Interval robotics

Chapter 8: Intervals and graphs

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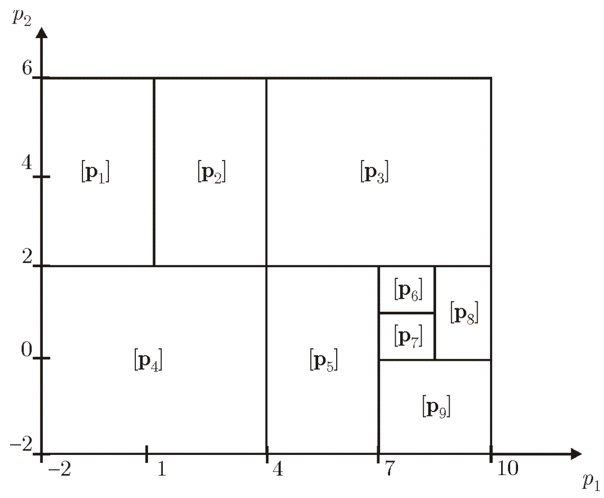
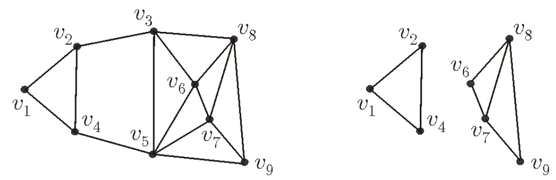
1 Path planning

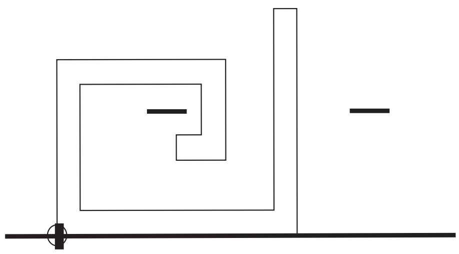


p_2
6
4
2
0
−2
$[\mathbf{p}_1]$
$[\mathbf{p}_2]$
$[\mathbf{p}_3]$
$[\mathbf{p}_4]$
$[\mathbf{p}_5]$
$[\mathbf{p}_6]$
$[\mathbf{p}_7]$
$[\mathbf{p}_8]$
$[\mathbf{p}_9]$
−2
1
4
7
10
p_1

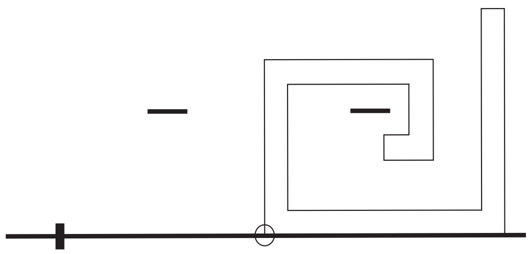


v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
v_1
v_2
v_4
v_6
v_7
v_8
v_9

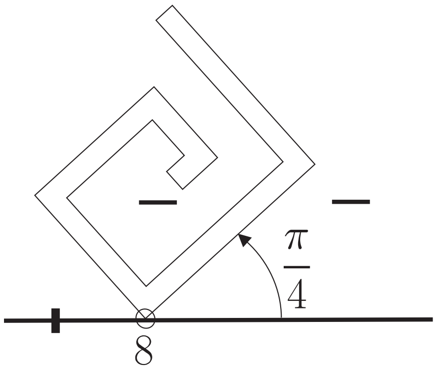
(a) (b)



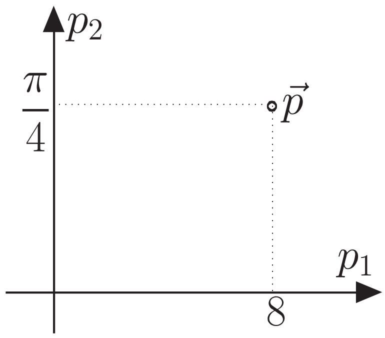
Initial configuration: $\vec{p} = (0\ 0)^{\mathrm{T}}$



