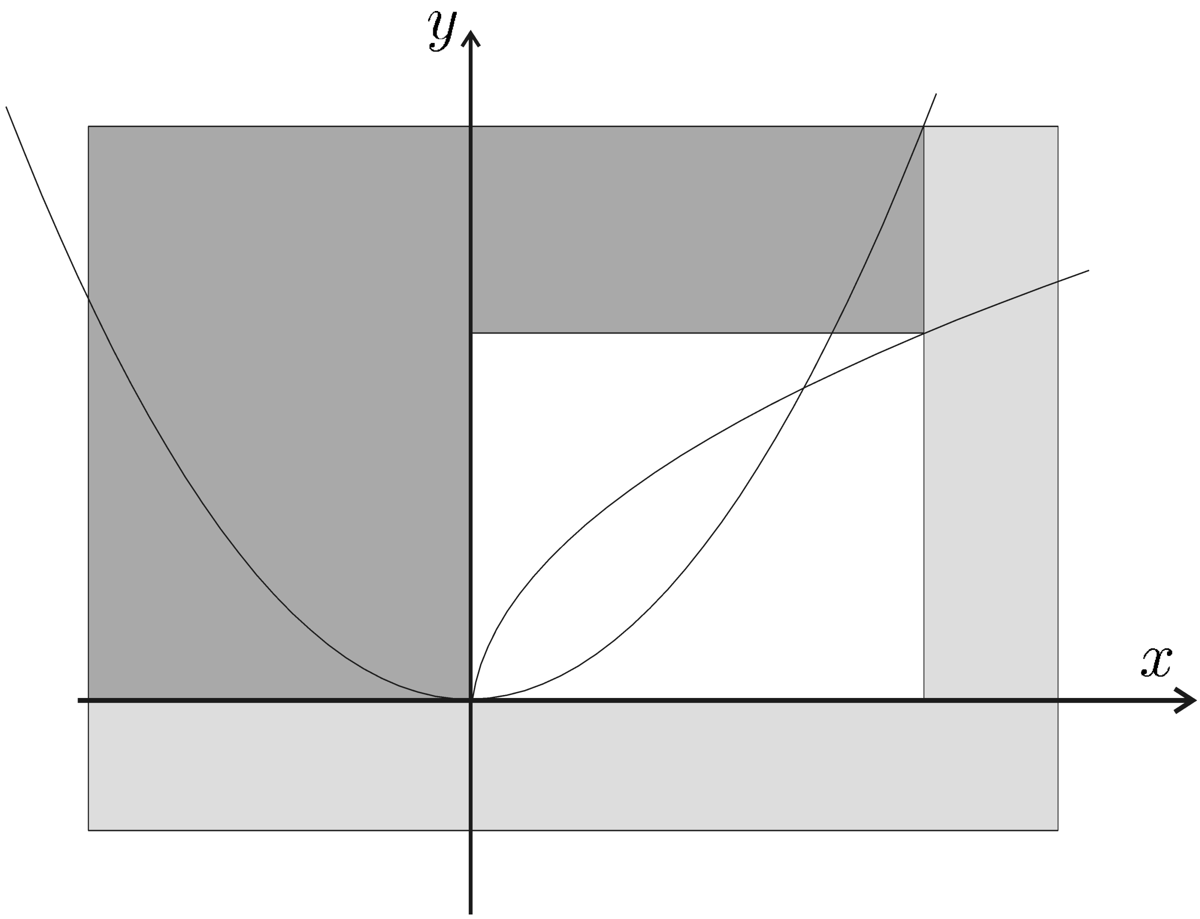


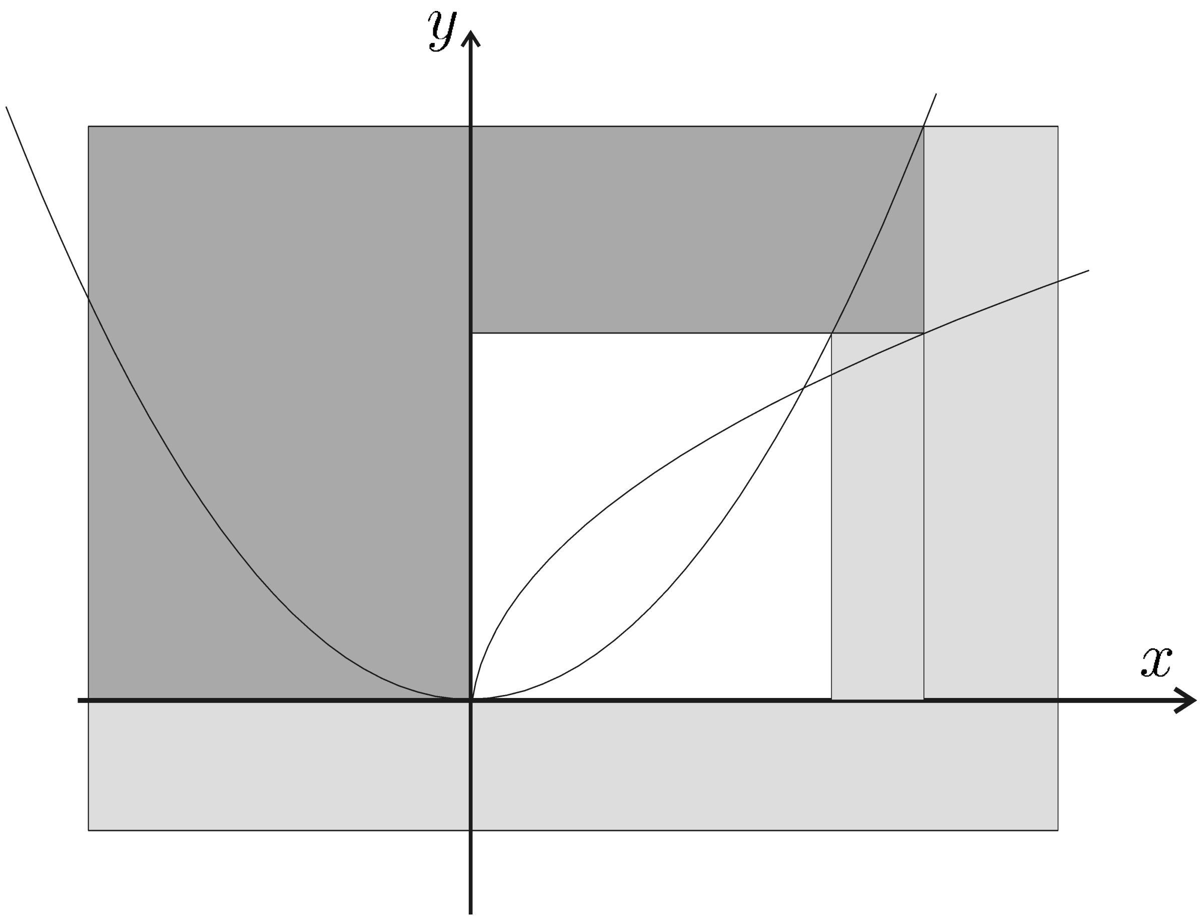
Contractor graph


y
x

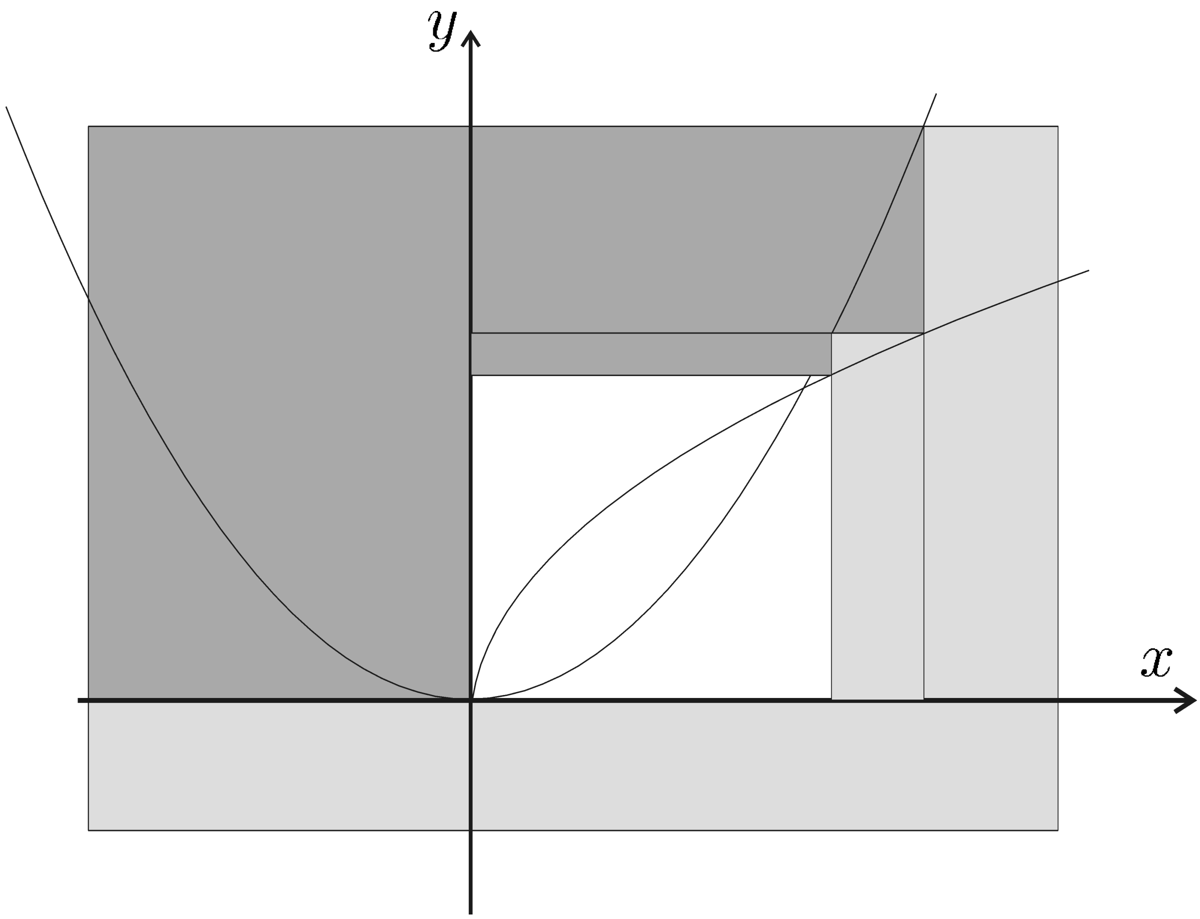


y
x

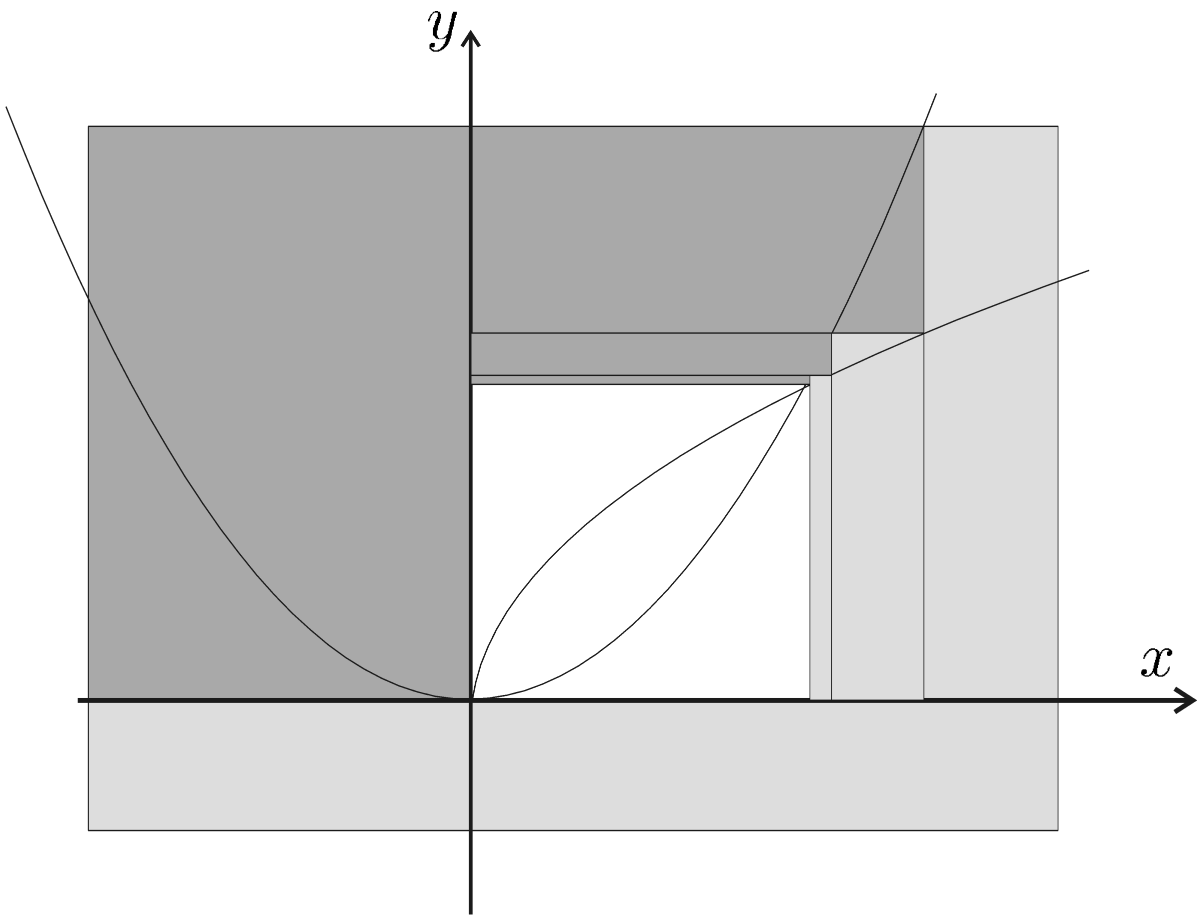

y
x



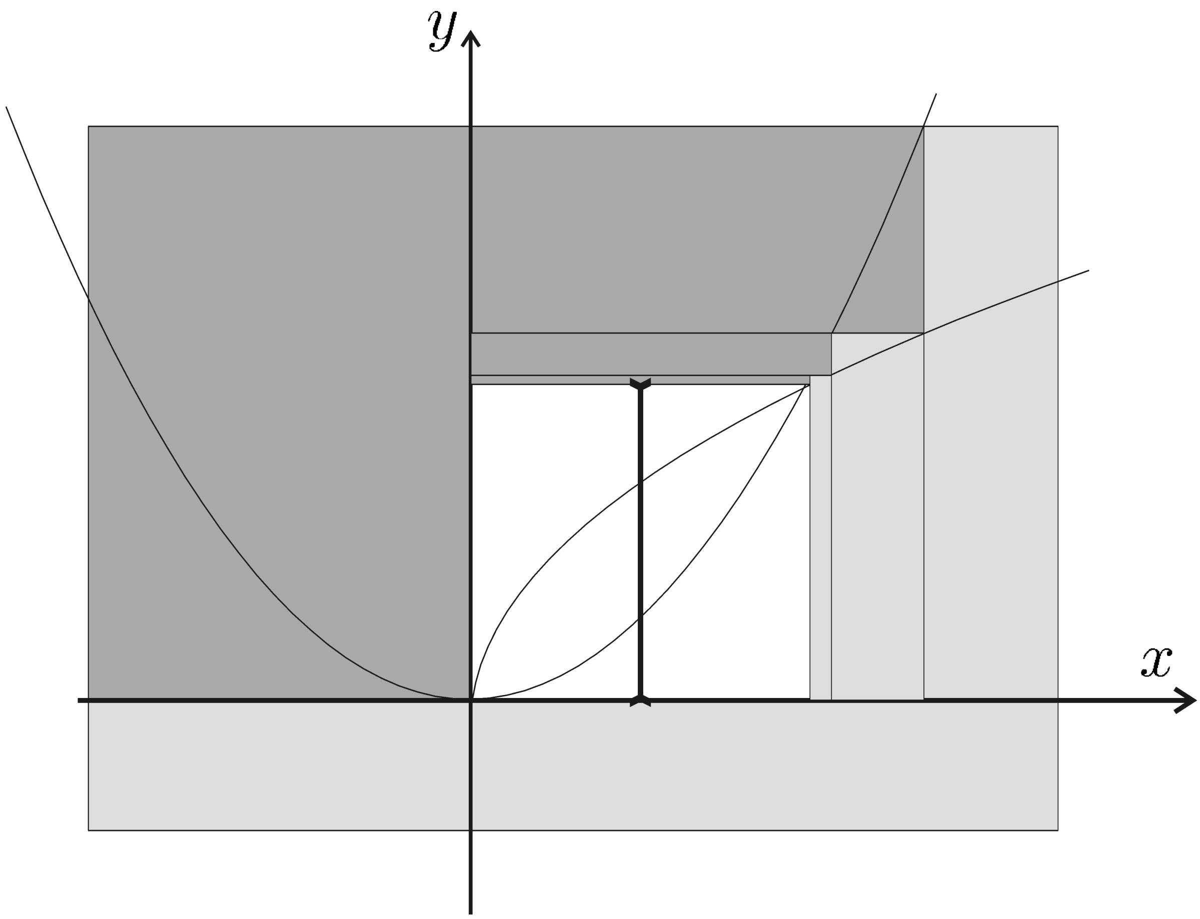
y
x

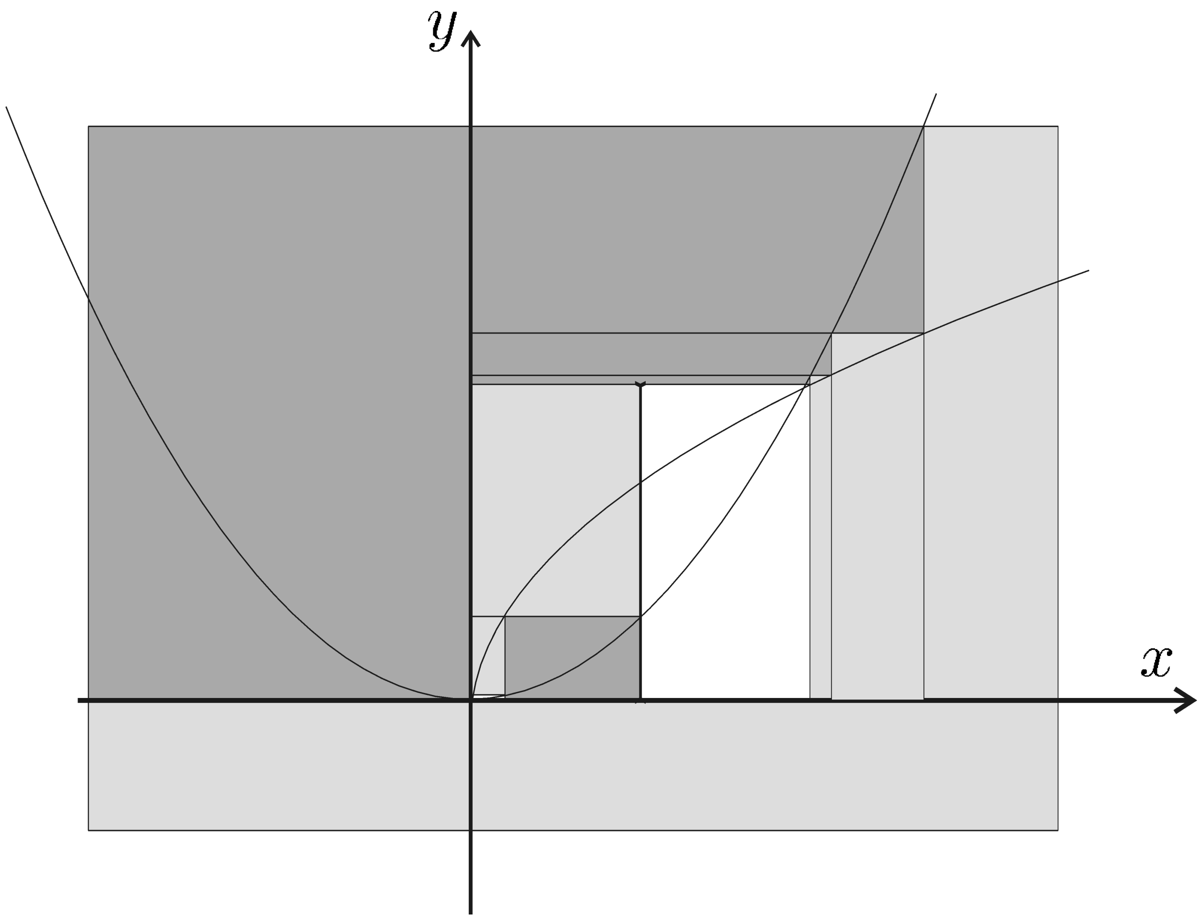


y
x

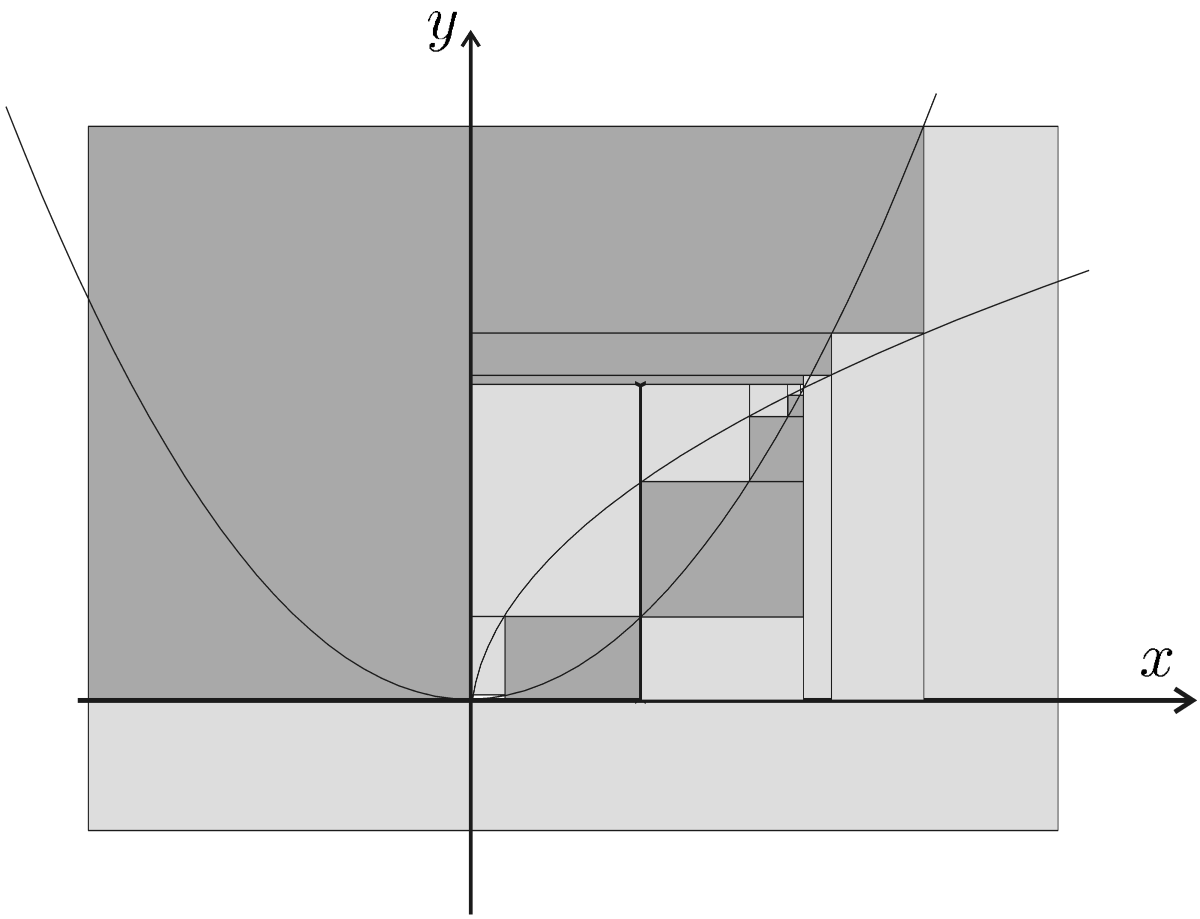


y
x


y
x



y
x


y
x

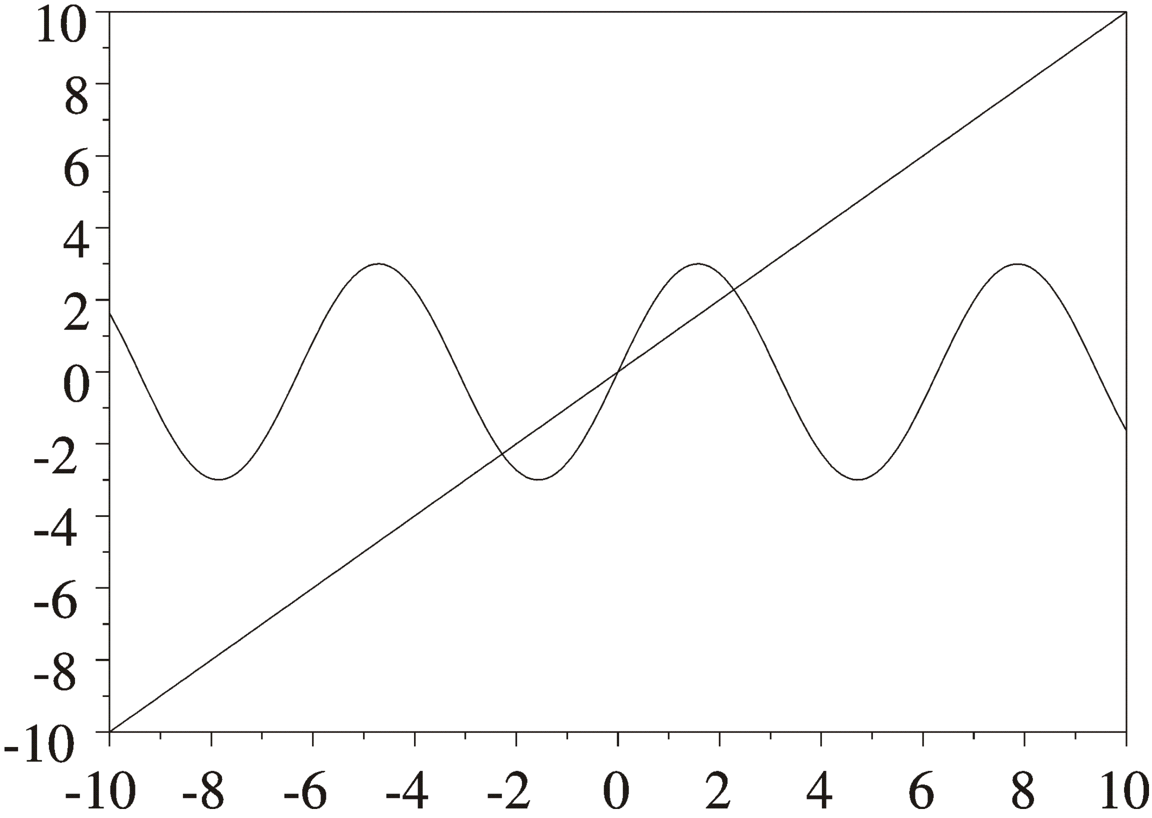
If $\mathcal{C}^*_{\mathbb{S}_1}$ and $\mathcal{C}^*_{\mathbb{S}_2}$ are two minimal contractors for $\mathbb{S}_1$ and $\mathbb{S}_2$ then

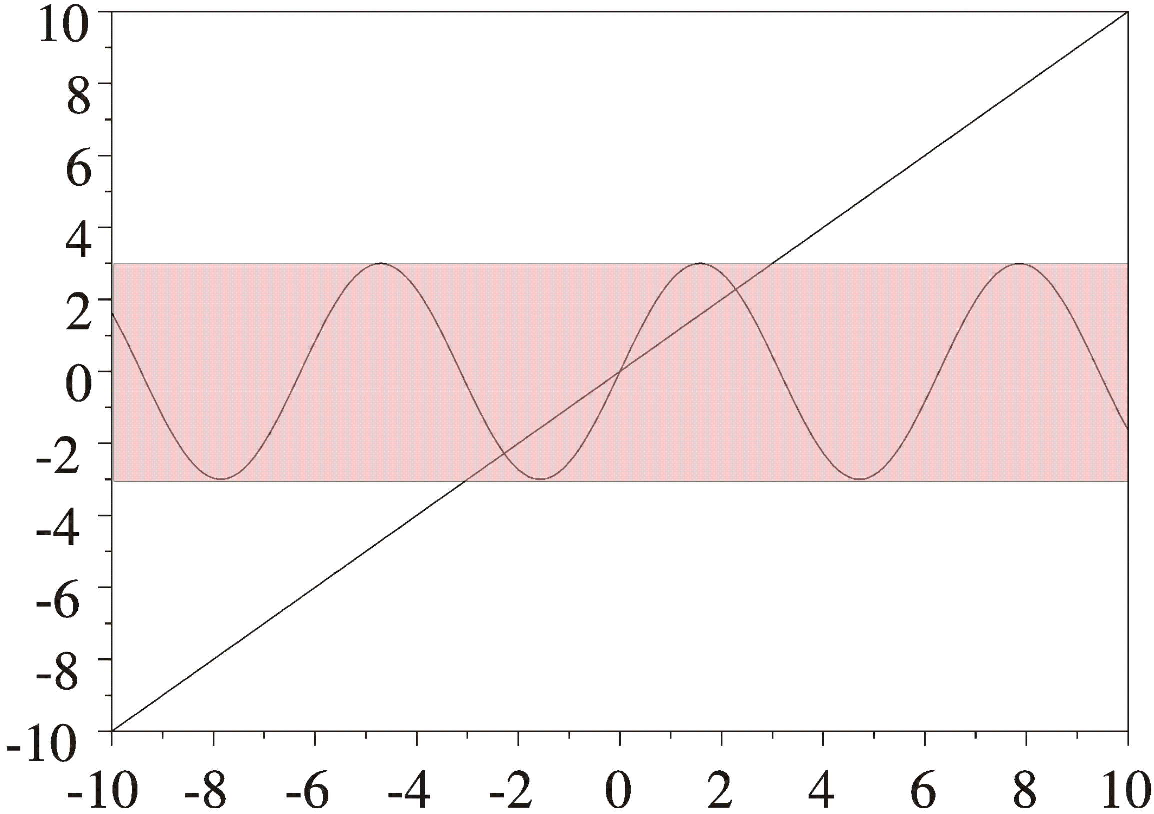
