Interval robotics

Chapter 2: Subpavings

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1 Subpavings

1.1 Definitions

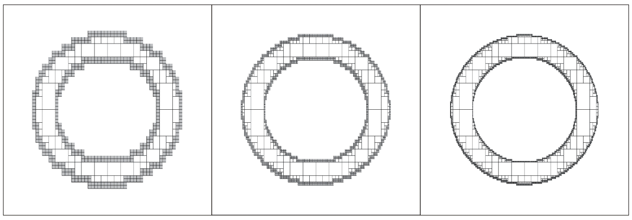
A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.

Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \,\big|\, x_1^2 + x_2^2 \in [1, 2]\}.$$



Set operations such as $\mathbb{Z} := \mathbb{X} + \mathbb{Y}$, $\mathbb{X} := \mathbf{f}^{-1}(\mathbb{Y})$, $\mathbb{Z} := \mathbb{X} \cap \mathbb{Y} \ldots$ can be approximated by subpaving operations.

2 Set inversion

If $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$.

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

Stack-queue

A *queue* is a list on which two operations are allowed:

- add an element at the end (*push*)
- remove the first element (*pull*).

A *stack* is a list on which two operations are allowed:

- add an element at the beginning of the list (*stack*)

- remove the first element (*pop*).

Example: Let $\mathcal{L}$ be an empty queue.

k	operation	result
0		$\mathcal{L} = \emptyset$
1	push $(\mathcal{L}, a)$	$\mathcal{L} = \{a\}$
2	push $(\mathcal{L}, b)$	$\mathcal{L} = \{a, b\}$
3	$x :=$ pull $(\mathcal{L})$	$x = a, \mathcal{L} = \{b\}$
4	$x :=$ pull $(\mathcal{L})$	$x = b, \mathcal{L} = \emptyset$.

If $\mathcal{L}$ is a stack, the table becomes

k	operation	result
0		$\mathcal{L} = \emptyset$
1	stack $(\mathcal{L}, a)$	$\mathcal{L} = \{a\}$
2	stack $(\mathcal{L}, b)$	$\mathcal{L} = \{a, b\}$
3	$x :=$ pop $(\mathcal{L})$	$x = b, \mathcal{L} = \{a\}$
4	$x :=$ pop $(\mathcal{L})$	$x = a, \mathcal{L} = \emptyset.$

.

Algorithm Sivia(in: $[\mathbf{x}](0), \mathbf{f}, \mathbb{Y}$)	
1	$\mathcal{L} := \{[\mathbf{x}](0)\}$;
2	pull $[\mathbf{x}]$ from $\mathcal{L}$;
3	if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$, 'red');
4	elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$, 'blue');
5	elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$, 'yellow')};
6	else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$;
7	if $\mathcal{L} \neq \emptyset$, go to 2

If $\Delta\mathbb{X}$ denotes the union of yellow boxes and if $\mathbb{X}^-$ is the union of red boxes then :

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^- \cup \Delta\mathbb{X}.$$

3 Image evaluation

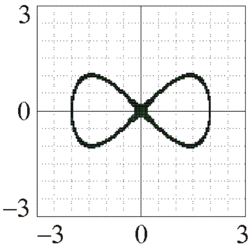
Define

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} (x_1 - 1)^2 - 1 + x_2 \\ -x_1^2 + (x_2 - 1)^2 \end{pmatrix},$$

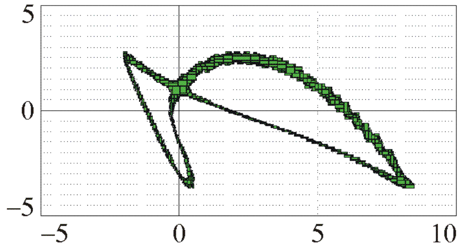
and

$$\mathbb{X}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^4 - x_1^2 + 4x_2^2 \in [-0.1, 0.1] \right\}.$$

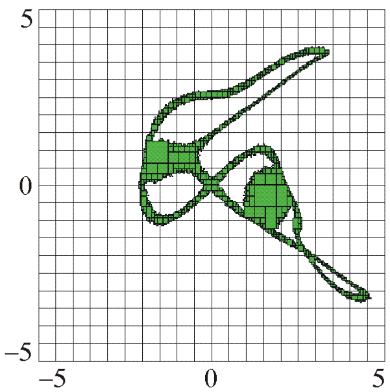
We shall compute $\mathbb{X}_1$, $\mathbf{f}(\mathbb{X}_1)$ and $\mathbf{f}^{-1} \circ \mathbf{f}(\mathbb{X}_1)$.



(a): $\mathbb{X}_1$



(b): $\mathbf{f}(\mathbb{X}_1)$



(c): $\mathbf{f}^{-1}(\mathbf{f}(\mathbb{X}_1))$

4 Paver

A paver is an algorithm which generates boxes by bisections and classifies them.