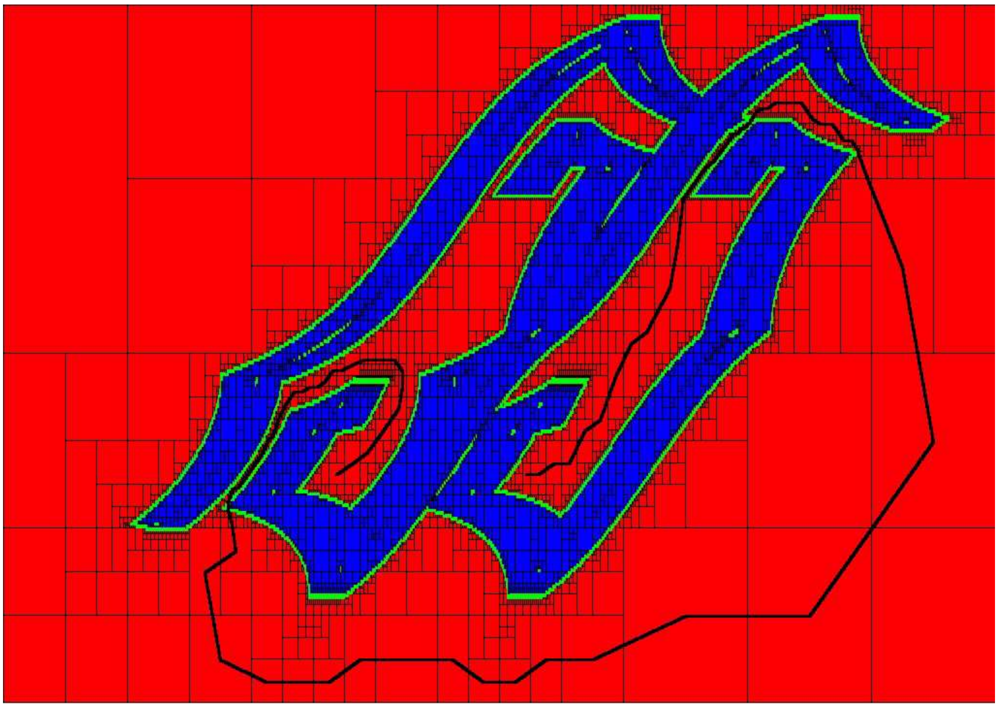
Goal configuration: $\vec{p} = (17\ 0)^{\mathrm{T}}$