$$\mathcal{C}_{\mathbb{S}} = \mathcal{C}^*_{\mathbb{S}_1} \circ \mathcal{C}^*_{\mathbb{S}_2} \circ \mathcal{C}^*_{\mathbb{S}_1} \circ \mathcal{C}^*_{\mathbb{S}_2} \circ \dots$$

is a contractor for $\mathbb{S} = \mathbb{S}_1 \cap \mathbb{S}_2$, but it is not always optimal. This is the *local consistency effect.*

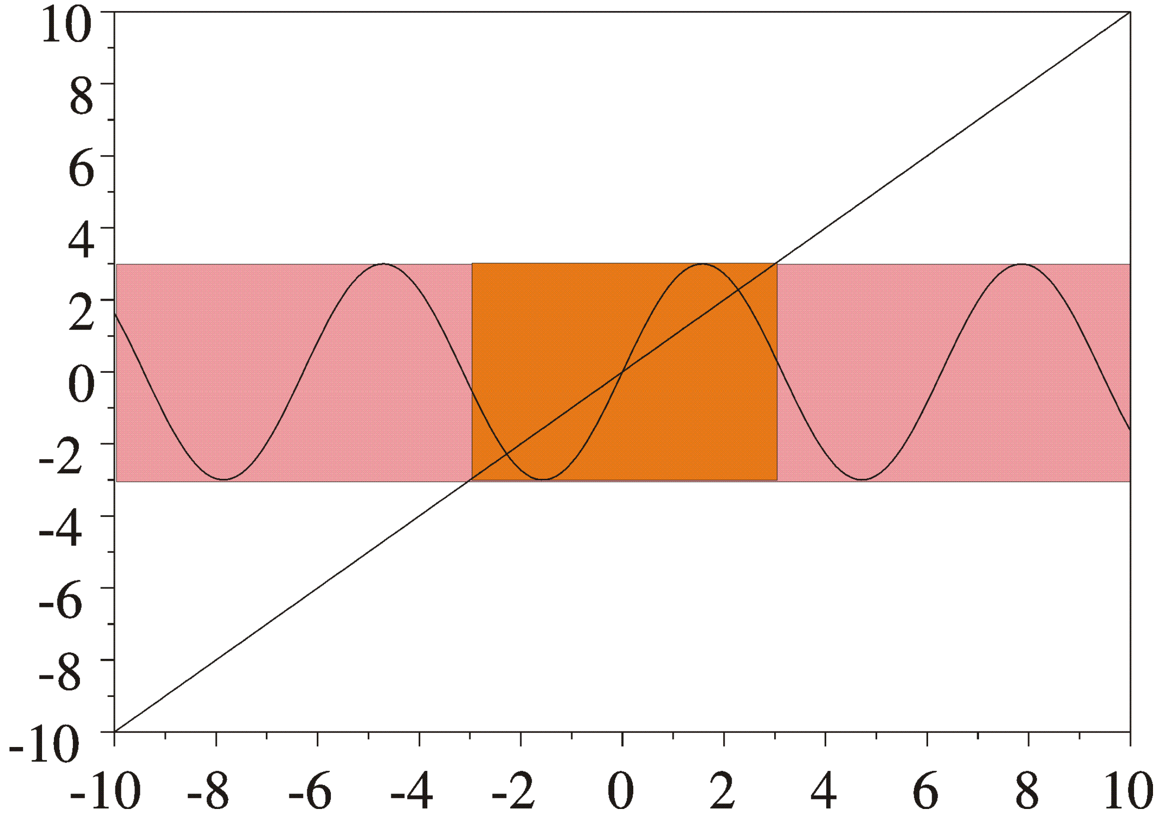
Example 2 (local consistency). Consider the system

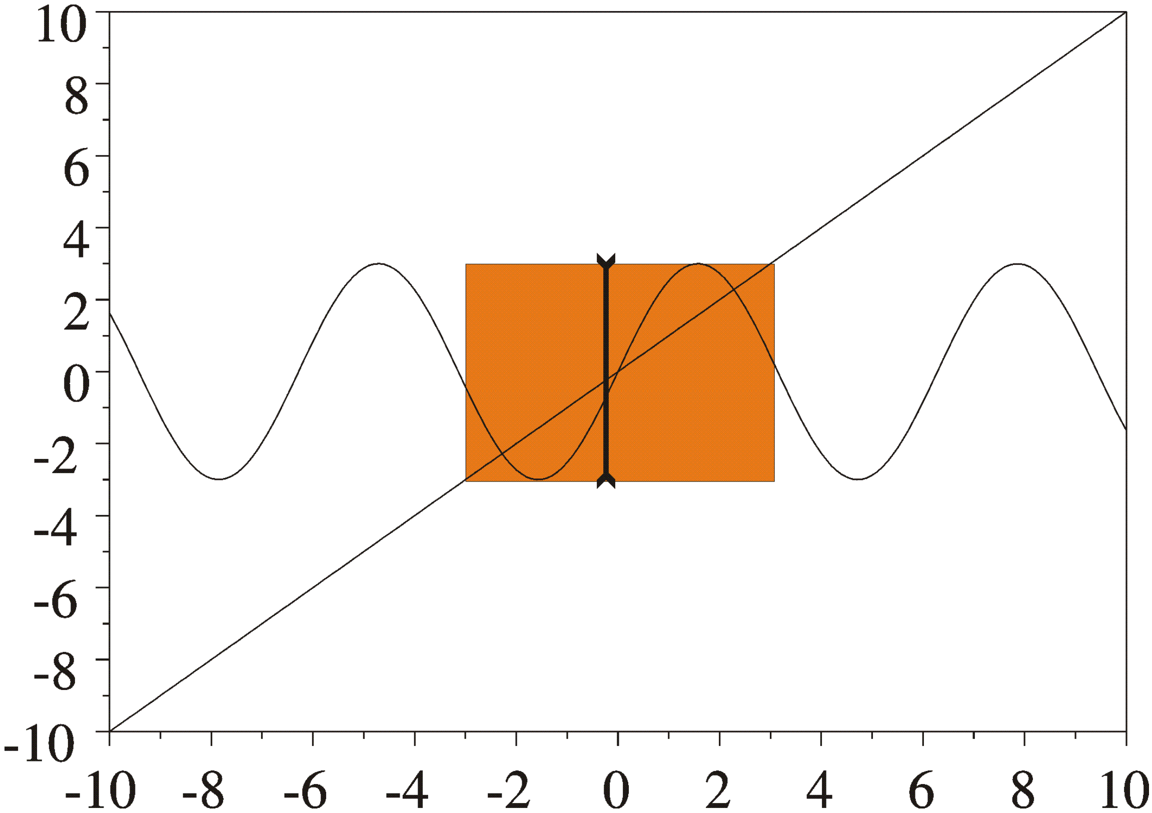
$$\left\{\begin{array}{rcl} y & = & 3\sin(x) \\ y & = & x \end{array}\right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10

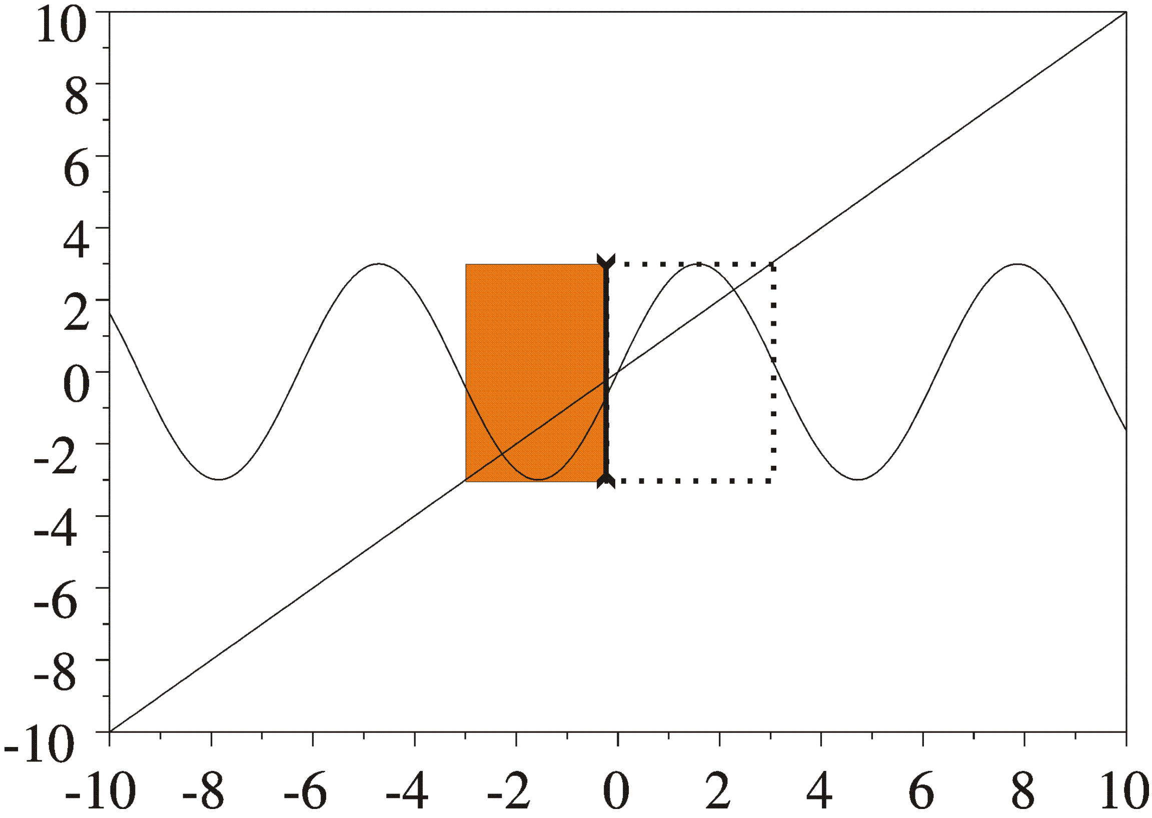


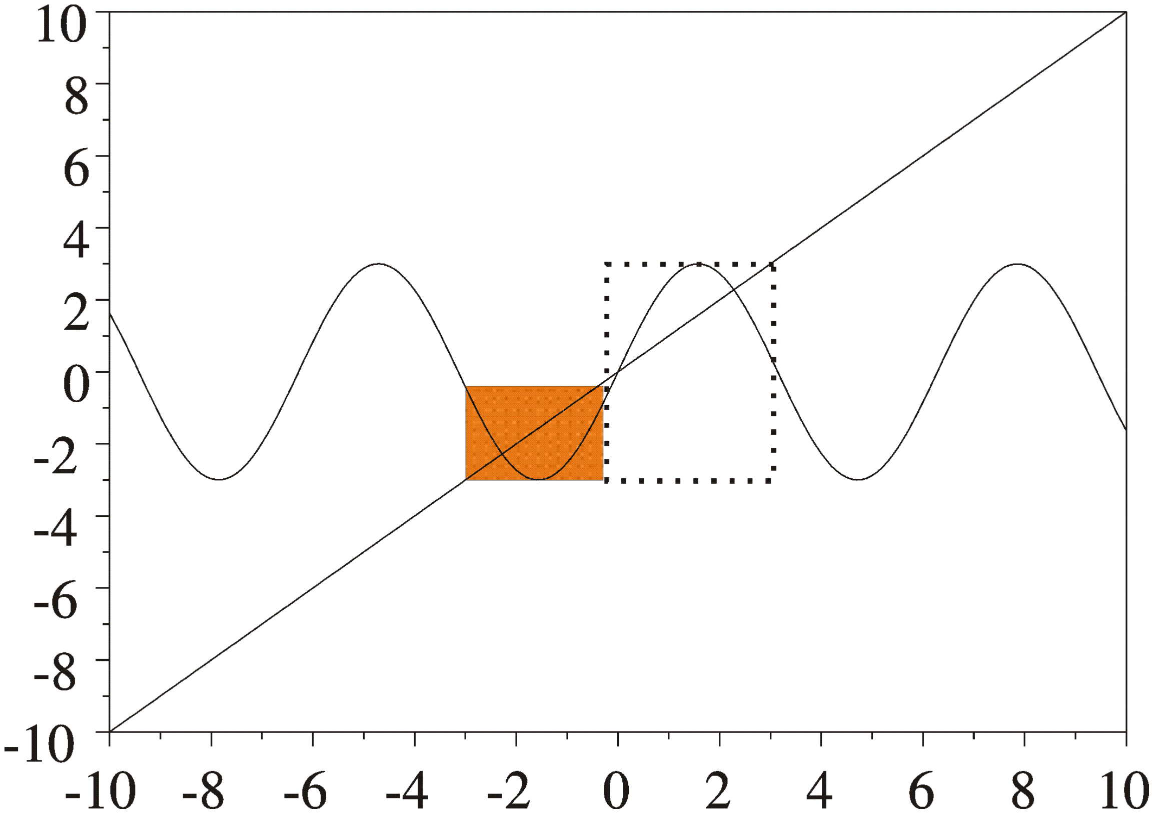
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10



10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10


10
8
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-4
-6
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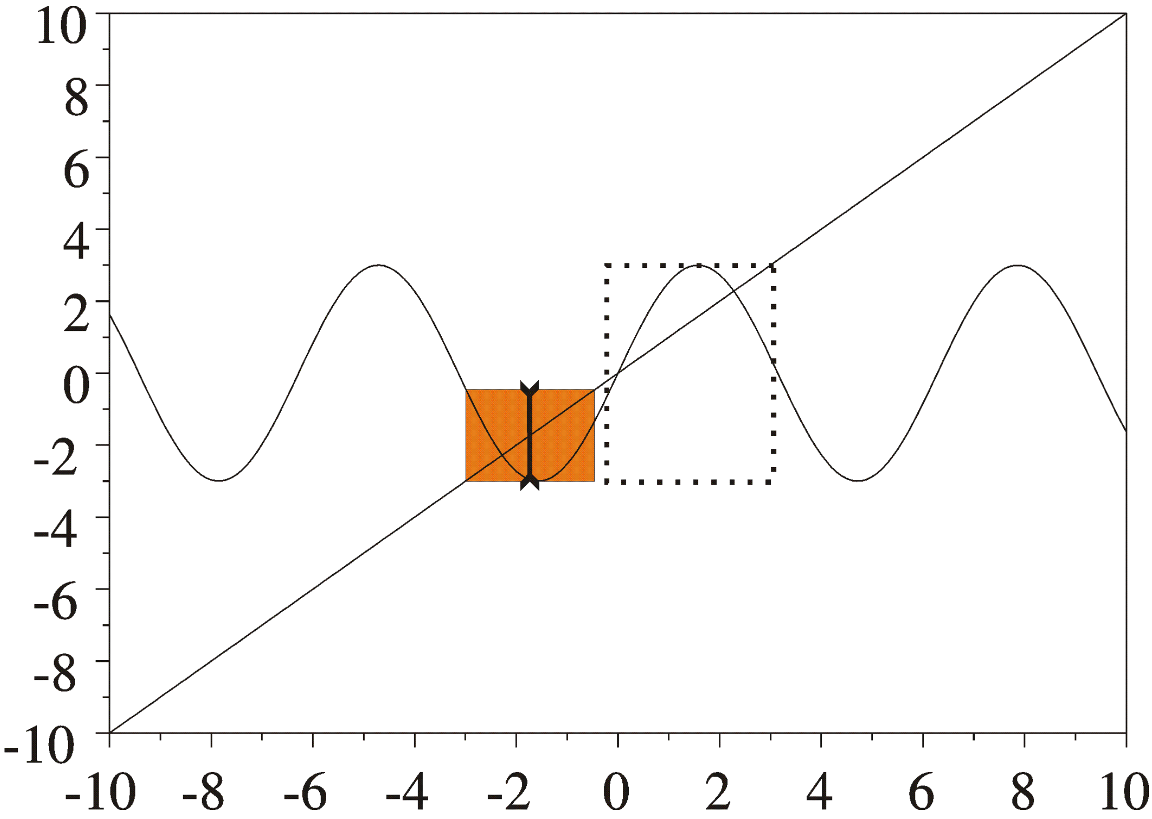
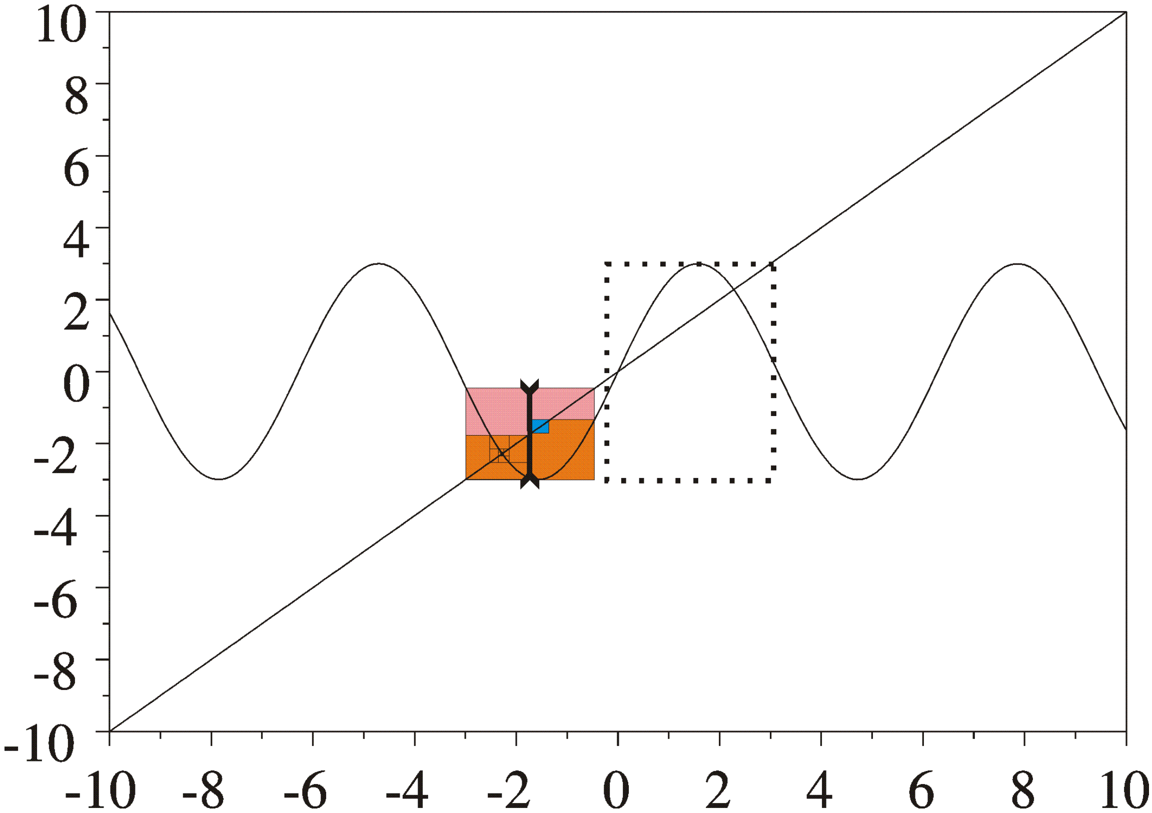
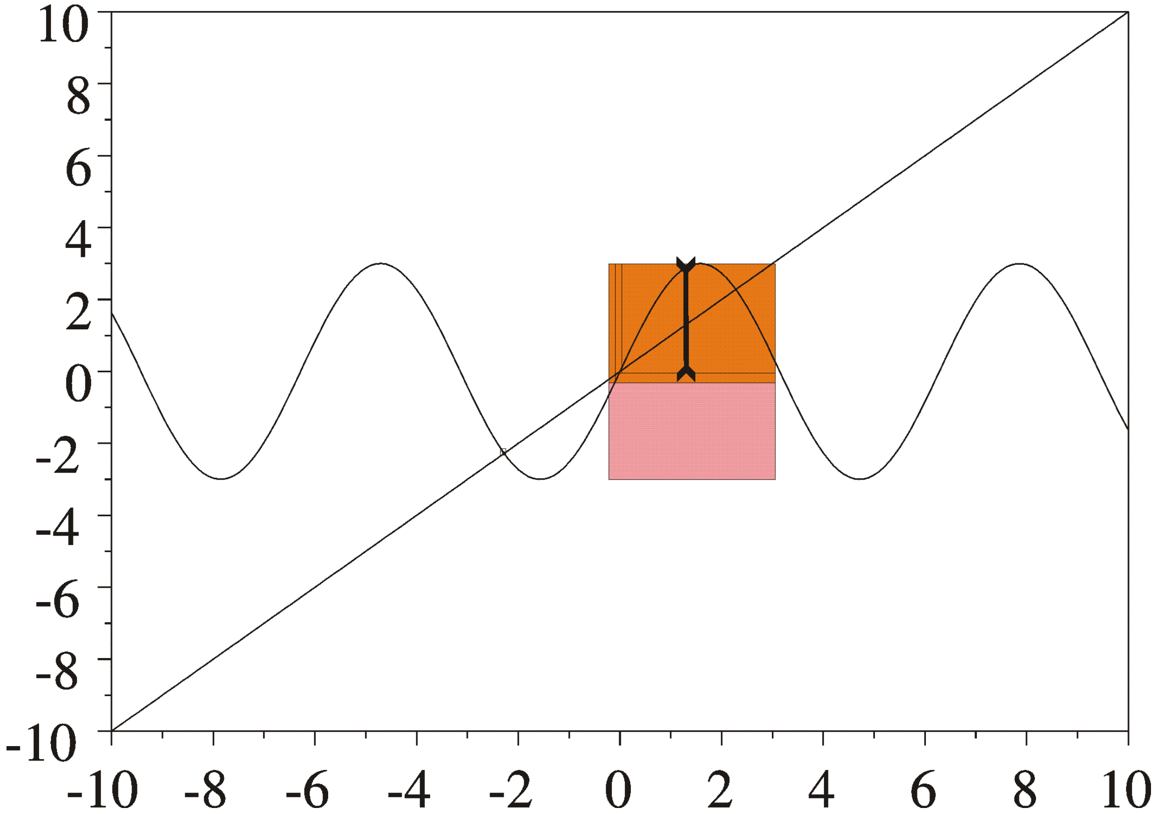
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10

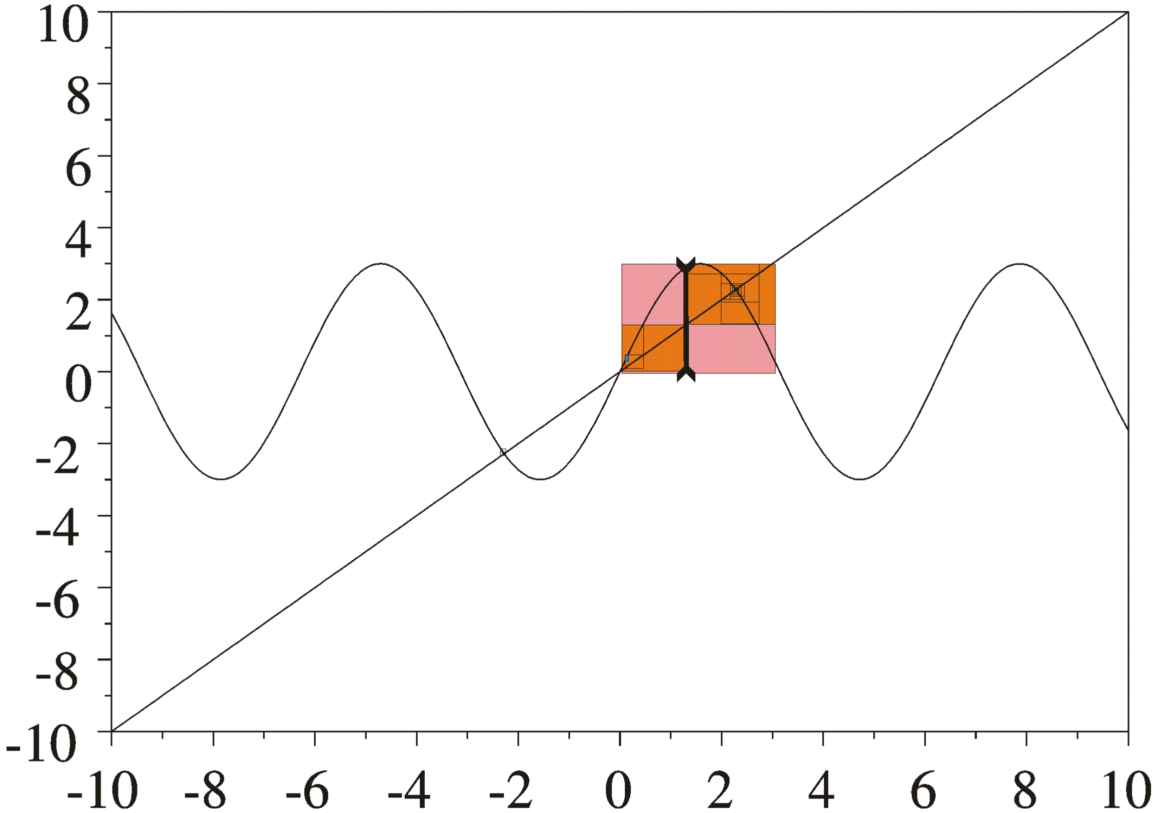

10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10



10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10



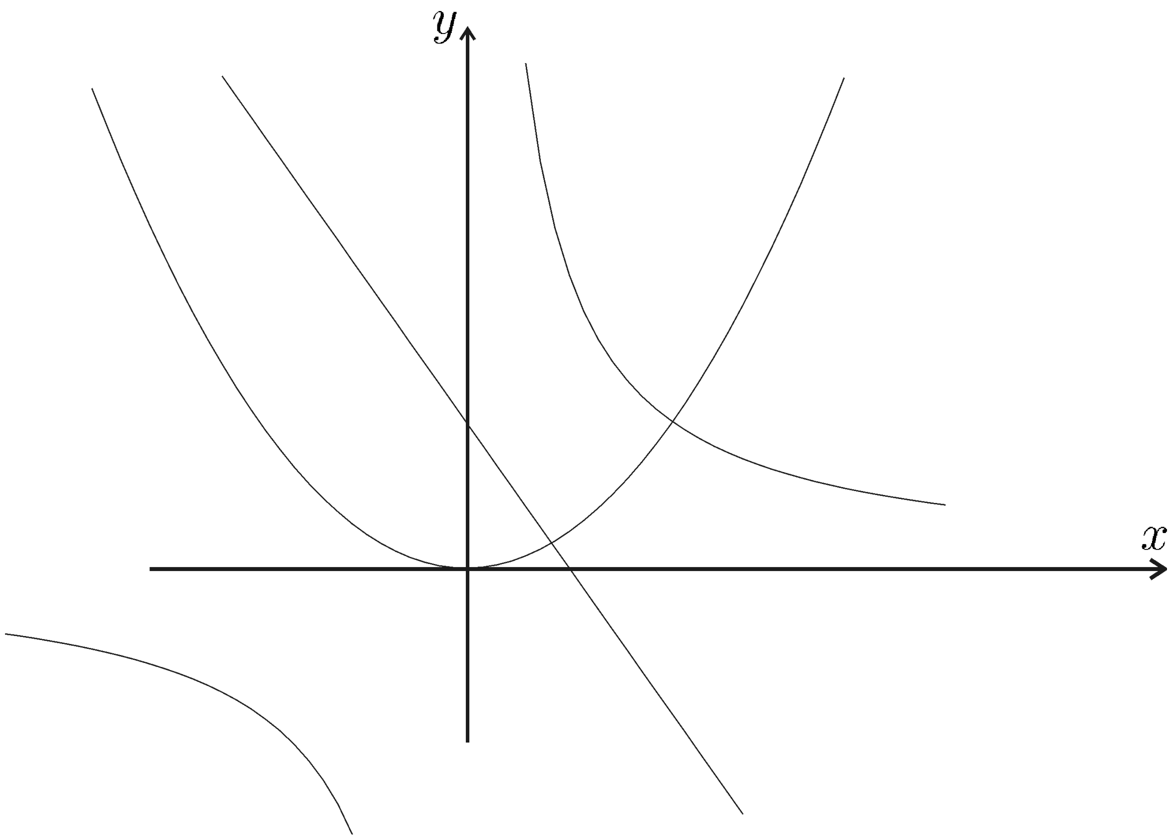
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10

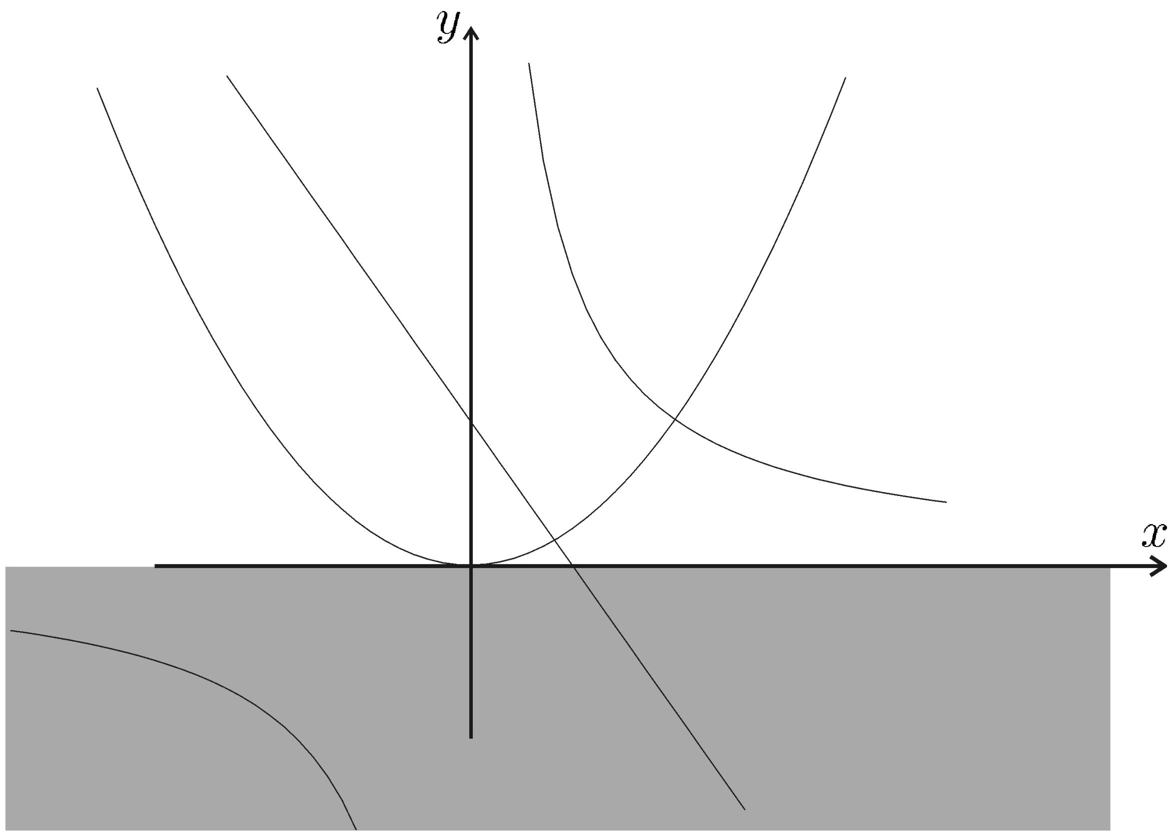


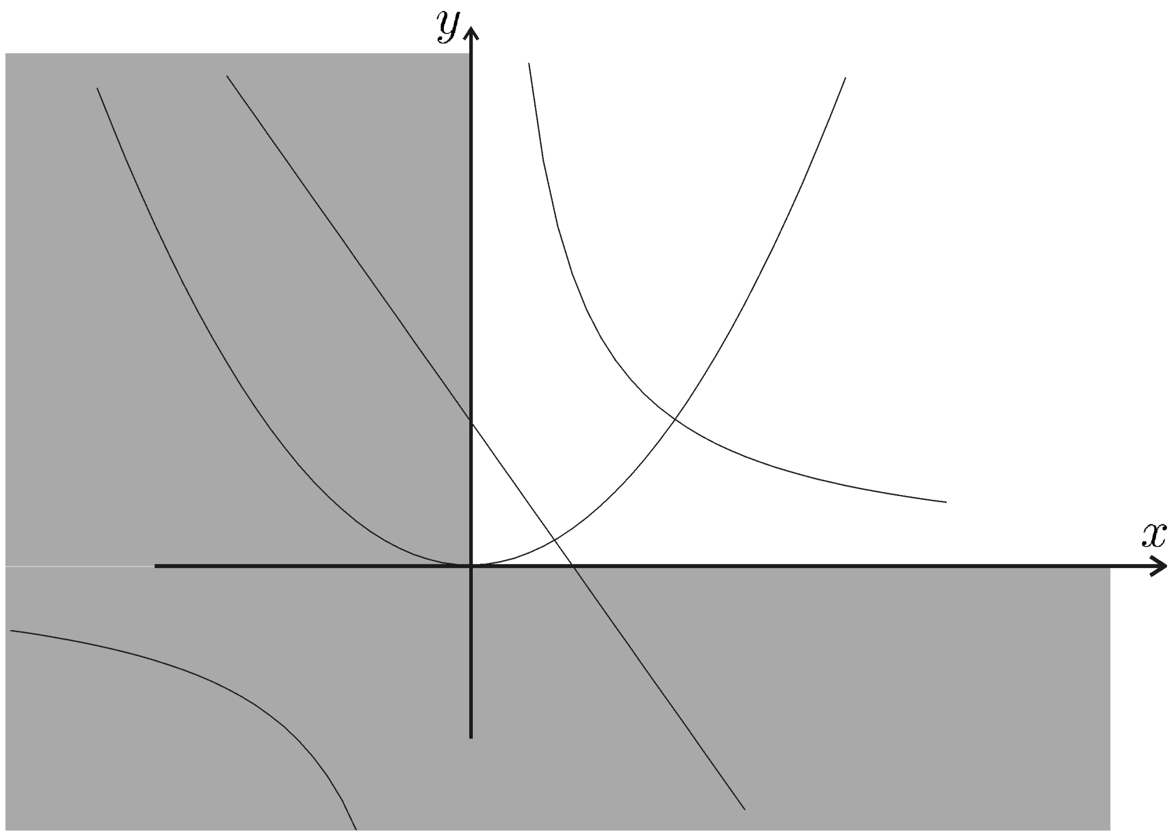
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10

Example 3 (more equations than unknowns).