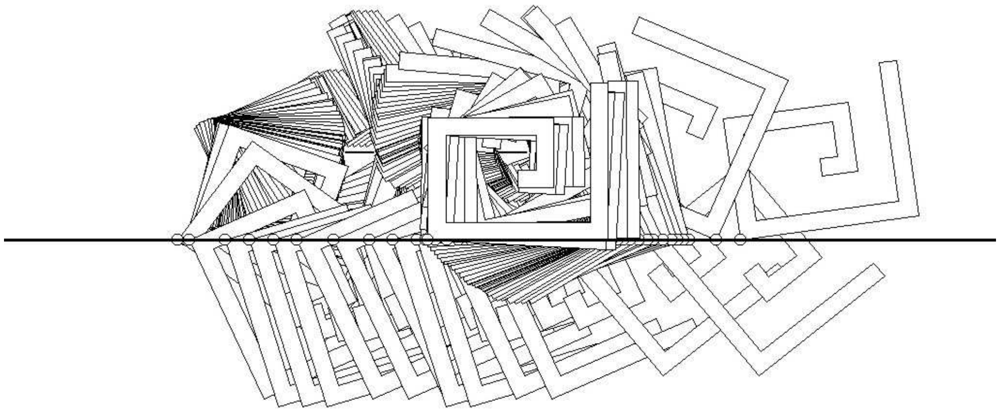


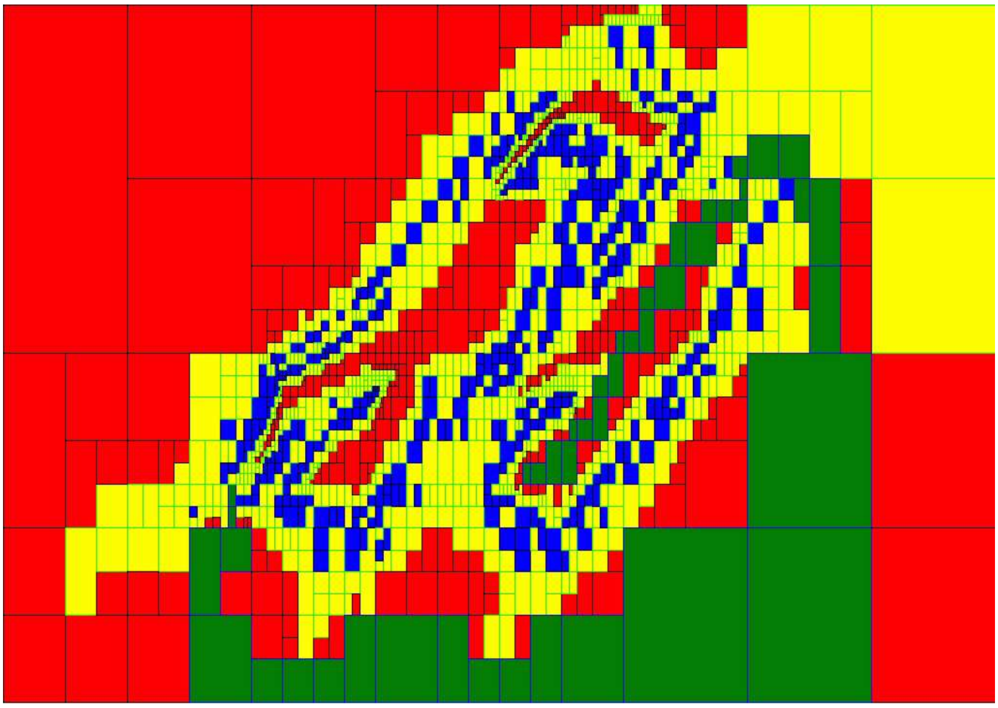
Room

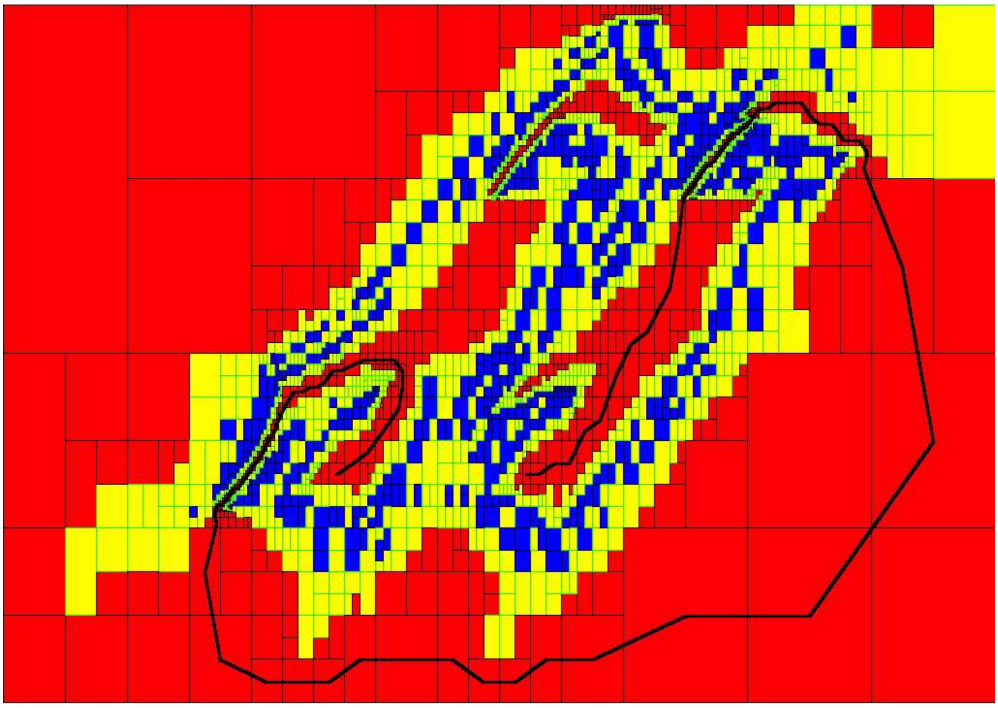


Configuration space









2 Counting connected components

(Collaboration with N. Delanoue and B. Cottenceau)