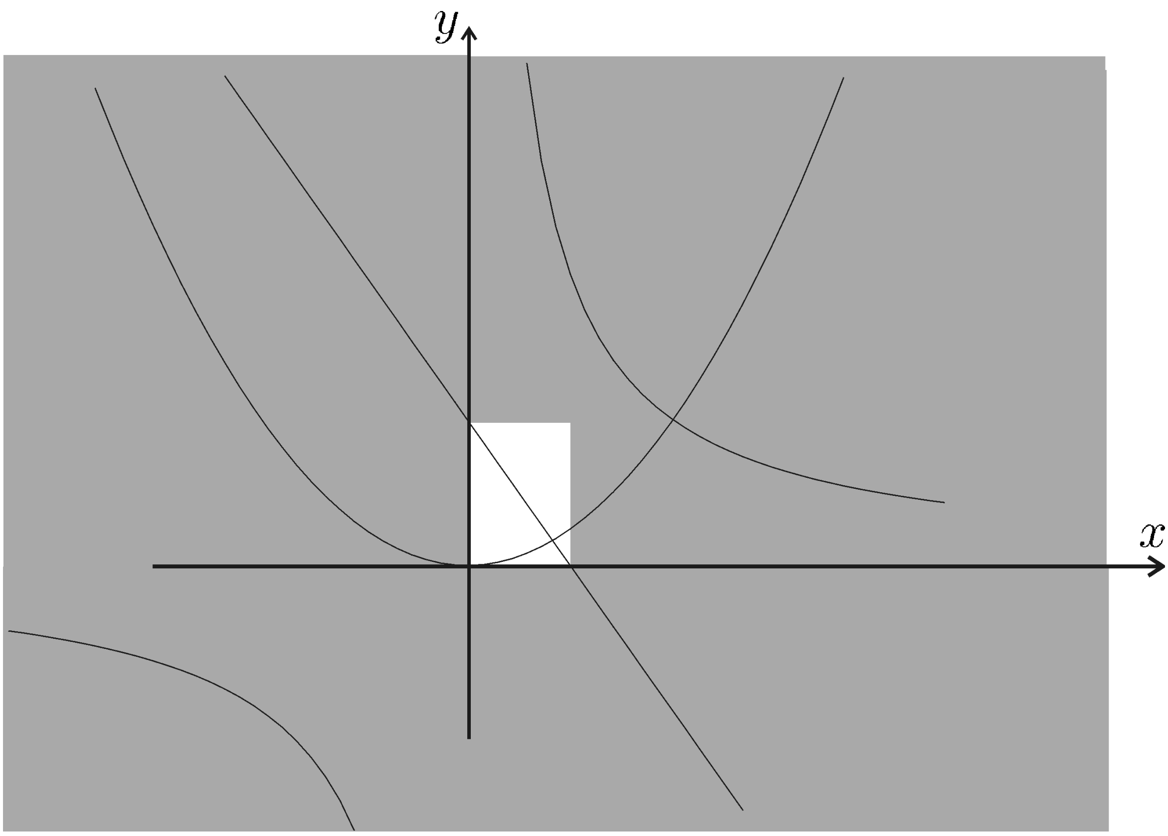
$$\begin{cases} (C_1): & y = x^2 \\ (C_2): & xy = 1 \\ (C_3): & y = -2x + 1 \end{cases}$$


y
x

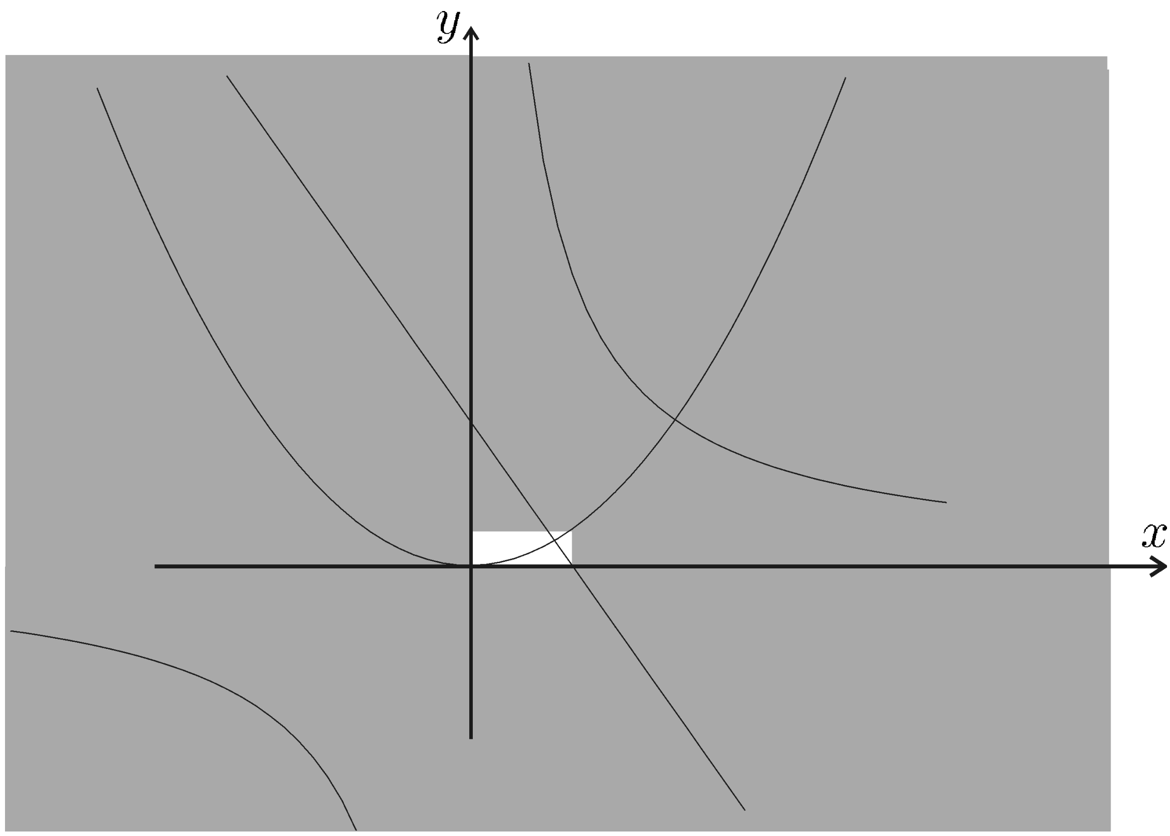

y
x



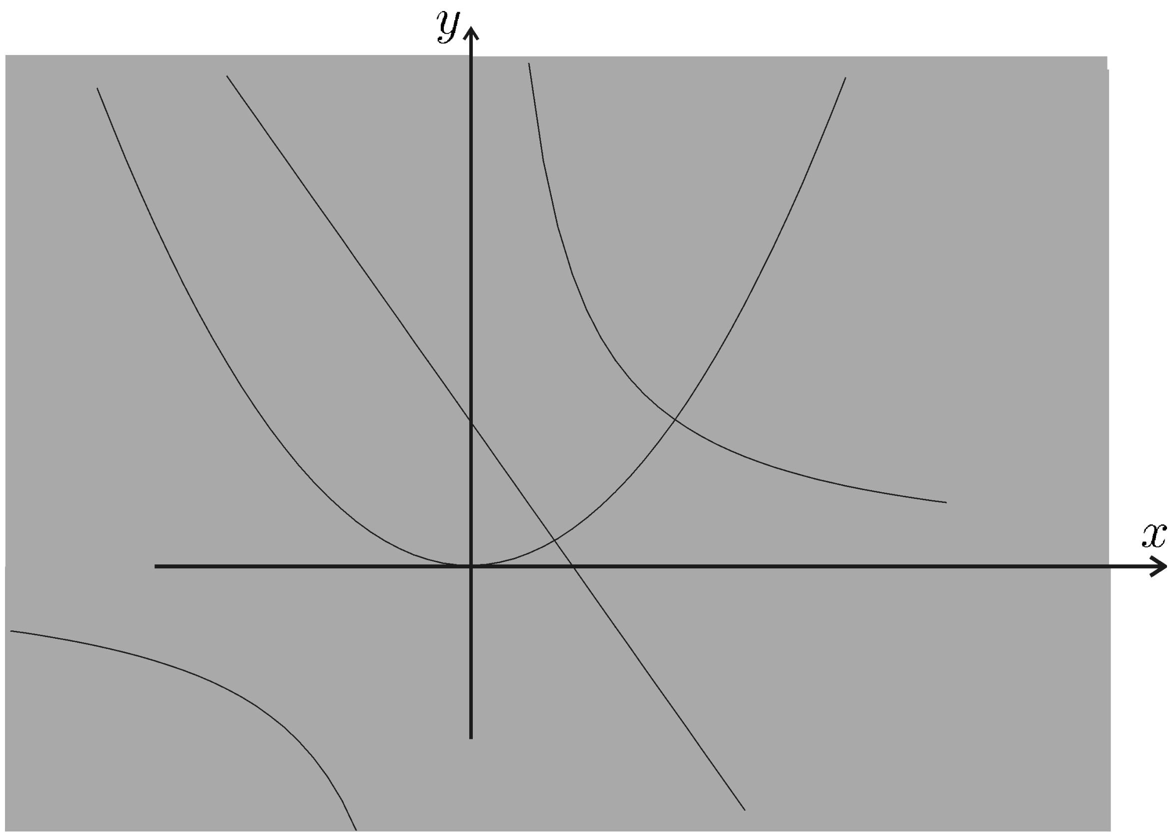
y
x



y
x



y
x


y
x

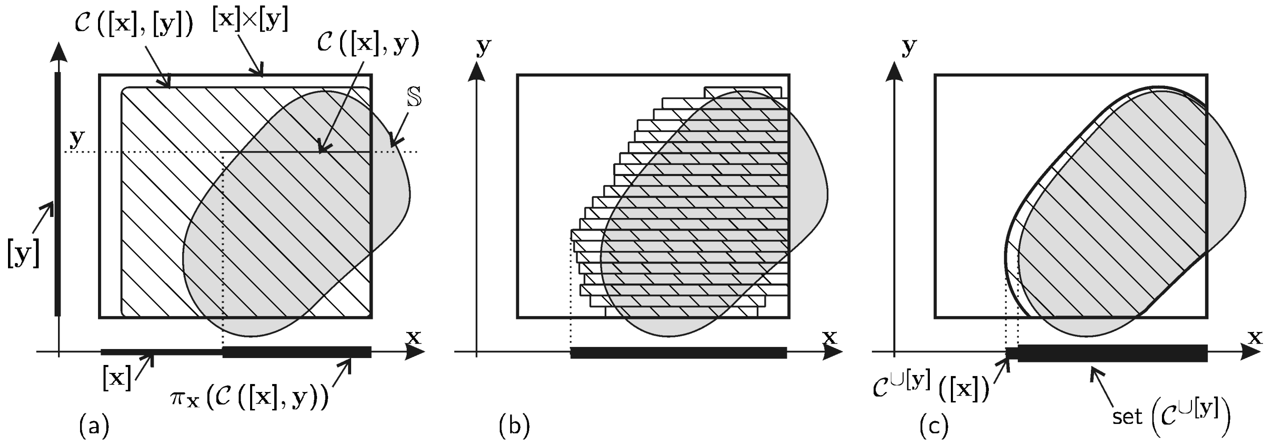
$$
\begin{array}{lll}
(C_1) \Rightarrow & y \in & [-\infty, \infty]^2 \;=\; [0, \infty] \\
(C_2) \Rightarrow & x \in & 1/[0, \infty] \;= [0, \infty] \\
(C_3) \Rightarrow & y \in & [0, \infty] \;\cap\; ((-2)\,.[0, \infty] + 1) \\
 & & = [0, \infty] \;\cap\; ([-\infty, 1]) = [0, 1] \\
 & x \in & [0, \infty] \;\cap\; (-[0, 1]/2 + 1/2) = [0, \frac{1}{2}] \\
(C_1) \Rightarrow & y \in & [0, 1] \cap [0, 1/2]^2 = [0, 1/4] \\
(C_2) \Rightarrow & x \in & [0, 1/2] \;\cap\; 1/[0, 1/4] = \emptyset \\
 & y \in & [0, 1/4] \cap 1/\emptyset = \emptyset
\end{array}
$$

3.5 Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
repetition	$\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$
repeat intersection	$\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$
repeat union	$\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^\infty$

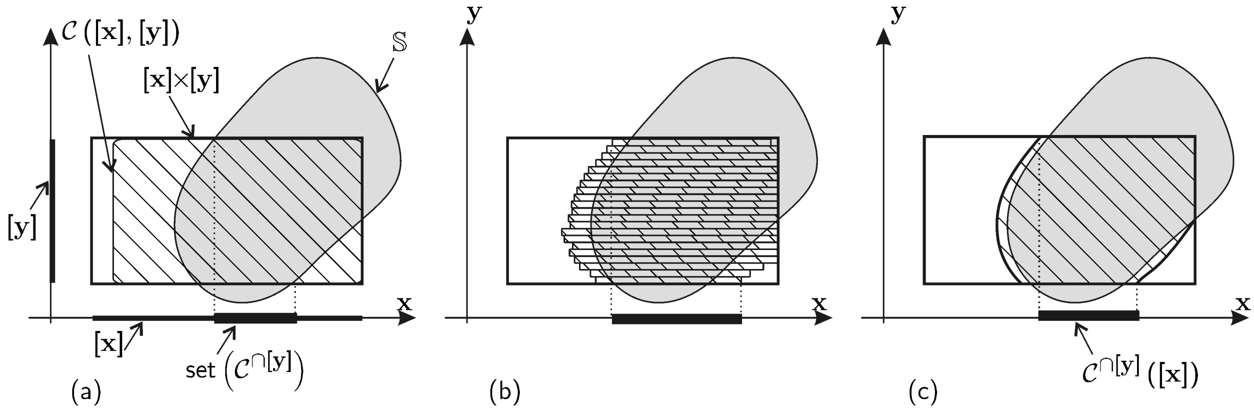
Consider the contractor $\mathcal{C}([\mathbf{x}],[\mathbf{y}])$, where $[\mathbf{x}] \in \mathbb{IR}^n, [\mathbf{y}] \in \mathbb{IR}^p$. We define the contractor

$$\mathcal{C}^{\cup[\mathbf{y}]}([\mathbf{x}]) = \left[\bigcup_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}(\mathcal{C}([\mathbf{x}],\mathbf{y}))\right] \quad \text{(projected union)}$$



Define the contractor

$$\mathcal{C}^{\cap[\mathbf{y}]}([\mathbf{x}]) \;=\; \bigcap_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}\left(\mathcal{C}\left([\mathbf{x}],\mathbf{y}\right)\right), \qquad \text{(projected intersection)}$$

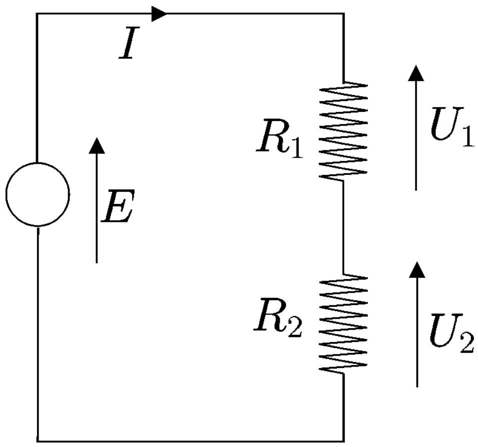


We have

$$\mathsf{set}\left(\mathcal{C}^{\cup[\mathbf{y}]}\right) = \{\mathbf{x}, \exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \mathsf{set}\left(\mathcal{C}\right)\}$$
$$\mathsf{set}\left(\mathcal{C}^{\cap[\mathbf{y}]}\right) = \{\mathbf{x}, \forall \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \mathsf{set}\left(\mathcal{C}\right)\}.$$

3.6 Circuits

Example 1



Domains

$$\begin{aligned}
E &\in [23V, 26V]; I \in [4A, 8A]; \\
U_1 &\in [10V, 11V]; U_2 \in [14V, 17V]; \\
P &\in [124W, 130W]; R_1 \in [0, \infty[\text{ and } R_2 \in [0, \infty[.
\end{aligned}$$

Constraints

(i) $P = EI$, (ii) $E = (R_1 + R_2)\, I$, (iii) $U_1 = R_1 I$, (iv) $U_2 = R_2 I$, (v) $E = U_1 + U_2$.

Solution set

$$\mathbb{S} = \left\{ \begin{pmatrix} E \\ R_1 \\ R_2 \\ I \\ U_1 \\ U_2 \\ P \end{pmatrix} \in \begin{pmatrix} [23, 26] \\ [0, \infty[\\ [0, \infty[\\ [4, 8] \\ [10, 11] \\ [14, 17] \\ [124, 130]; \end{pmatrix}, \left\{ \begin{array}{l} P = EI \\ E = (R_1 + R_2)\, I \\ U_1 = R_1 I \\ U_2 = R_2 I \\ E = U_1 + U_2 \end{array} \right. \right\}$$

```
variables
 E in [23 ,26];
 I in [4,8];
 U1 in [10,11];
 U2 in [14 ,17];
 P in [124,130];
 R1 in [0 ,1e08 ];
 R2 in [0 ,1e08 ];
contractor_list L
 P=E*I;
 E=(R1+R2)*I;
 U1=R1*I;
 U2=R2*I;
 E=U1+U2;
end
```

```
contractor C
   compose(L);
end
contractor epsilon
   precision(1);
end
```

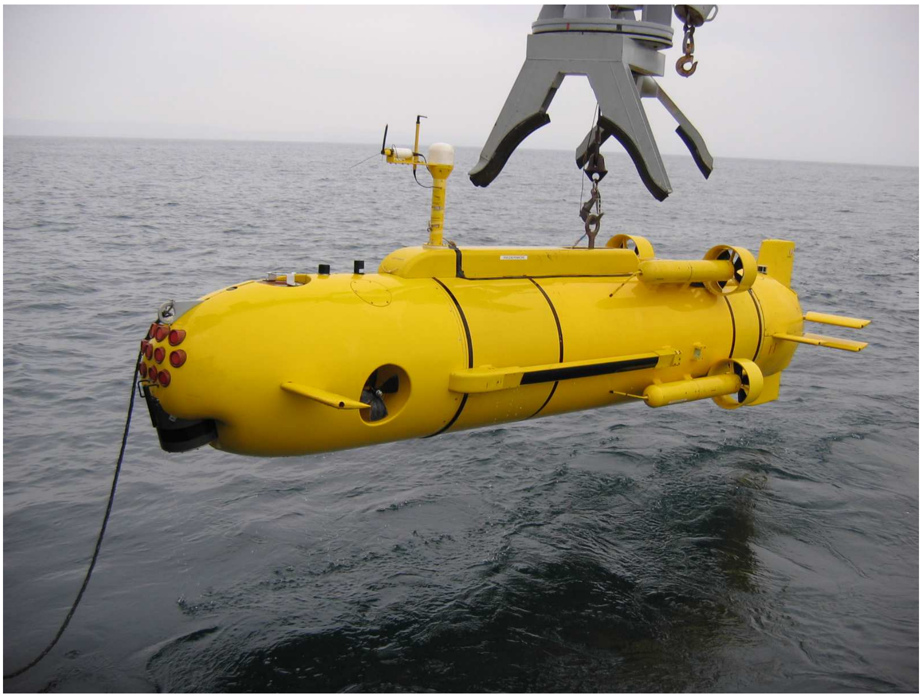
Quimper returns

$$
\begin{aligned}
&[24;26]\times[1.846;2.307]\times[2.584;3.355]\\
&\times[4.769;5.417]\times[10;11]\times[14;16]\times[124;130]\,,
\end{aligned}
$$

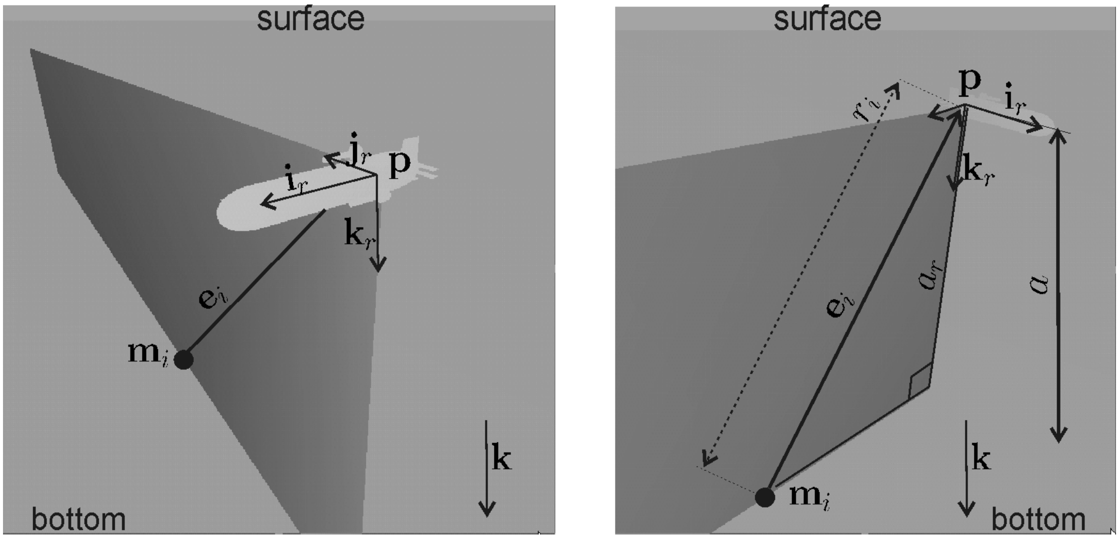
i.e.,

$$
\begin{array}{ll}
E\in[24;26]\,, & R_1\in[1.846;2.307]\,,\\
R_2\in[2.584;3.355], & I\in[4.769;5.417]\,,\\
U_1\in[10;11]\,, & U_2\in[14;16]\,,\\
P\in[124;130]\,. &
\end{array}
$$

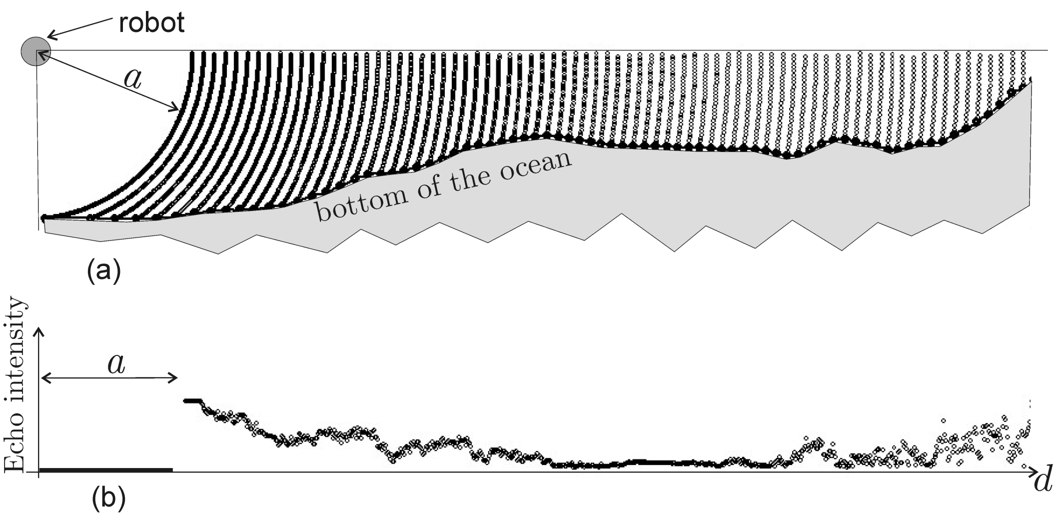
4 SLAM

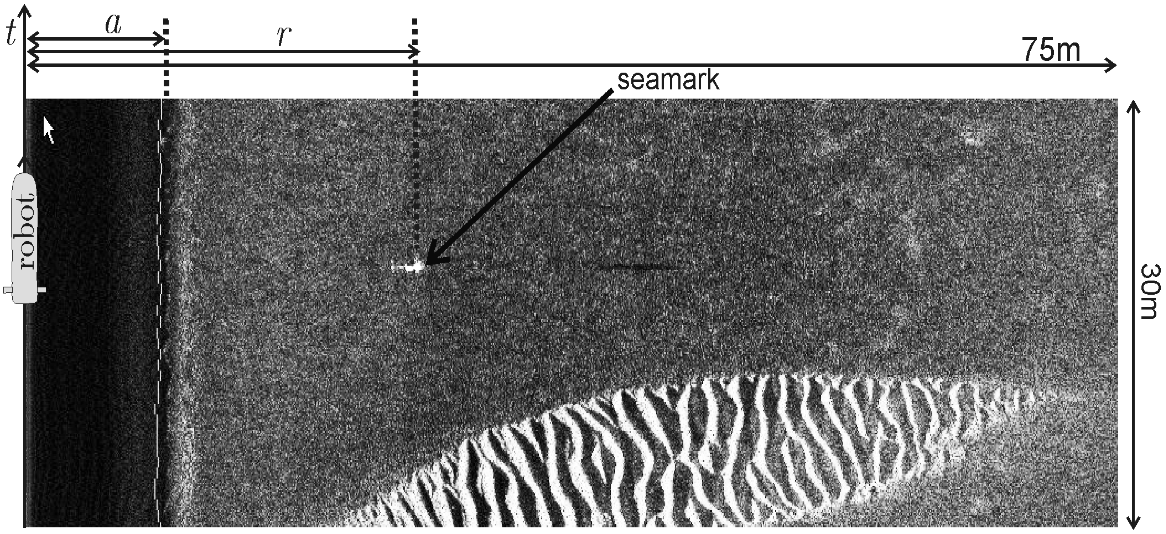


Show the video



surface
bottom
surface
bottom


robot
a
bottom of the ocean
(a)
Echo intensity
a
d
(b)



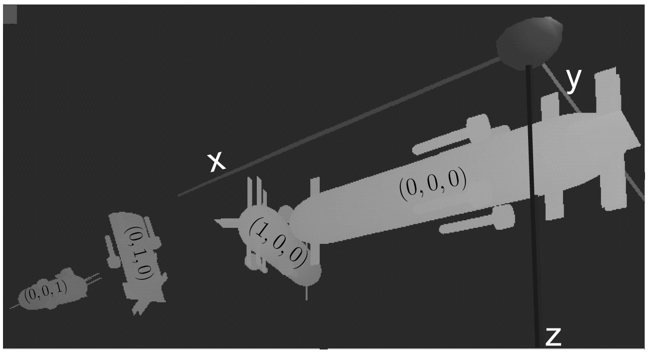
Mine detection with SonarPro

Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \tilde{\mathbf{v}}_r + 0.004 * [-1, 1] \, . \tilde{\mathbf{v}}_r + 0.004 * [-1, 1]$$

Inertial central (Octans III from IXSEA).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.\,[-1,1] \\ 1.75 \times 10^{-4}.\,[-1,1] \\ 5.27 \times 10^{-3}.\,[-1,1] \end{pmatrix}.$$



Six mines have been detected.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	7054	7092	7374	7748	9038	9688	10024	10817	11172	11232	11279	11688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

4.1 Constraints

$$t \in \{6000.0, 6000.1, 6000.2, \ldots, 11999.4\},$$

$$i \in \{0, 1, \ldots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \cdot \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) \cdot \frac{\pi}{180}\right) & 0 \end{pmatrix} \cdot \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix},$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix},$$

$$\mathbf{R}_\varphi(t) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{array} \right),$$

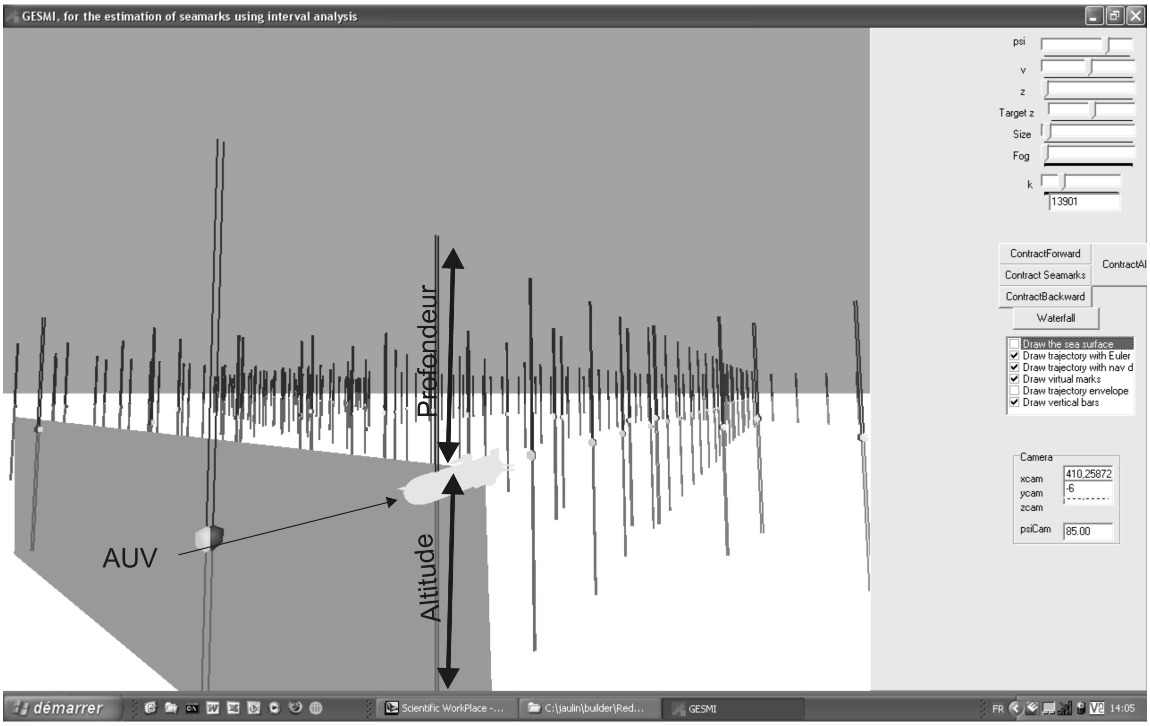
$$\mathbf{R}(t) = \mathbf{R}_\psi(t) \cdot \mathbf{R}_\theta(t) \cdot \mathbf{R}_\varphi(t),$$

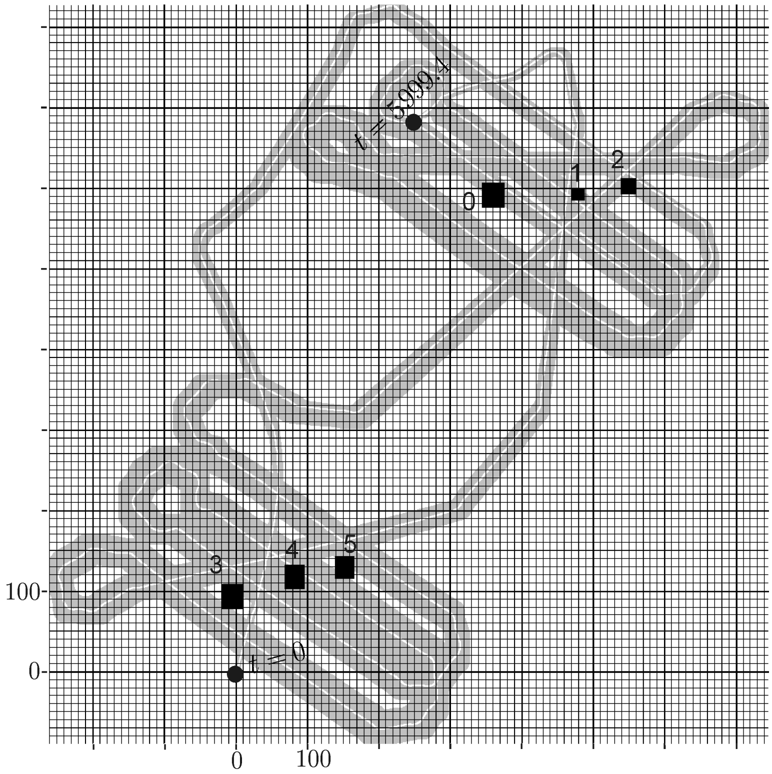
$$\dot{\mathbf{p}}(t) = \mathbf{R}(t) \cdot \mathbf{v}_r(t),$$

$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| \ = r(i),$$

$$\mathbf{R}^\mathsf{T}(\tau(i)) \cdot (\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^{\times 2}.$$

4.2 GESMI


GESMI, for the estimation of seamarks using interval analysis
Profondeur
Altitude
AUV
psi
v
z
Target z
Size
Fog
k
13901
ContractForward
Contract Seamarks
ContractBackward
ContractAl
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw trajectory with nav d
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
Camera
xcam
410,25872
ycam
-6
zcam
psiCam
85.00
démarrer
Scientific WorkPlace -...
C:\jaulin\builder\Red...
GESMI
FR
14:05



t = 5900.4
0
1
2
3
4
5
t = 0
100
0
0
100

5 Robust parameter estimation

Exercise. A robot measures its own distance to three marks. The distances and the coordinates of the marks are

mark	x_i	y_i	d_i
1	0	0	$[22, 23]$
2	10	10	$[10, 11]$
3	30	-30	$[53, 54]$

1) Define the set $\mathbb{X}$ al all feasible positions.
2) Build the contractor associated with $\mathbb{X}$.
2) Build the contractor associated with $\overline{\mathbb{X}}$.

Solution.

$$\mathbb{X} = \bigcap_{i \in \{1,2,3\}} \underbrace{\left\{(x, y) \mid (x - x_i)^2 + (y - y_i)^2 \in \left[d_i^-, d_i^+\right]\right\}}_{\mathbb{X}_i}$$

$$
\begin{aligned}
\overline{\mathbb{X}} &= \overline{\bigcap_{i\in\{1,2,3\}} \mathbb{X}_i} = \bigcup_{i\in\{1,2,3\}} \overline{\mathbb{X}_i} \\
&= \bigcup_{i\in\{1,2,3\}} \Big\{(x,y) \mid (x-x_i)^2 + (y-y_i)^2 \in \big[-\infty, d_i^-\big]\Big\} \\
&\qquad \cup \Big\{(x,y) \mid (x-x_i)^2 + (y-y_i)^2 \in \big[d_i^+, \infty\big]\Big\}
\end{aligned}
$$

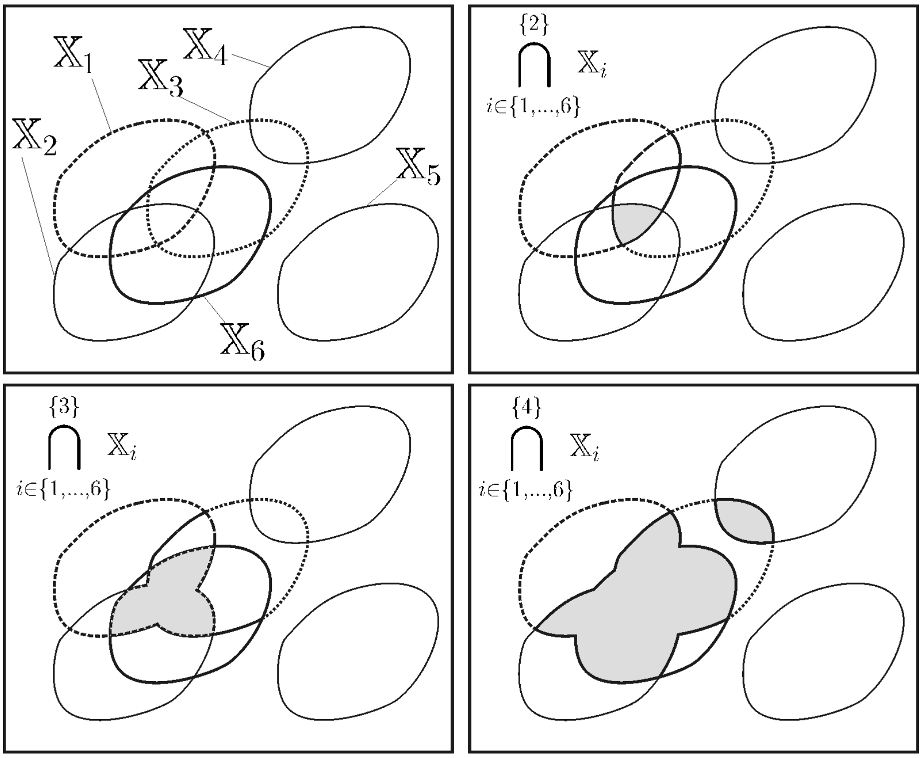
$$
\begin{aligned}
\mathcal{C} &= \bigcap_{i \in \{1,2,3\}} \mathcal{D}_{\left[d_i^-, d_i^+\right]} \\
\overline{\mathcal{C}} &= \bigcup_{i \in \{1,2,3\}} \left(\mathcal{D}_{\left[-\infty, d_i^-\right]}\right) \cup \left(\mathcal{D}_{\left[d_i^+, \infty\right]}\right)
\end{aligned}
$$

5.1 Relaxed intersection

Dealing with outliers

$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Consider m sets $\mathbb{X}_{\mathbf{1}}, \ldots, \mathbb{X}_m$ of $\mathbb{R}^n$. The q-relaxed intersection $\bigcap^{\{q\}} \mathbb{X}_i$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most.



$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\bigcap_{i\in\{1,\dots,6\}}^{\{2\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\dots,6\}}^{\{3\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\dots,6\}}^{\{4\}} \mathbb{X}_i$

Exercise. Consider 8 intervals: $\mathbb{X}_1 = [1, 4]$, $\mathbb{X}_2 = [2, 4]$, $\mathbb{X}_3 = [2, 7]$, $\mathbb{X}_4 = [6, 9]$, $\mathbb{X}_5 = [3, 4]$, $\mathbb{X}_6 = [3, 7]$. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?, \quad \bigcap^{\{1\}} \mathbb{X}_i = ?, \quad \bigcap^{\{2\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{3\}} \mathbb{X}_i = ?, \quad \bigcap^{\{4\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?, \quad \bigcap^{\{6\}} \mathbb{X}_i = ?.$$

Solution. For $\mathbb{X}_1 = [1,4], \mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$, we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset, \ \bigcap^{\{1\}} \mathbb{X}_i = [3,4], \ \bigcap^{\{2\}} \mathbb{X}_i = [3,4], \\
\bigcap^{\{3\}} \mathbb{X}_i &= [2,4] \cup [6,7], \ \bigcap^{\{4\}} \mathbb{X}_i = [2,7], \\
\bigcap^{\{5\}} \mathbb{X}_i &= [1,9], \ \bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}.
\end{aligned}$$

If $\mathbb{X}_i$'s are intervals, the relaxed intersection can be computed with a complexity of $n \log n$.

Take all bounds of all intervals with their brackets.

Bounds	1	4	2	4	2	7	6	9	3	4	3	7
Brackets	[	]	[	]	[	]	[	]	[	]	[	]

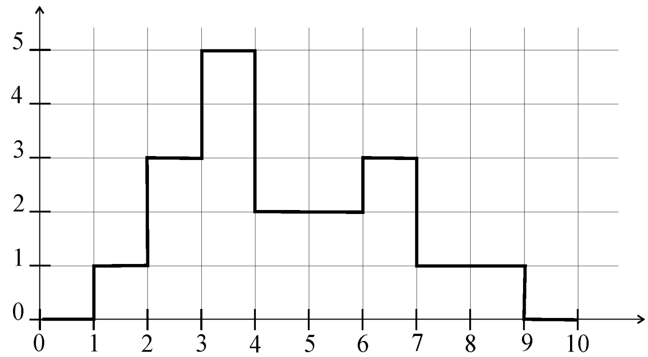
Sort the columns with respect the bounds:

Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]

Scan from left to right, counting $+1$ for '[' and -1 for ']':

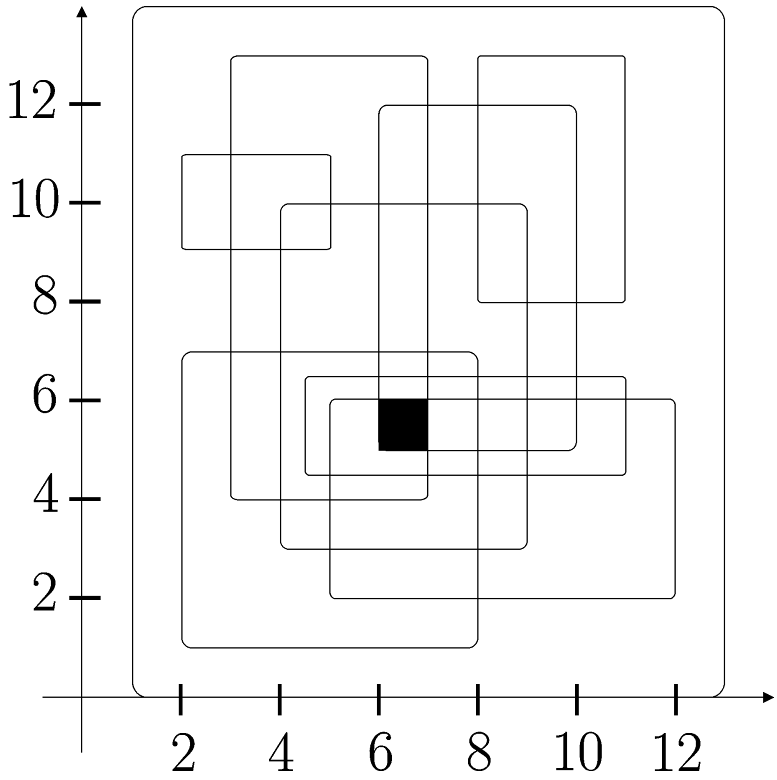
Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]
Sum	1	2	3	4	5	4	3	2	3	2	1	0

Read the q-intersections



Set-membership function associated with the 6 intervals

Computing the q relaxed intersection of m boxes is tractable.