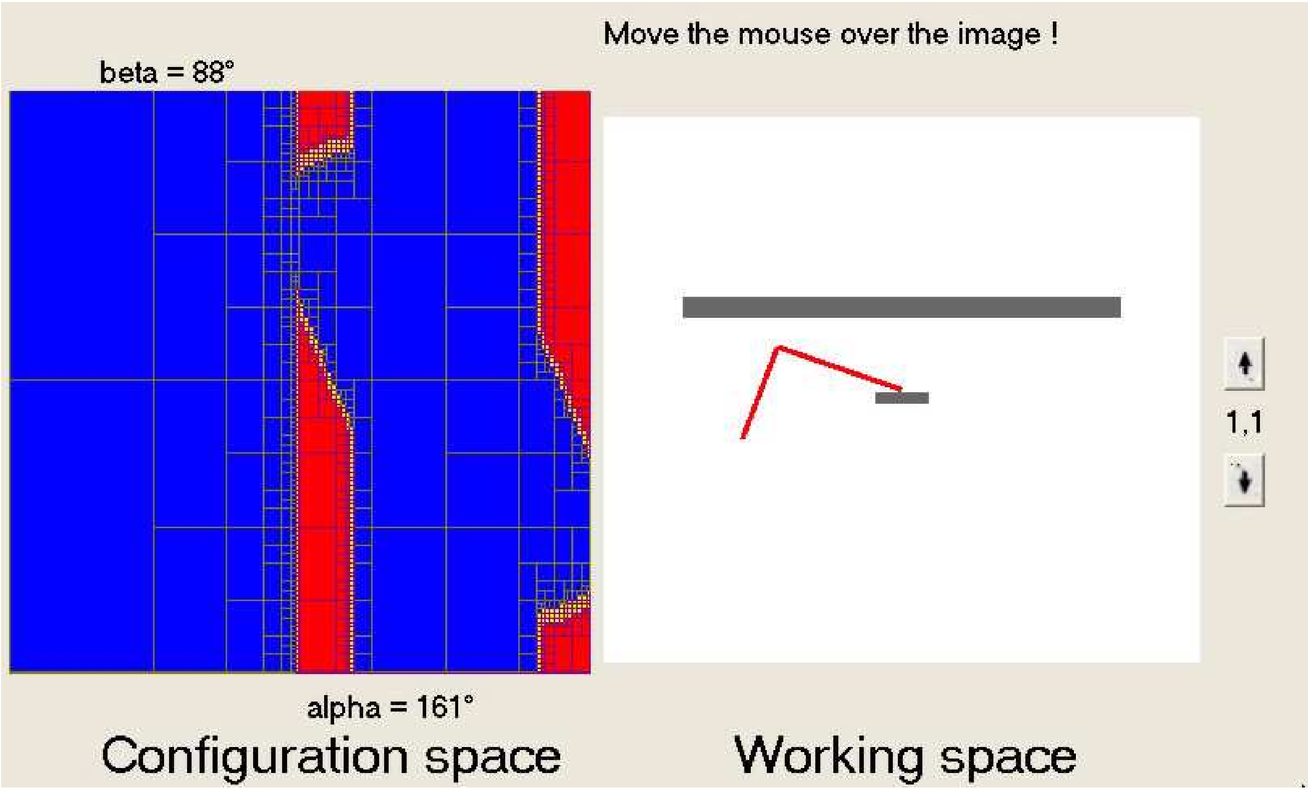
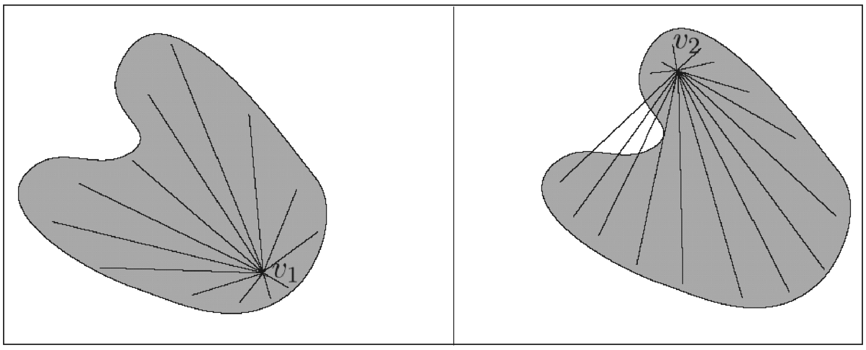


Figure 1:

The point $\mathbf{v}$ is a *star* for $\mathbb{S} \subset \mathbb{R}^n$ if $\forall \mathbf{x} \in \mathbb{S}, \forall \alpha \in [0, 1]$, $\alpha\mathbf{v} + (1 - \alpha)\mathbf{x} \in \mathbb{S}$. .