The black box is the 2-intersection of 9 boxes

Formal definition

$$\bigcap^{\{q\}} \mathbb{X}_i = \bigcup_{\{\sigma_1,\ldots,\sigma_{n-q}\}} \mathbb{X}_{\sigma_1} \cap \cdots \cap \mathbb{X}_{\sigma_{n-q}}$$

$$\bigcup^{\{q\}} \mathbb{X}_i = \bigcap_{\{\sigma_1,\ldots,\sigma_{n-q}\}} \mathbb{X}_{\sigma_1} \cup \cdots \cup \mathbb{X}_{\sigma_{n-q}}$$

Remark

$$\bigcap^{\{0\}} \mathbb{X}_i = \bigcap \mathbb{X}_i$$

$$\bigcup^{\{0\}} \mathbb{X}_i = \bigcup \mathbb{X}_i$$

Dual rule

$$\bigcap^{\{q\}} \mathbb{X}_i = \bigcup^{\{n-q-1\}} \mathbb{X}_i$$

De Morgan's law

$$
\begin{aligned}
\overline{\bigcap^{\{q\}} \mathbb{X}_i} &= \bigcup^{\{q\}} \overline{\mathbb{X}_i} \\
\overline{\bigcup^{\{q\}} \mathbb{X}_i} &= \bigcap^{\{q\}} \overline{\mathbb{X}_i}.
\end{aligned}
$$

From the De Morgan's law and the dual rules, we get

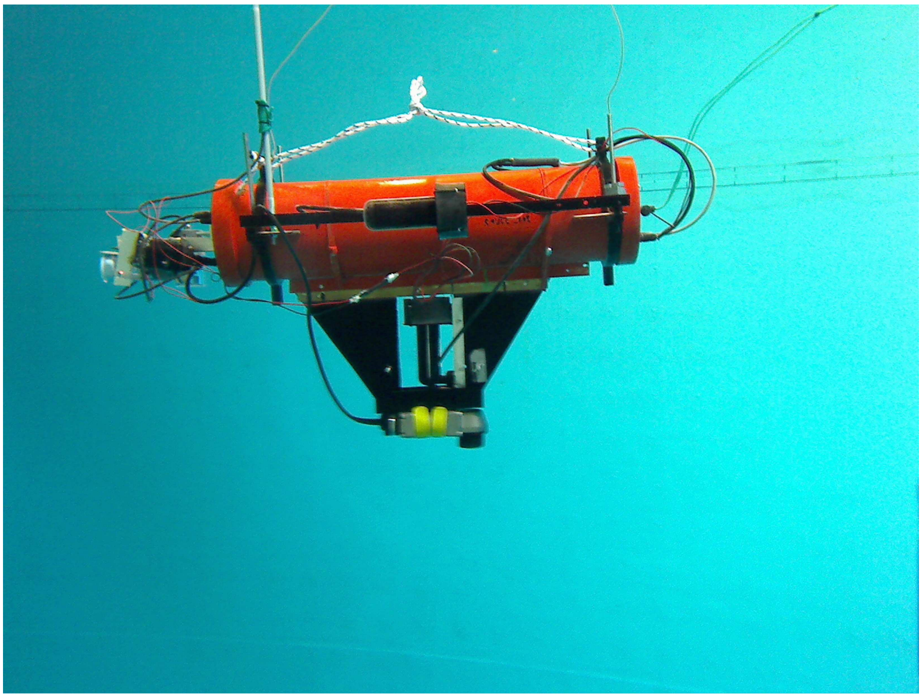
$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} = \overline{\bigcup^{\{n-q-1\}} \mathbb{X}_i} = \bigcap^{\{n-q-1\}} \overline{\mathbb{X}_i}$$

Relaxation of contractors

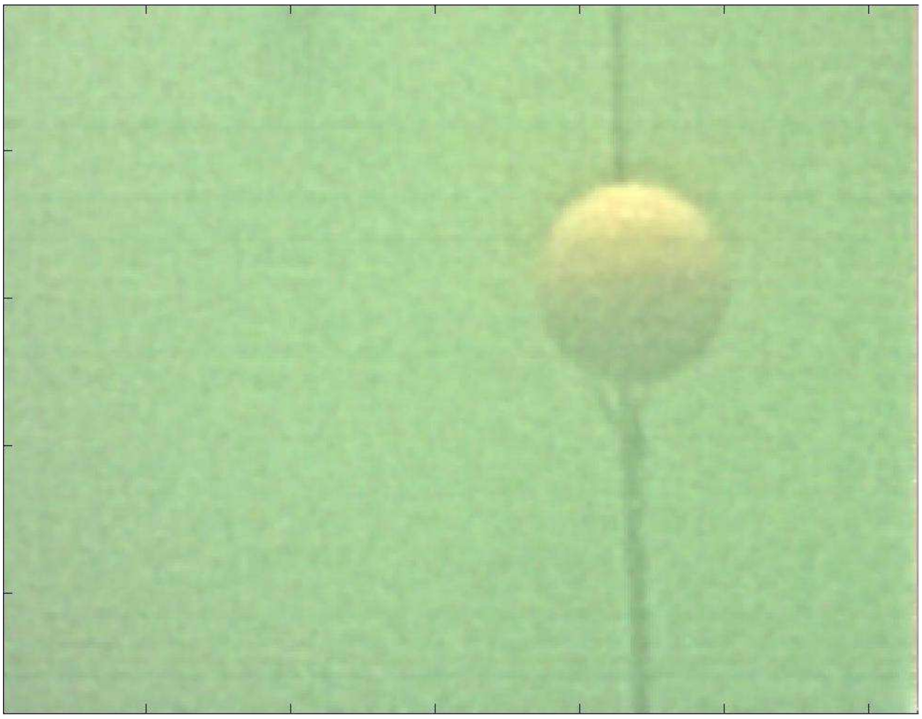
We define the q-relaxed intersection between m contractors

$$\mathcal{C} = \left(\bigcap_{i \in \{1,\ldots,m\}}^{\{q\}} \mathcal{C}_i \right) \Leftrightarrow \forall \, [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\left([\mathbf{x}]\right) = \bigcap^{\{q\}} \mathcal{C}_i\left([\mathbf{x}]\right).$$

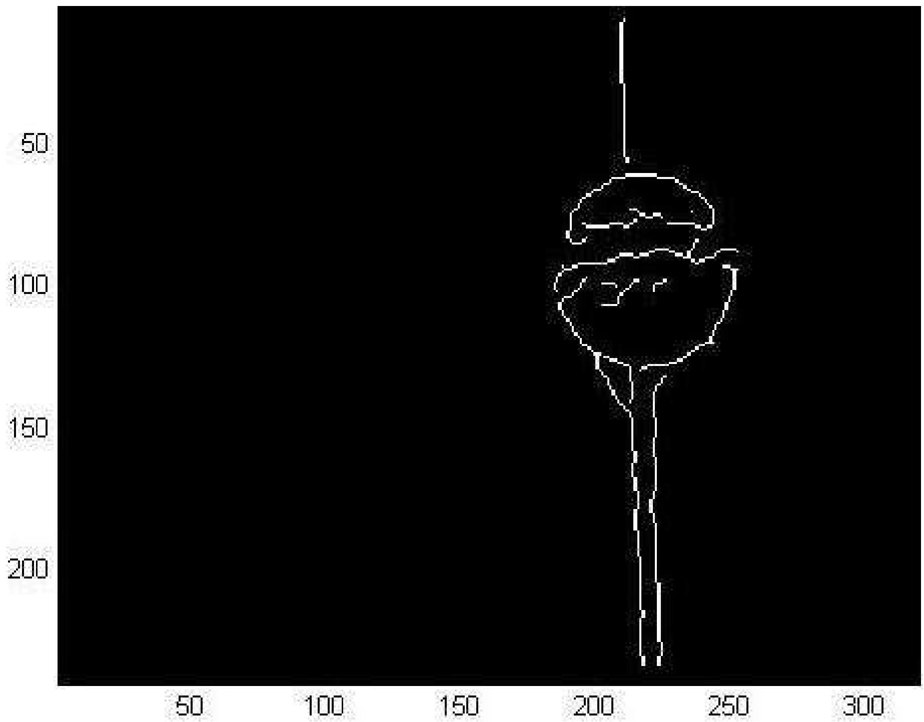
5.2 Shape detection



Sauc'isse robot swimming inside a pool



A spheric buoy seen by Sauc'isse



50
100
150
200
50
100
150
200
250
300

50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
600
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1

An *implicit parameter set estimation problem* amounts to characterizing

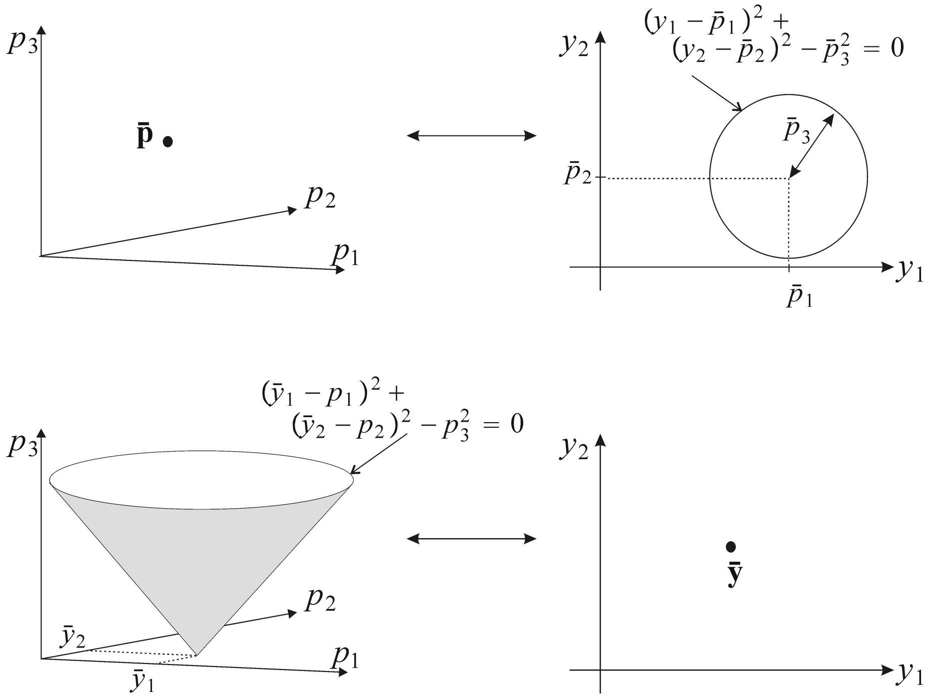
$$\mathbb{P} = \bigcap_{i \in \{1,\ldots,m\}} \underbrace{\{\mathbf{p} \in \mathbb{R}^n, \exists \mathbf{y} \in [\mathbf{y}](i), \mathbf{f}(\mathbf{p}, \mathbf{y}) = \mathbf{0}\}}_{\mathbb{P}_i}$$

where $\mathbf{p}$ is the parameter vector, $[\mathbf{y}](i)$ is the ith measurement box and $\mathbf{f}$ is the model function.

Consider the *shape function* $\mathbf{f}(\mathbf{p}, \mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^2$ corresponds to a pixel and $\mathbf{p}$ is the shape vector.

Example (circle):

$$f(\mathbf{p}, \mathbf{y}) = (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2.$$



p_3
$\bar{\mathbf{p}}$
p_2
p_1
$(y_1 - \bar{p}_1)^2 +$
$(y_2 - \bar{p}_2)^2 - \bar{p}_3^2 = 0$
y_2
$\bar{p}_3$
$\bar{p}_2$
y_1
$\bar{p}_1$
$(\bar{y}_1 - p_1)^2 +$
$(\bar{y}_2 - p_2)^2 - p_3^2 = 0$
p_3
p_2
$\bar{y}_2$
p_1
$\bar{y}_1$
y_2
$\bar{\mathbf{y}}$
y_1

The *shape* associated with $\mathbf{p}$ is

$$\mathcal{S}(\mathbf{p}) \overset{\text{def}}{=} \left\{\mathbf{y} \in \mathbb{R}^2, \mathbf{f}(\mathbf{p}, \mathbf{y}) = \mathbf{0}\right\}.$$

Consider a set of (small) boxes in the image

$$\mathcal{Y} = \{[\mathbf{y}](1), \ldots, [\mathbf{y}](m)\}.$$

Each box is assumed to intersect the shape we want to extract.

In our buoy example,

- $\mathcal{Y}$ corresponds to edge pixel boxes.

- $f(\mathbf{p}, \mathbf{y}) = (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2$.

- $\mathbf{p} = (p_1, p_2, p_3)^{\mathsf{T}}$ where p_1, p_2 are the coordinates of the center of the circle and p_3 its radius.

The q relaxed feasible set is

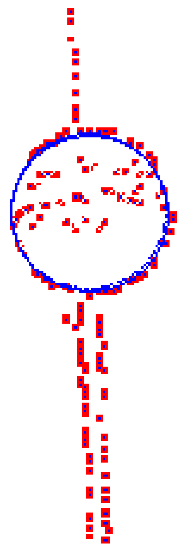
$$\mathbb{P}^{\{q\}} \stackrel{\mathsf{def}}{=} \bigcap_{i\in\{\mathbf{1},\ldots,m\}}^{\{q\}} \left\{\mathbf{p}\in\mathbb{R}^n, \exists\mathbf{y}\in[\mathbf{y}](i), \mathbf{f}\left(\mathbf{p},\mathbf{y}\right)=\mathbf{0}\right\}.$$

An optimal contractor for the set

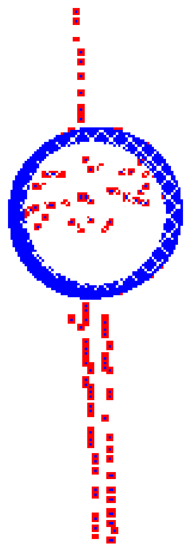
$$\left\{\mathbf{p} \in [\mathbf{p}], \exists \mathbf{y} \in [\mathbf{y}], (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2 = 0\right\}.$$

FB(in: $[\mathbf{y}], [\mathbf{p}]$, out: $[\mathbf{p}]$)

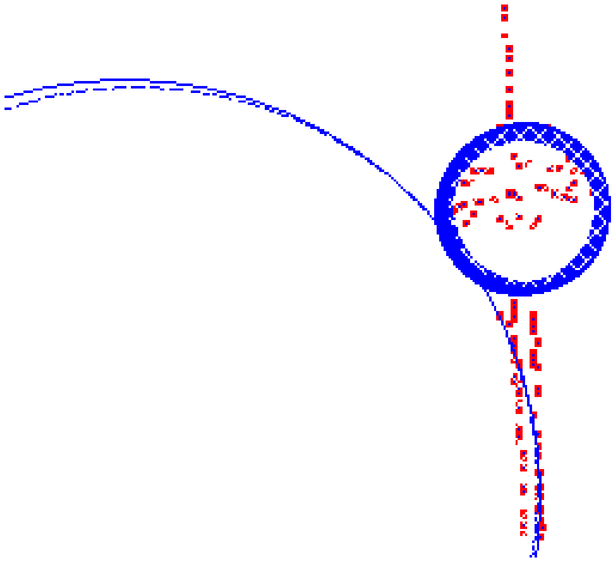
$[d_1] := [y_1] - [p_1]\,;$
$[d_2] := [y_2] - [p_2]\,;$
$[c_1] := [d_1]^2\,;$
$[c_2] := [d_2]^2\,;$
$[c_3] := [p_3]^2\,;$
$[e] := [0,0] \cap ([c_1] + [c_2] - [c_3])\,;$
$[c_1] := [c_1] \cap ([e] - [c_2] + [c_3])\,;$
$[c_2] := [c_2] \cap ([e] - [c_1] + [c_3])\,;$
$[c_3] := [c_3] \cap ([c_1] + [c_2] - [e])\,;$
$[\bar{p}_3] := [p_3] \cap \sqrt{[c_3]};$
$[d_2] := [d_2] \cap \sqrt{[c_2]};$
$[d_1] := [d_1] \cap \sqrt{[c_1]};$
$[p_2] := [p_2] \cap ([y_2] - [d_2])\,;$
$[p_1] := [p_1] \cap ([y_1] - [d_1])\,;$



$q = 0.70\ m$ (i.e. 70% of the data can be outlier)



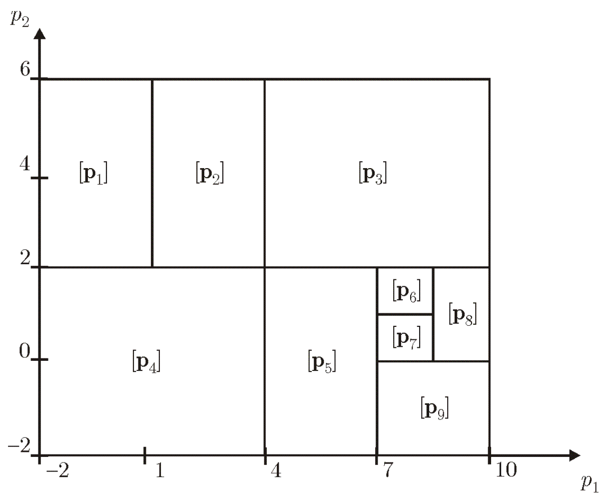
$q = 0.80\ m$ (i.e. 80% of the data can be outlier)



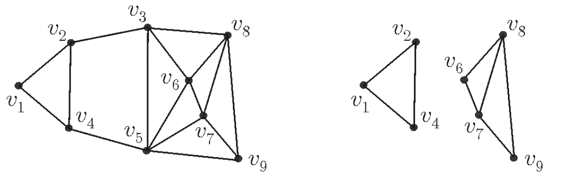
$q = 0.81\ m$ (i.e. 81% of the data can be outlier)

6 Intervals and graphs

6.1 Path planning

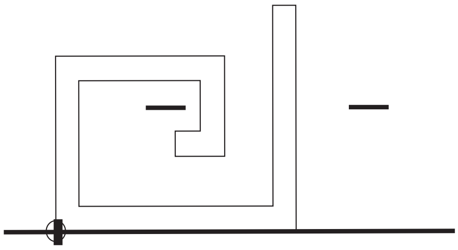


p_2
6
4
2
0
−2
$[\mathbf{p}_1]$
$[\mathbf{p}_2]$
$[\mathbf{p}_3]$
$[\mathbf{p}_4]$
$[\mathbf{p}_5]$
$[\mathbf{p}_6]$
$[\mathbf{p}_7]$
$[\mathbf{p}_8]$
$[\mathbf{p}_9]$
−2
1
4
7
10
p_1

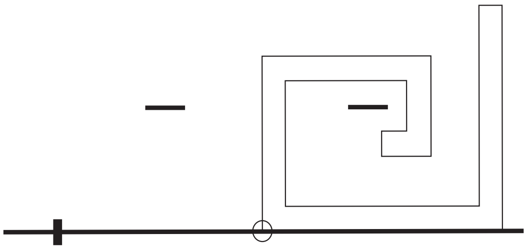


v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
v_1
v_2
v_4
v_6
v_7
v_8
v_9

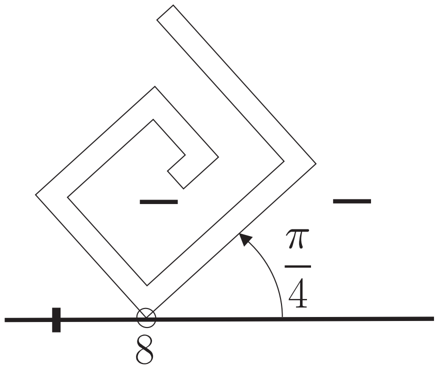
(a) (b)



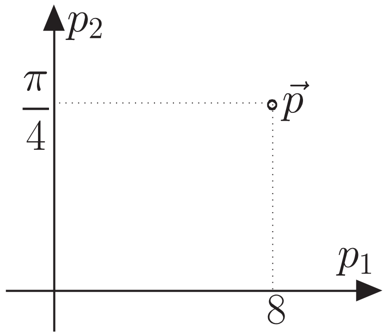
Initial configuration: $\vec{p} = (0\ 0)^{\mathrm{T}}$



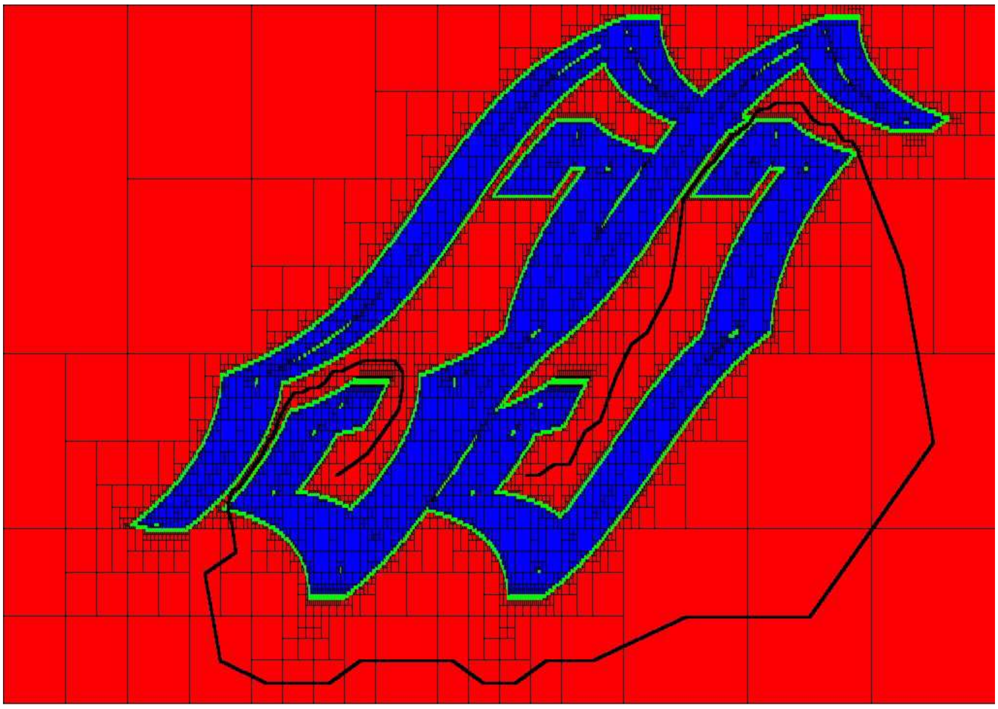
Goal configuration: $\vec{p} = (17\ 0)^{\mathrm{T}}$

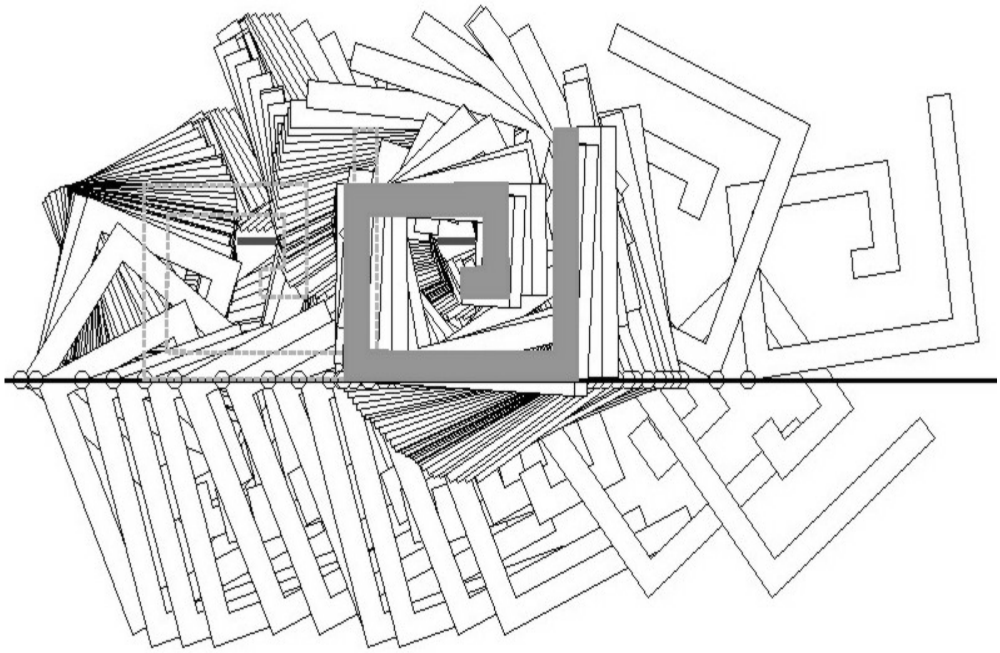


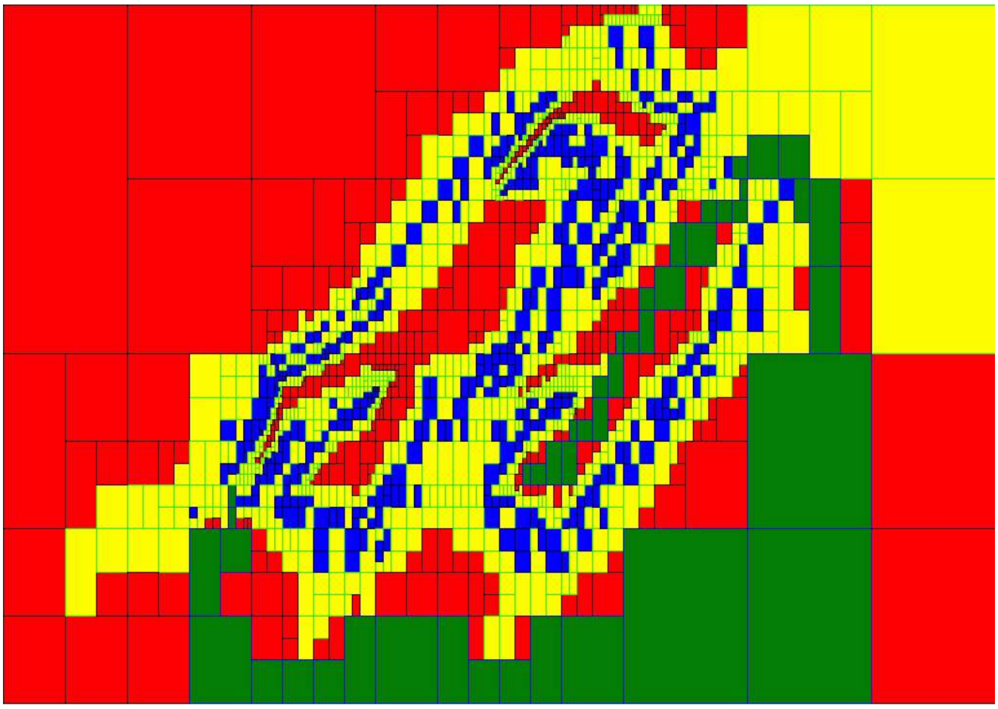
Room

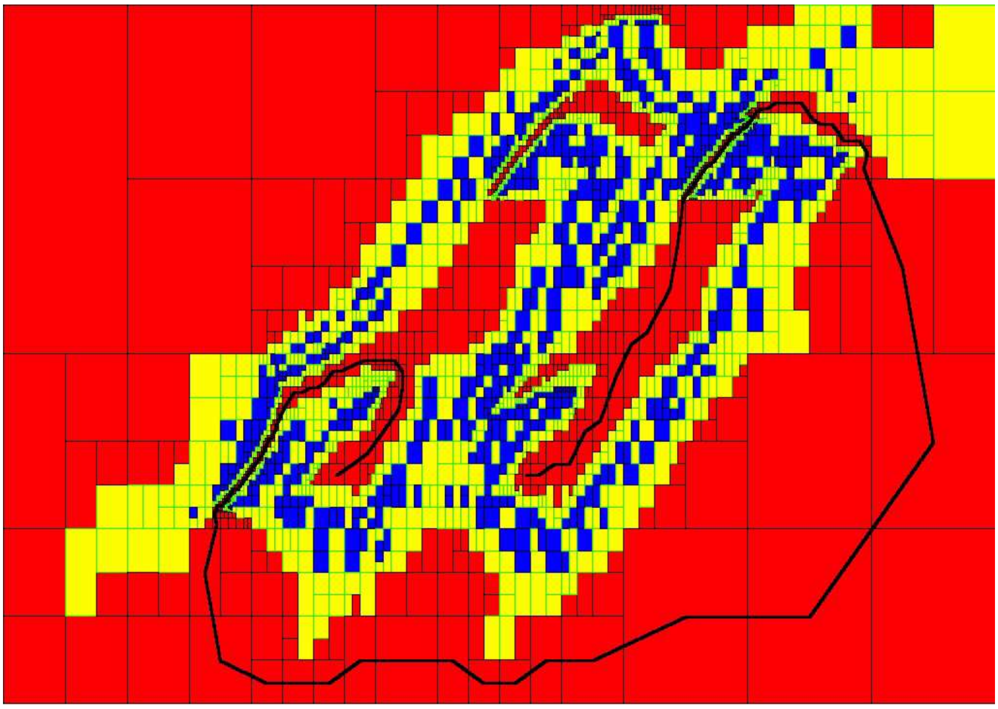


Configuration space



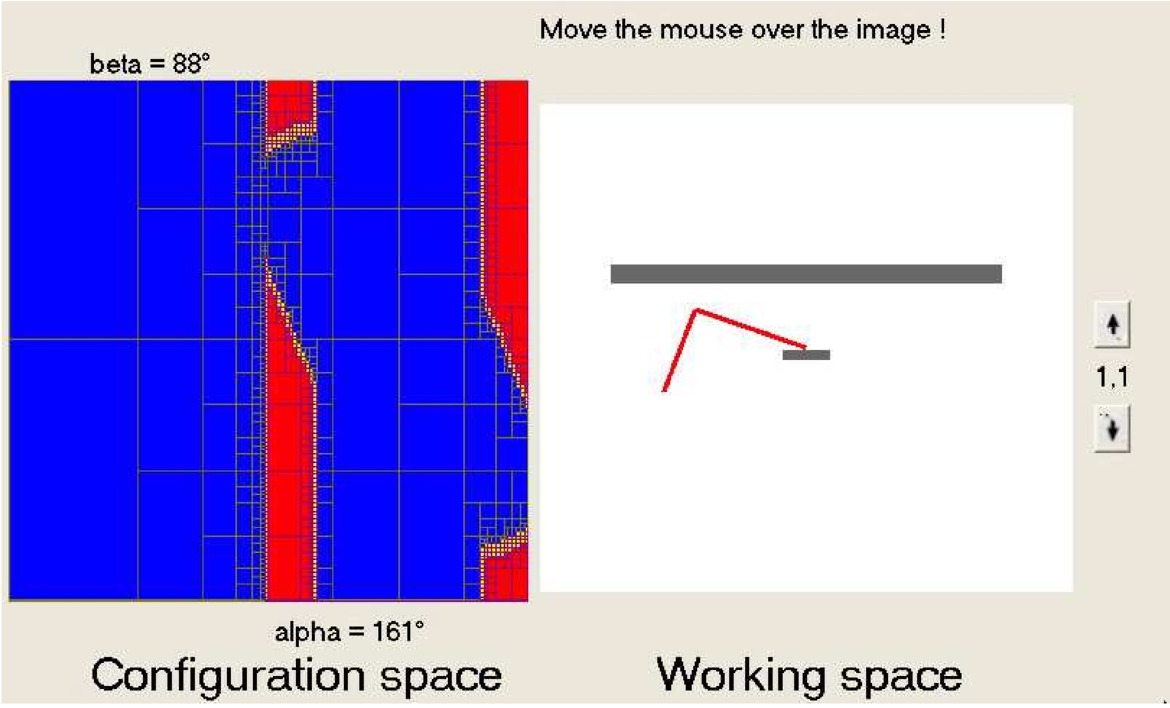






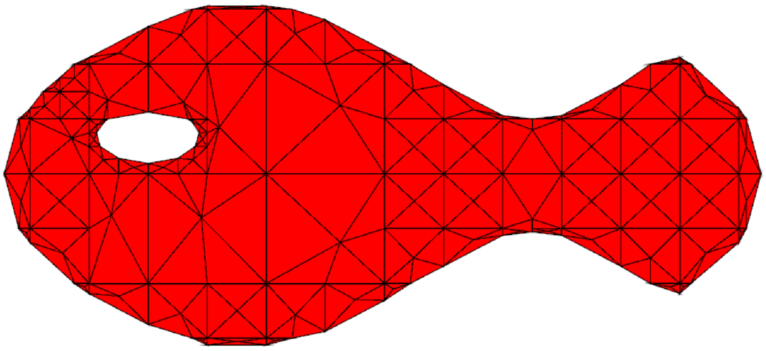
6.2 Charaterizing the topology

(Collaboration with N. Delanoue and B. Cottenceau)



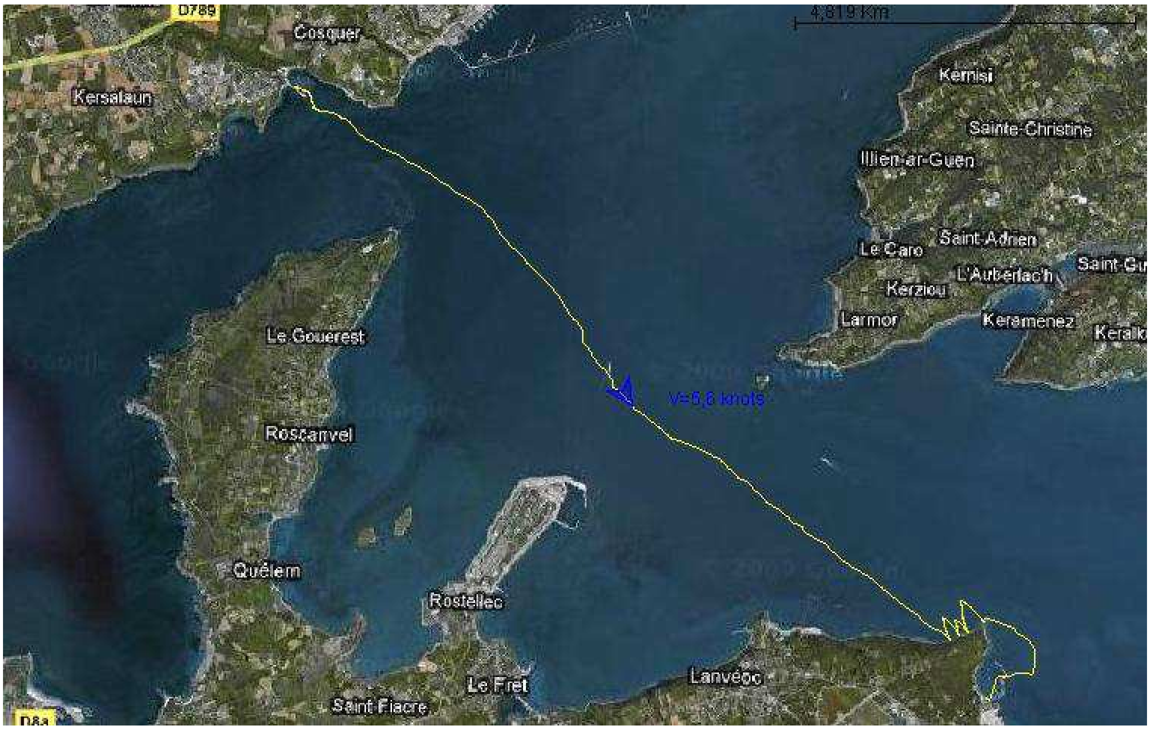
Move the mouse over the image !
beta = 88°
1,1
alpha = 161°
Configuration space
Working space

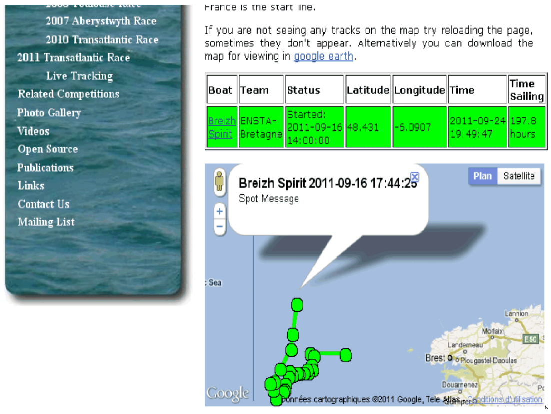
An approach has also been developed with N. Delanoue to compute a triangulation homeomorphic to $\mathbb{S}$.

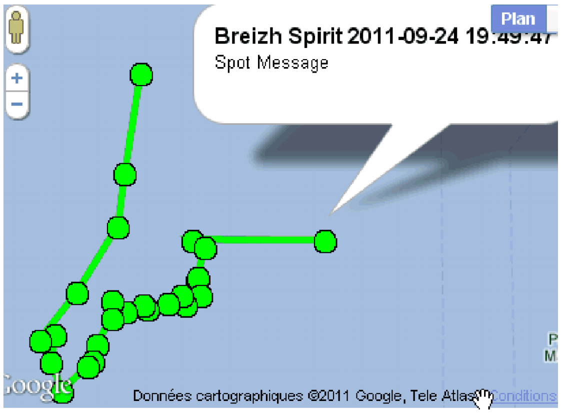


7 Saiboat robotics


BREIZH
ENSIETA
SPIRIT
ENSIETA
ENSIETA


D789
Cosquer
Kersalaun
4,819 Km
Kernisi
Sainte-Christine
Illien-ar-Guen
Le Caro
Saint-Adrien
L'Auberlac'h
Kerziou
Larmor
Keramenez
Le Gouerest
V=5,6 knots
Roscanvel
Quélern
Rostellec
Le Fret
Lanvéoc
Saint-Fiacre
D8a


2007 Aberystwyth Race
2010 Transatlantic Race
2011 Transatlantic Race
Live Tracking
Related Competitions
Photo Gallery
Videos
Open Source
Publications
Links
Contact Us
Mailing List
France is the start line.
If you are not seeing any tracks on the map try reloading the page, sometimes they don't appear. Alternatively you can download the map for viewing in google earth.
Boat | Team | Status | Latitude | Longitude | Time | Time Sailing
Breizh Spirit | ENSTA-Bretagne | Started: 2011-09-16 14:00:00 | 48.431 | -6.0907 | 2011-09-24 19:49:47 | 197.8 hours
Breizh Spirit 2011-09-16 17:44:25
Spot Message
Plan
Satellite
Sea
Lannion
Morlaix
Landerneau
Brest
Plougastel-Daoulas
Douarnenez
Google
Données cartographiques ©2011 Google, Tele Atlas


Breizh Spirit 2011-09-24 19:49:47
Spot Message
Plan
Google
Données cartographiques ©2011 Google, Tele Atlas
Conditions

7.1 Vaimos

Vaimos (IFREMER and ENSTA)

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}\left(\mathbf{x}\right)$, the robot satisfies an equation of the form

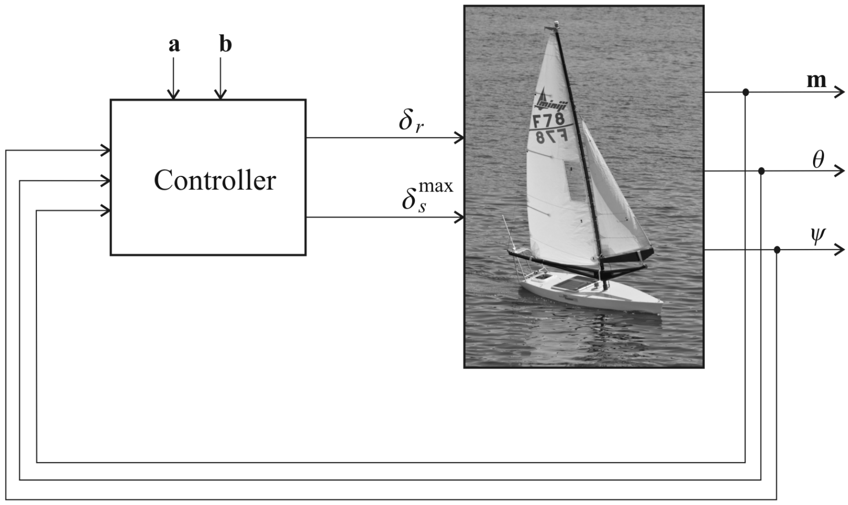
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right).$$

With all uncertainties, the robot satisfies.

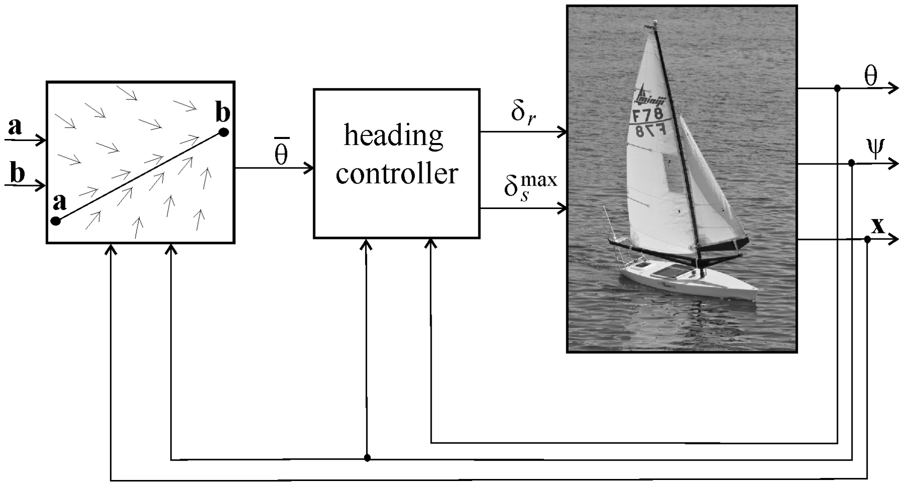
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

7.2 Line following



Controller of a sailboat robot



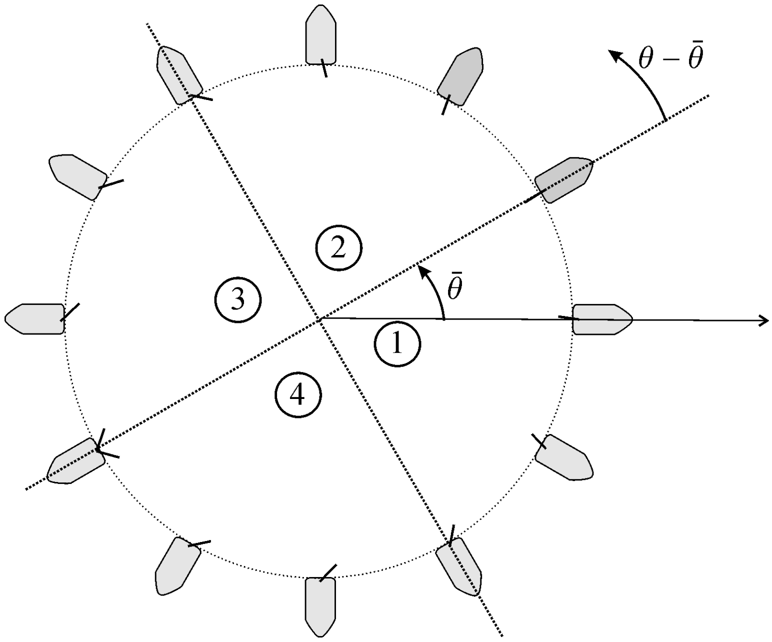
a
b
a
b
$\bar{\theta}$
heading controller
δ_r
δ_s^{max}
F78
θ
ψ
x

Heading controller

$$\left\{\begin{array}{lll}\delta_r & = & \left\{\begin{array}{ll}\delta_r^{\mathsf{max}}.\,\mathsf{sin}\left(\theta-\bar{\theta}\right) & \text{if }\ \mathsf{cos}\left(\theta-\bar{\theta}\right)\geq 0\\ \delta_r^{\mathsf{max}}.\mathsf{sign}\left(\mathsf{sin}\left(\theta-\bar{\theta}\right)\right) & \text{otherwise}\end{array}\right.\\ \delta_s^{\mathsf{max}} & = & \frac{\pi}{2}.\left(\frac{\mathsf{cos}(\psi-\bar{\theta})+1}{2}\right).\end{array}\right.$$

Rudder

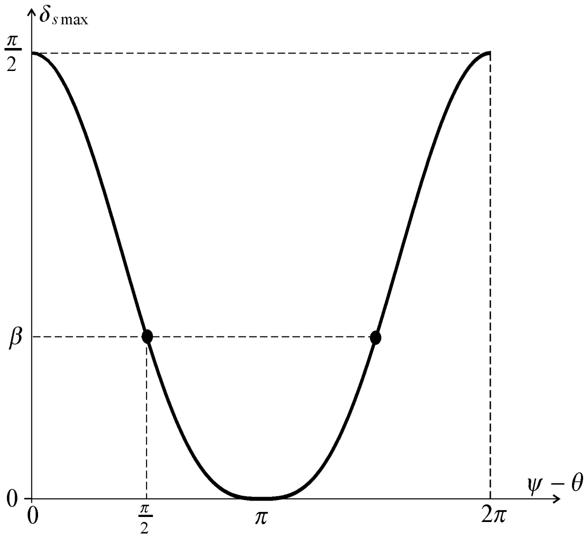
$$\delta_r = \begin{cases} \delta_r^{\mathsf{max}}.\,\mathsf{sin}\left(\theta - \bar{\theta}\right) & \text{if } \mathsf{cos}\left(\theta - \bar{\theta}\right) \geq 0 \\ \delta_r^{\mathsf{max}}.\mathsf{sign}\left(\mathsf{sin}\left(\theta - \bar{\theta}\right)\right) & \text{otherwise} \end{cases}$$

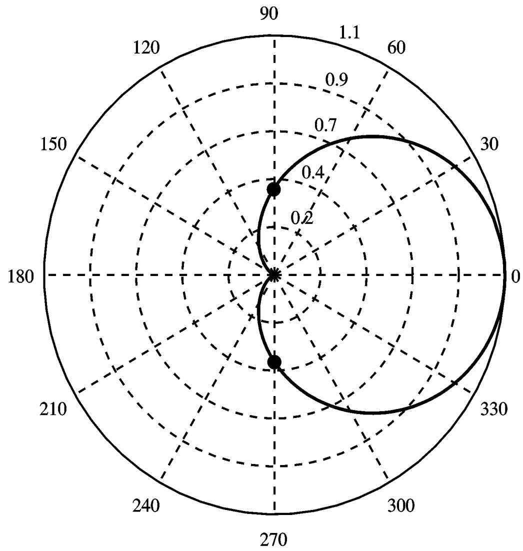


$\theta - \bar{\theta}$
2
$\bar{\theta}$
3
1
4

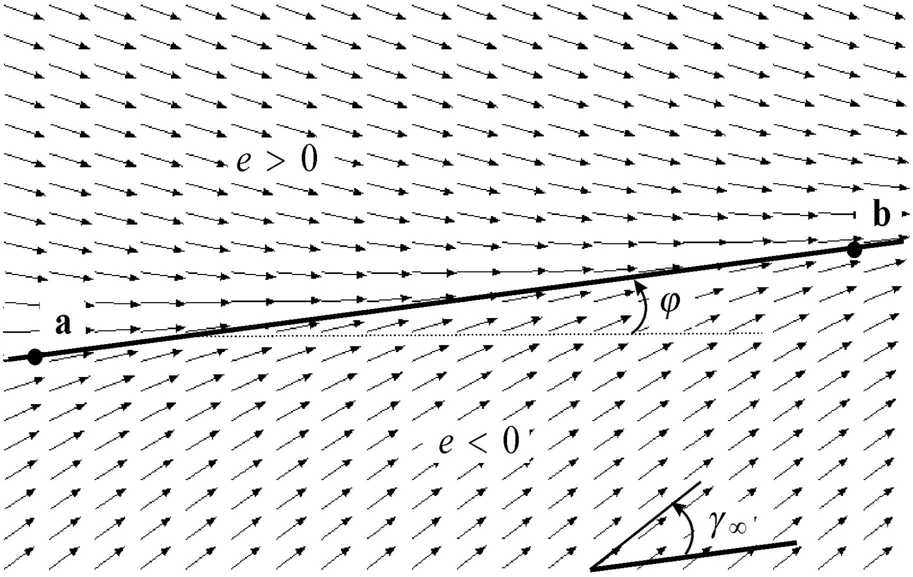
Sail

$$\delta_s^{\mathsf{max}} = \frac{\pi}{2} \cdot \left(\frac{\cos\left(\psi - \bar{\theta}\right) + 1}{2} \right)$$