$\mathbf{v}_1$ is a star for $\mathbb{S}$ whereas $\mathbf{v}_2$ is not

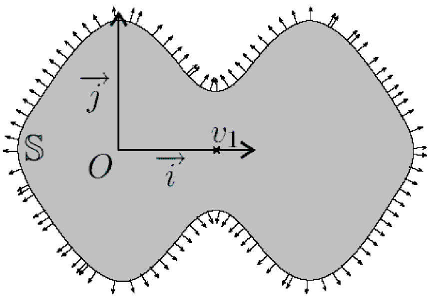
The set $\mathbb{S} \subset \mathbb{R}^n$ is *star-shaped* is there exists $\mathbf{v}$ such that $\mathbf{v}$ is a star for $\mathbb{S}$.

Theorem: Define the set

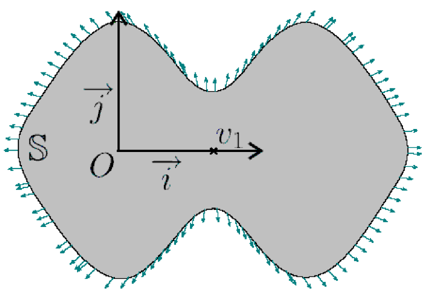
$$\mathbb{S} \stackrel{\text{def}}{=} \{\mathbf{x} \in [\mathbf{x}] | f(\mathbf{x}) \leq 0\} \tag{1}$$

where f is differentiable. We have the following implication

$$\left\{\mathbf{x} \in [\mathbf{x}] \mid f(\mathbf{x}) = 0, \frac{df}{d\mathbf{x}}(\mathbf{x}).(\mathbf{x} - \mathbf{v}) \leq 0\right\} = \emptyset \Rightarrow \mathbf{v} \text{ is a star} \tag{2}$$



$Df(x)$



$x - v_1$

If $\mathbf{v}$ is a star for $\mathbb{S}_1$ and a star for $\mathbb{S}_2$ then it is a star for $\mathbb{S}_1 \cap \mathbb{S}_2$ and for $\mathbb{S}_1 \cup \mathbb{S}_2$.

Consider a subpaving $\mathcal{P} = \{[\mathbf{p}_1], [\mathbf{p}_2], \ldots\}$ covering $\mathbb{S}$. The relation $\mathcal{R}$ defined by

$$[\mathbf{p}]\mathcal{R}[\mathbf{q}] \Leftrightarrow \mathbb{S} \cap [\mathbf{p}] \cap [\mathbf{q}] \neq \emptyset$$

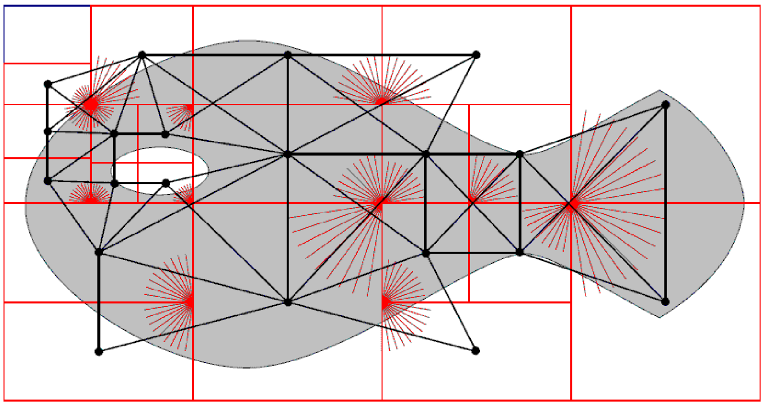
is *star-spangled graph* of the set $\mathbb{S}$ if

$$\forall [\mathbf{p}] \in \mathcal{P}, \mathbb{S} \cap [\mathbf{p}] \text{ is star-shaped.}$$