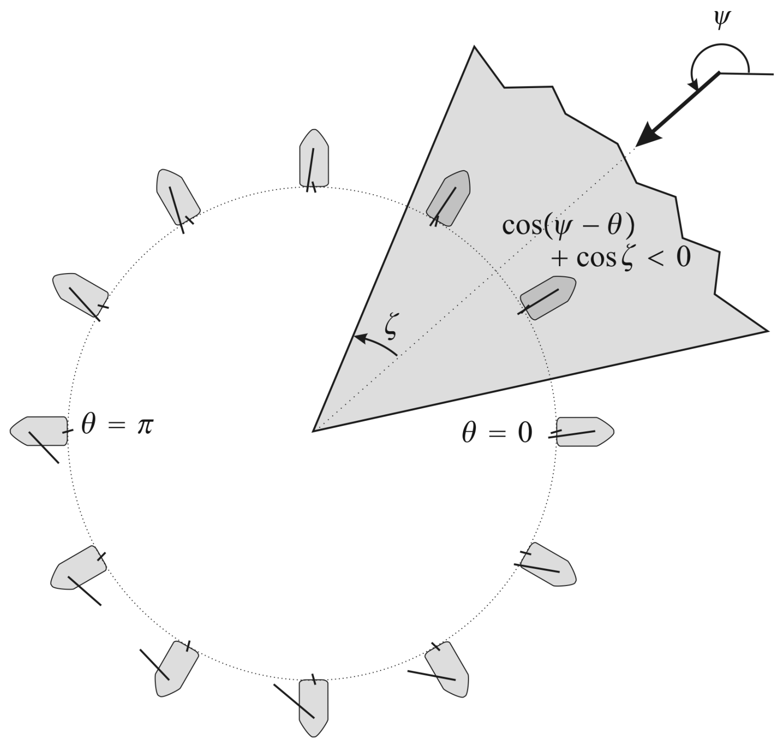


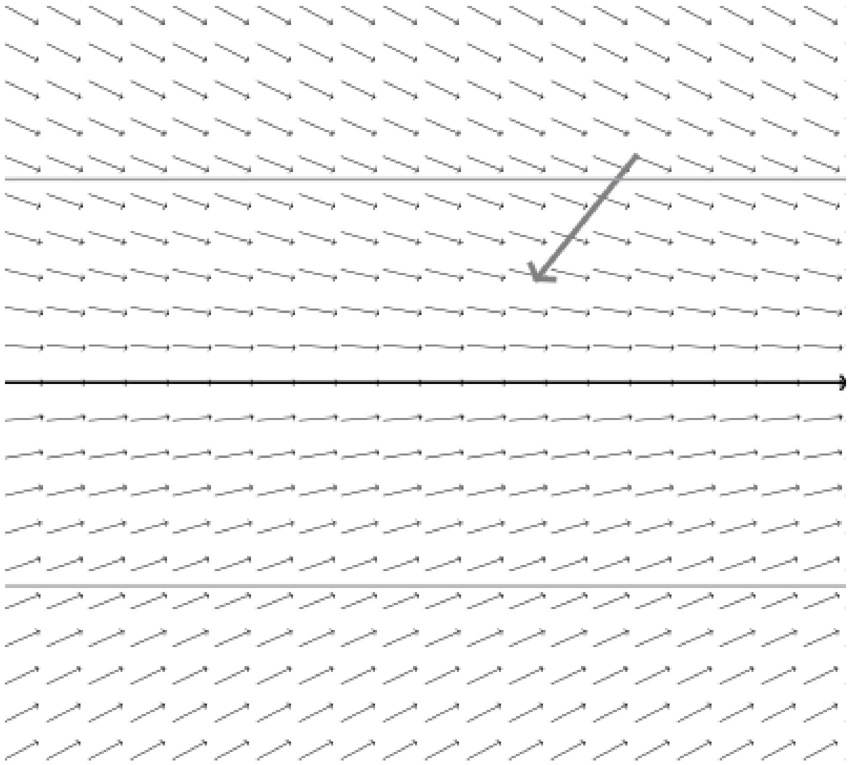
7.3 Vector field



Nominal vector field: $\theta^* = \varphi - \frac{1}{2}.\text{atan}\left(\frac{e}{r}\right).$

A course θ^* may be unfeasible





$$\theta^* = -\frac{2.\gamma_\infty}{\pi}.\mathsf{atan}\left(\frac{e}{r}\right)$$

7.4 Controller

Function in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\max}$; inout: q

$e = \frac{\det(\mathbf{b}-\mathbf{a},\mathbf{m}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|}$

if $|e| > r$ then $q = \mathrm{sign}(e)$

$\varphi = \mathrm{atan2}(\mathbf{b} - \mathbf{a})$

$\bar{\theta} = \varphi - \frac{1}{2}.\mathrm{atan}\left(\frac{e}{r}\right)$

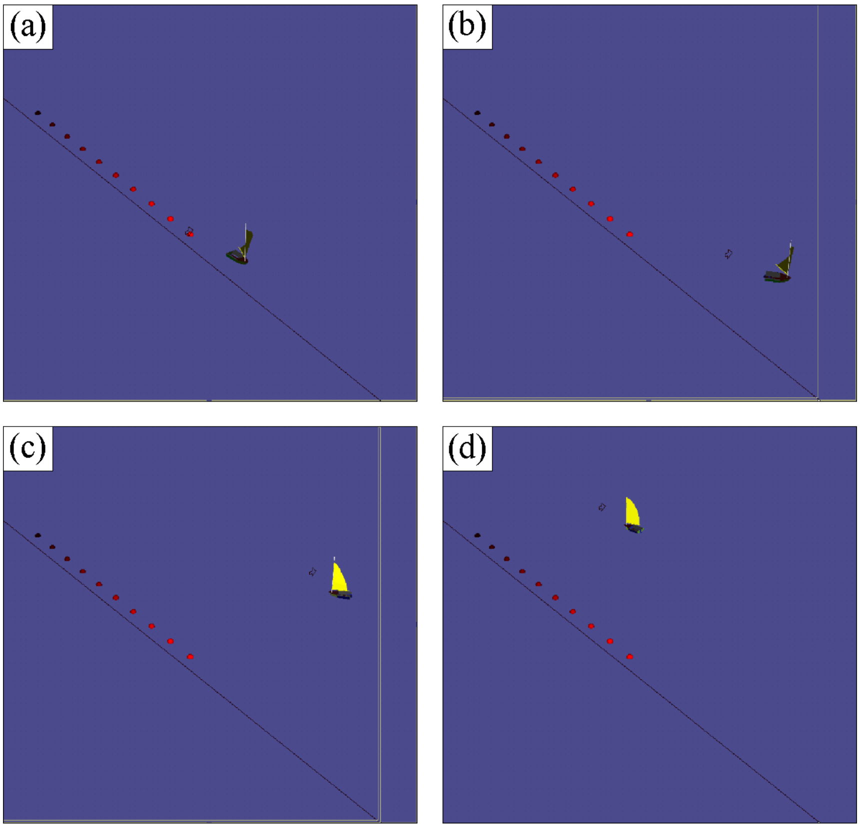
if $\cos\left(\psi - \bar{\theta}\right) + \cos\zeta < 0$ then $\bar{\theta} = \pi + \psi - q.\zeta$.

if $\cos\left(\theta - \bar{\theta}\right) \geq 0$ then $\delta_r = \delta_r^{\max}.\sin\left(\theta - \bar{\theta}\right)$

else $\delta_r = \delta_r^{\max}.\mathrm{sign}\left(\sin\left(\theta - \bar{\theta}\right)\right)$

$\delta_s^{\max} = \frac{\pi}{2}.\left(\frac{\cos(\psi-\bar{\theta})+1}{2}\right)$.

7.5 Validation by simulation


(a)
(b)
(c)
(d)

7.6 Theoretical validation

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

The system

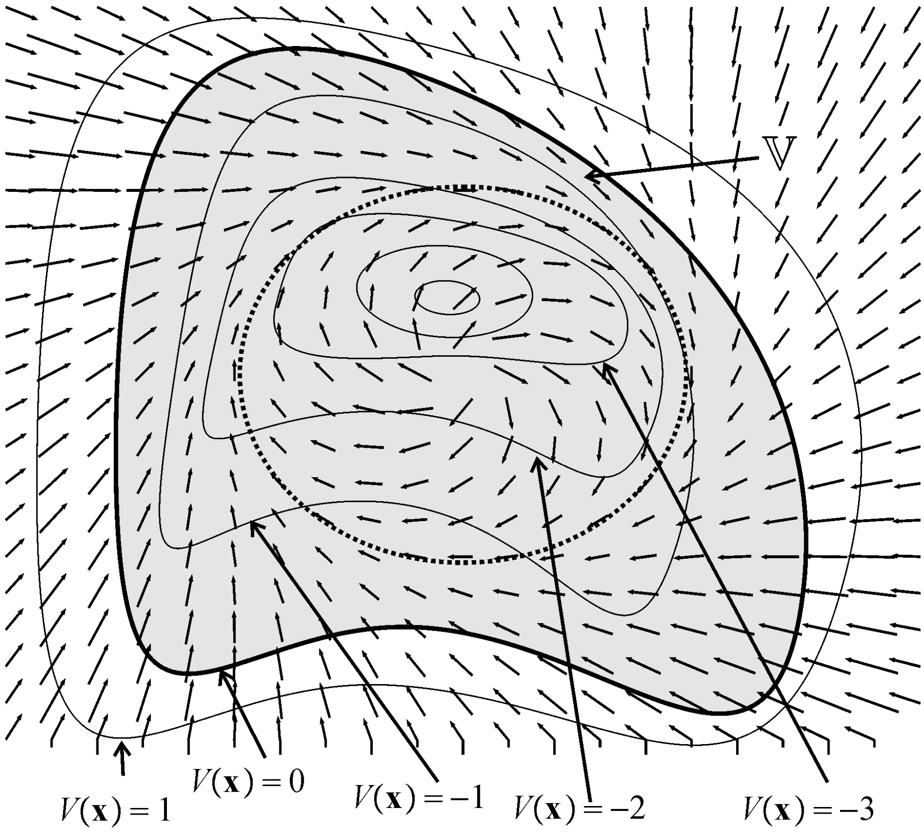
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$$

is Lyapunov-stable (1892) is there exists $V\left(\mathbf{x}\right) \geq 0$ such that

$$\begin{aligned} \dot{V}\left(\mathbf{x}\right) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}, \\ V\left(\mathbf{x}\right) &= 0 \text{ iff } \mathbf{x} = \mathbf{0}. \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

$$\Big(V\left(\mathbf{x}\right) \geq 0 \;\Rightarrow\; \dot{V}\left(\mathbf{x}\right) < 0\Big)\,.$$



$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$\begin{aligned}
&\left(V(\mathbf{x}) \geq 0 \ \Rightarrow \ \dot{V}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\ &\left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\ &\forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow\ &\neg\left(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) \geq 0 \text{ and } V(\mathbf{x}) \geq 0\right)
\end{aligned}$$

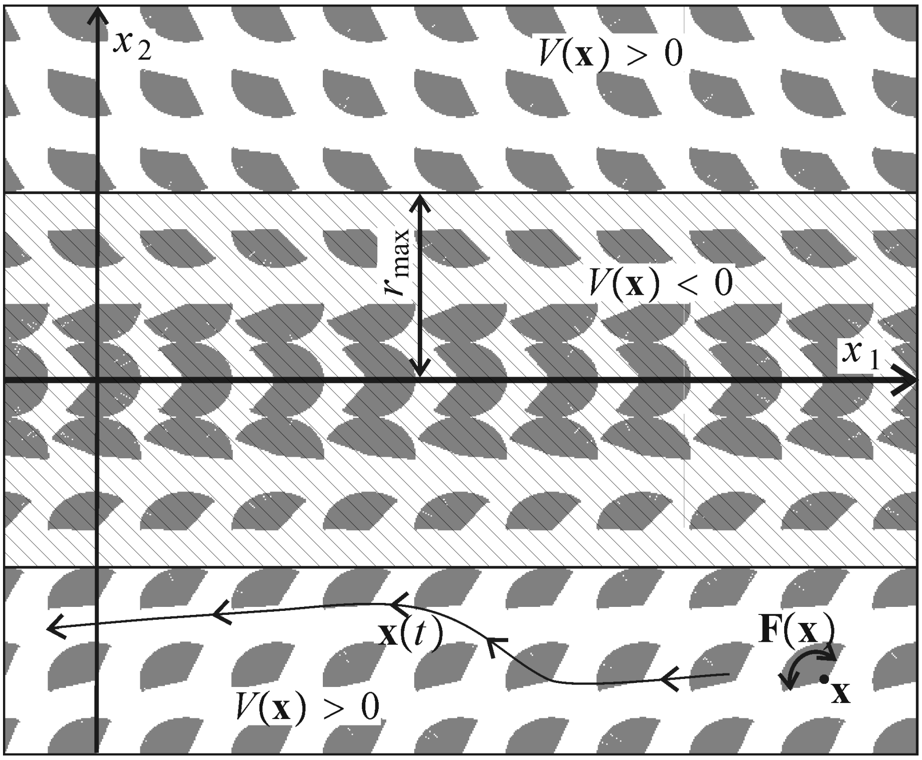
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

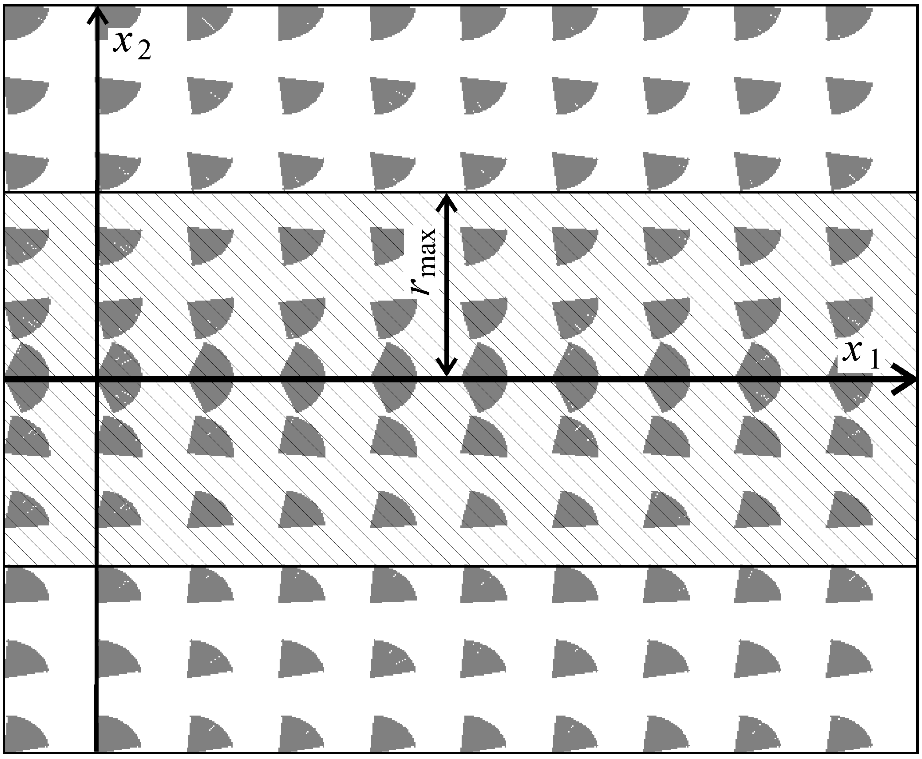
Theorem. We have

$$\left\{\begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a} \geq 0 \\ \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ V(\mathbf{x}) \geq 0 \end{array}\right. \quad \text{inconsistent} \;\Leftrightarrow\; \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) \text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$

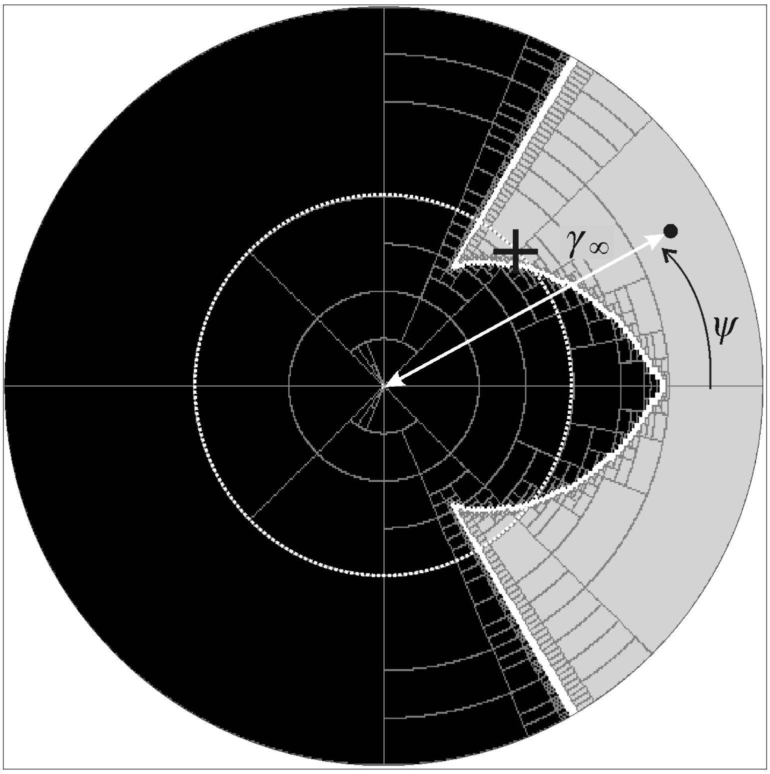

x_2
$r_{\max}$
x_1

7.7 Parametric case

Consider the differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{F}\left(\mathbf{x}, \mathbf{p}\right).$$

We characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V-stable.



+
γ_∞
ψ

7.8 Experimental validation

Brest

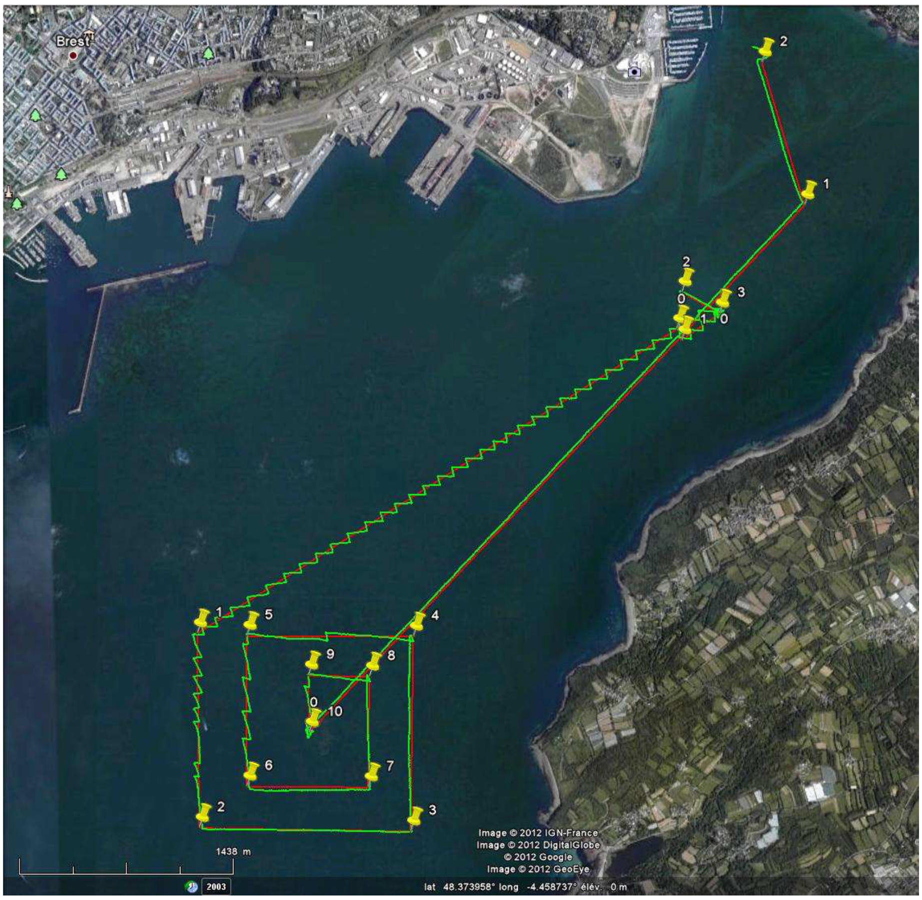

Brest
1438 m
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Image © 2012 DigitalGlobe
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Image © 2012 GeoEye
2003
lat 48.373958° long -4.458737° élév. 0 m

Show Dashboard

Brest-Douarnenez. January 17, 2012, 8am


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Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google

Montrer la mise à l'eau

Middle of Atlantic ocean

350 km made by Vaimos in 53h, September 6-9, 2012.

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.

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