For instance, a star-spangled graph for the set

$$\mathbb{S} \stackrel{\text{def}}{=} \left\{ (x,y) \in \mathbb{R}^2 \mid \begin{pmatrix} x^2 + 4y^2 - 16 \\ 2\sin x - \cos y + y^2 - \frac{3}{2} \\ -(x+\frac{5}{2})^2 - 4(y-\frac{2}{5})^2 + \frac{3}{10} \end{pmatrix} \leq 0 \right\}$$

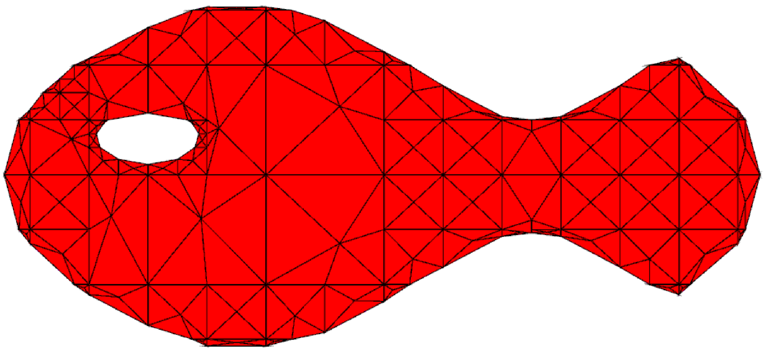
is



For each $[\mathbf{p}]$ of the paving $\mathcal{P}$, a common star located at the corner of $[\mathbf{p}]$ (represented in red) has been found for all three constraints.

Theorem: The number of connected components of the star-spangled graph of $\mathbb{S}$ is equal to that of $\mathbb{S}$.

An extension of this approach has also been developed with N. Delanoue to compute a triangulation homeomorphic to $\mathbb{S}$.



3 Capture basin

(With M. Lhommeau and L. Hardouin)

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \\ \mathbf{x}(0) & = & \mathbf{x}_0 \end{cases}$$

$\mathbf{u}(t) \in [\mathbf{u}] \in \mathbb{R}^m$ is the control, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector.

The solution of this ODE is denoted by $\varphi(t\,; \mathbf{x}_0, \mathbf{u}(.))$.

Define two compact sets $\mathbf{T}$ and $\mathbf{K}$ such that $\mathbf{T} \subset \mathbf{K} \subset \mathbb{R}^n$. $\mathbf{T}$ is the *target* and $\mathbf{K}$ is the *viable set*. Define the *capture basin* $\mathbf{C}$ as

$$\mathbf{C} = \{\mathbf{x}_0 \in \mathbf{K} \mid \exists t > 0, \exists \mathbf{u}\left(.\right), \mathbf{u}\left([0,t]\right) \subset [\mathbf{u}], \; . \\ \varphi(t, \mathbf{x}_0, \mathbf{u}\left(.\right)) \in \mathbf{T} \text{ and } \varphi\left([0,t], \mathbf{x}_0, \mathbf{u}\left(.\right)\right) \subset \mathbf{K}\}$$

Notation. If $[t] \in \mathbb{IR}$, $[\mathbf{x}_0] \in \mathbb{IR}^n$, $[\mathbf{u}] \in \mathbb{IR}^m$

$$\mathbf{\Phi}\left([t], [\mathbf{x}_0], [\mathbf{u}]\right) \stackrel{\text{def}}{=} \{\varphi\left(t, \mathbf{x}_0, \mathbf{u}\left(.\right)\right), \ t \in [t], \mathbf{x}_0 \in [\mathbf{x}_0], \mathbf{u}\left([0,\right.$$

Note that when $[t], [\mathbf{x}_0], [\mathbf{u}]$ are punctual, $\mathbf{\Phi}\left(t, \mathbf{x}_0, \mathbf{u}\right)$ is a point of $\mathbb{R}^n$ which corresponds to the integration of the ODE with a constant control $\mathbf{u}$.

We have

(i) $\mathbf{x}_0 \in \mathbf{T} \Rightarrow \mathbf{x}_0 \in \mathbf{C}$

(ii) $\mathbf{x}_0 \notin \mathbf{K} \Rightarrow \mathbf{x}_0 \notin \mathbf{C}$

(iii) $(\mathbf{u} \in [\mathbf{u}], \Phi(t, \mathbf{x}_0, \mathbf{u}) \in \mathbf{C}, \Phi([0,t], \mathbf{x}_0, \mathbf{u}) \subset \mathbf{K})$
$\Rightarrow \mathbf{x}_0 \in \mathbf{C}$

(iv) $(\Phi(t; \mathbf{x}_0, [\mathbf{u}]) \cap \mathbf{C} = \emptyset,\ \Phi([0,t], \mathbf{x}_0, [\mathbf{u}]) \cap \mathbf{T} = \emptyset)$
$\Rightarrow \mathbf{x}_0 \notin \mathbf{C}$

Thus

$$
\begin{array}{ll}
\text{(i)} & [\mathbf{x}_0] \subset \mathbf{T} \Rightarrow [\mathbf{x}_0] \subset \mathbf{C} \\
\text{(ii)} & [\mathbf{x}_0] \cap \mathbf{K} = \emptyset \Rightarrow [\mathbf{x}_0] \cap \mathbf{C} = \emptyset \\
\text{(iii)} & (\mathbf{u} \in [\mathbf{u}], \boldsymbol{\Phi}(t, [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{C}, \boldsymbol{\Phi}([0,t], [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{K}) \\
& \quad \Rightarrow [\mathbf{x}_0] \subset \mathbf{C} \\
\text{(iv)} & (\boldsymbol{\Phi}(t, [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{C} = \emptyset,\ \boldsymbol{\Phi}([0,t], [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{T} = \emptyset) \\
& \quad \Rightarrow [\mathbf{x}_0] \cap \mathbf{C} = \emptyset
\end{array}
$$

Algorithm (in: $\mathbf{K}, \mathbf{T}$; out: $\mathbf{C}^-, \mathbf{C}^+$)

1 $\mathbf{C}^- := \emptyset$; $\mathbf{C}^+$ is a union of boxes covering $\mathbf{K}$;

2 repeat

3 take a box $[\mathbf{x}_0]$ in $\mathbf{C}^+$ ($\mathbf{C}^+$ has not changed)

4 if $[\mathbf{x}_0] \subset \mathbf{T}$ then $\mathbf{C}^- := \mathbf{C}^- \cup [\mathbf{x}_0]$; goto 2;

5 if $[\mathbf{x}_0] \cap \mathbf{K} = \emptyset$, $\mathbf{C}^+ := \mathbf{C}^+ \backslash [\mathbf{x}_0]$; goto 2;

6 take $t \in \mathbb{R}^+$; and $\mathbf{u} \in [\mathbf{u}]$;

7 if $\mathbf{\Phi}(t, [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{C}^-$ and $\mathbf{\Phi}([0, t], [\mathbf{x}_0], \mathbf{u}) \subset \mathbf{K}$
then $\mathbf{C}^- := \mathbf{C}^- \cup [\mathbf{x}_0]$; goto 2;

8 if $\mathbf{\Phi}(t, [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{C}^+ = \emptyset$ and $\mathbf{\Phi}([0, t], [\mathbf{x}_0], [\mathbf{u}]) \cap \mathbf{T}$
then $\mathbf{C}^+ := \mathbf{C}^+ \backslash [\mathbf{x}_0]$; goto 2;

9 until no more change can be observed

After completion of the algorithm, we have

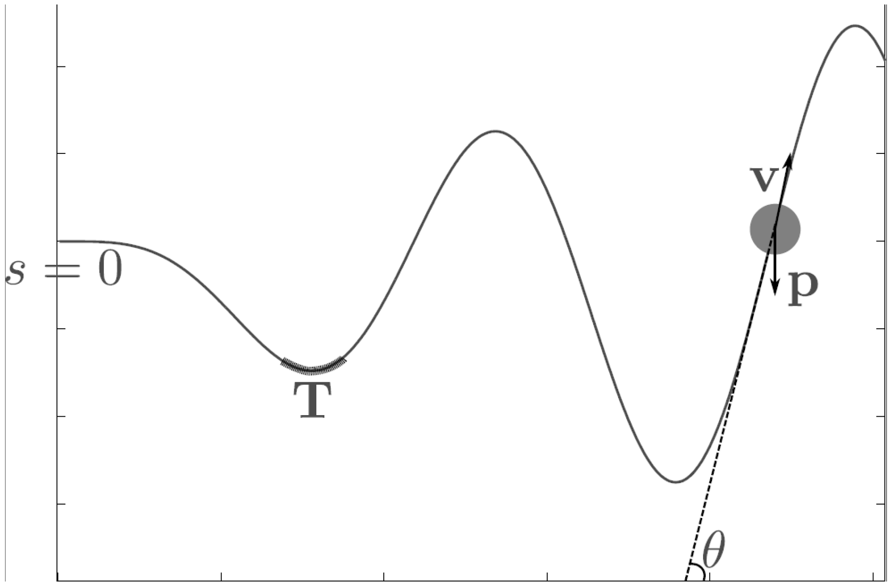
$$\mathbf{C}^{-} \subset \mathbf{C} \subset \mathbf{C}^{+}.$$

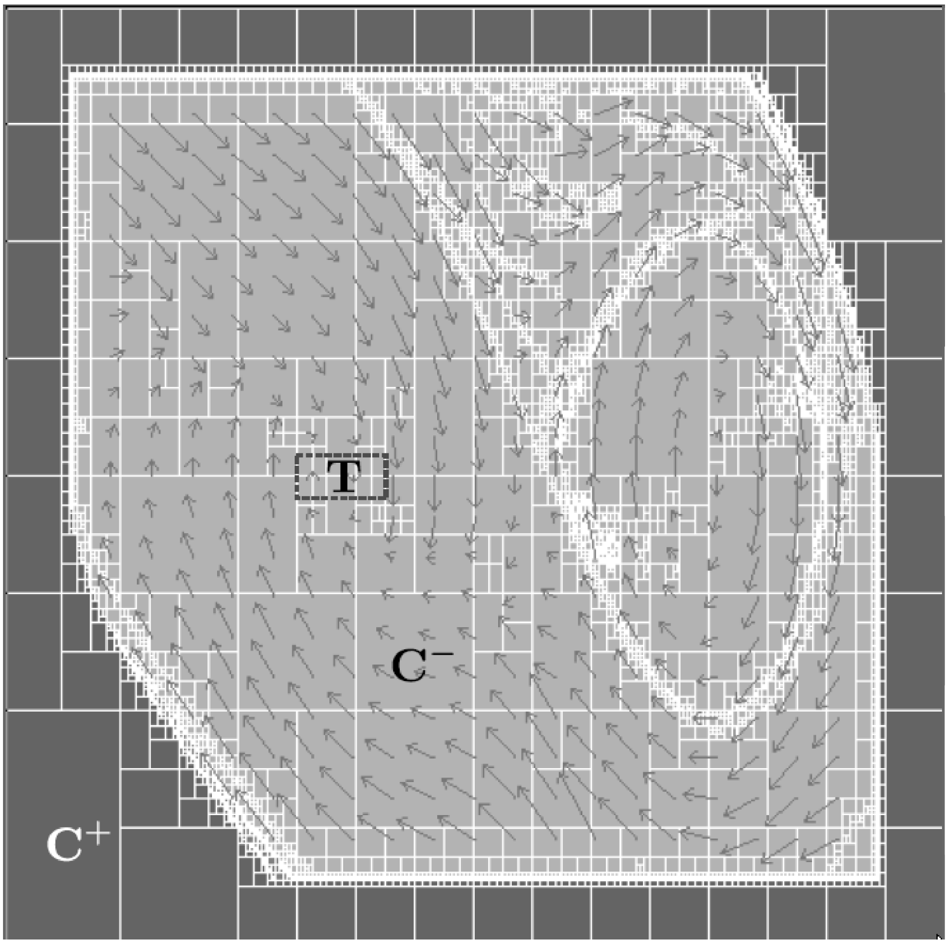
Consider a rolling ball described by

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin\left(\theta(x_1)\right) - x_2 + u \end{cases} \tag{3}$$

where x_1 is the curve position of the ball and x_2 is its speed. Moreover $[u] := [-2, 2]$, $\mathbf{K} = [0, 12] \times [-6, 6]$, $\mathbf{T} = [3.5, 4.5] \times [-1, 1]$ and

$$\theta(x) = \sin(1.1.x) - \frac{1}{2}\sin(x)$$





T
C−
C+