Take

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid t\,(\mathbf{x}) = 1\} = t^{-1}(1)$$

We want an enclosure of the form

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+$$

Algorithm Sivia(in: $[\mathbf{x}](0), [t]$)

1 $\mathcal{L} := \{[\mathbf{x}](0)\}$;
2 pull $([\mathbf{x}],\mathcal{L})$;
3 if $[t]([\mathbf{x}]) = 1$, draw($[\mathbf{x}]$, 'red');
4 elseif $[t]([\mathbf{x}]) = 0$, draw($[\mathbf{x}]$, 'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$, 'yellow')};
6 else bisect $[\mathbf{x}]$ into $[\mathbf{x}](1)$ and $[\mathbf{x}](2)$; push $([\mathbf{x}](1),[\mathbf{x}](2)$
7 if $\mathcal{L} \neq \emptyset$, go to 2

5 Projection

Consider the set

$$\mathbb{Z} = \{\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in [\mathbf{x}] \times [\mathbf{y}] \mid t(\mathbf{x}, \mathbf{y})\},$$

where $t(\mathbf{x}, \mathbf{y})$ is a test. The projection of $\mathbb{Z}$ onto $\mathbf{x}$ is

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], t(\mathbf{x}, \mathbf{y})\}.$$

The test $t_{\mathbf{X}}(\mathbf{x})$ defined by

$$t_{\mathbf{X}}(\mathbf{x}) \Leftrightarrow \exists \mathbf{y} \in [\mathbf{y}], t(\mathbf{x}, \mathbf{y})$$

is called the projection of t onto $\mathbf{x}$.

Algorithm $[t_{\mathbf{x}}]$(in: $[\mathbf{x}],[\mathbf{y}],[t]$)

1 $\mathcal{L} := \{[\mathbf{y}]\}$;
2 while $\mathcal{L} \neq \emptyset$,
3   pull $([\mathbf{y}],\mathcal{L})$;
4   if $[t]([\mathbf{x}]$,center$([\mathbf{y}])) = 1$, return (1);
5   if $[t]([\mathbf{x}],[\mathbf{y}]) = 0$, goto 2;
6   if $w([\mathbf{y}]) < w([\mathbf{x}])$, return ([0, 1]);
      bisect $[\mathbf{y}]$ into $[\mathbf{y}](1)$ and $[\mathbf{y}](2)$; push $([\mathbf{y}](1)$,$[\mathbf{y}](2)$,
7 end while;
8 return 0.

6 Dealing with quantifiers

Example. Characterize

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x},\mathbf{y}) \leq \mathbf{0}\}.$$

i.e.,

$$\begin{array}{rcl} \mathbb{Z} & = & \{(\mathbf{x},\mathbf{y}) \in [\mathbf{x}] \times [\mathbf{y}] \mid \mathbf{f}(\mathbf{x},\mathbf{y}) \leq \mathbf{0}\} \\ \mathbb{X} & = & \mathsf{proj}_{\mathbf{x}}(\mathbb{Z}). \end{array}$$

We decompose as follows

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid t_2(\mathbf{x})\} \quad \text{where} \quad \begin{array}{l} t_2(\mathbf{x}) \Leftrightarrow (\exists \mathbf{y} \in [\mathbf{y}],\ t_1(\mathbf{x}, \mathbf{y}) \\ t_1(\mathbf{x}, \mathbf{y}) \Leftrightarrow (\mathbf{f}(\mathbf{x}, \mathbf{y}) \leq \mathbf{0})\,. \end{array}$$

To use Sivia, build $[t_{\mathbb{Z}}]([\mathbf{x}], [\mathbf{y}])$ and then build $[t_{\mathbb{X}}]([\mathbf{x}])$.

Example. Consider

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], \forall \mathbf{z} \in [\mathbf{z}], \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \mathbf{0}\}.$$

We have

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], \neg (\exists \mathbf{z} \in [\mathbf{z}], \neg (\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \mathbf{0}))\}$$

We decompose as follows

$$\begin{array}{l} t_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Leftrightarrow \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \mathbf{0}. \\ t_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Leftrightarrow \neg t_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\ t_3(\mathbf{x}, \mathbf{y}) \Leftrightarrow (\exists \mathbf{z} \in [\mathbf{z}], t_2(\mathbf{x}, \mathbf{y}, \mathbf{z})) \\ t_4(\mathbf{x}, \mathbf{y}) = \neg t_3(\mathbf{x}, \mathbf{y}) \\ t_5(\mathbf{x}) \Leftrightarrow (\exists \mathbf{y} \in [\mathbf{y}], t_4(\mathbf{x}, \mathbf{y})) \end{array}$$

Thus

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid t_5(\mathbf{x})\}$$

Example. Consider

$$a = \min_{\mathbf{x} \in [\mathbf{x}]} f(\mathbf{x}).$$

For a given y, we have

$$\begin{aligned} y &\geq a \Leftrightarrow \exists \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \leq y \\ y &\leq a \Leftrightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq y. \end{aligned}$$

Thus the global minimum belongs to the singleton

$$\begin{array}{rcl} \{a\} & = & \{y \mid y \geq a\} \cap \{y \mid y \leq a\} \\ & = & \{y \mid \exists \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) - y \leq 0\} \cap \{y \mid \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x} \end{array}$$

To use Sivia, we decompose as follows

$$\{a\} = \{y \mid t_1(y) \wedge t_2(y)\} \quad \text{where} \quad \left\{ \begin{array}{l} t_1(y) \Leftrightarrow (\exists \mathbf{x} \in [\mathbf{x}],\ t \\ t_2(y) \Leftrightarrow \neg (\exists \mathbf{x} \in [\mathbf{x}], \end{array} \right.$$

$$\begin{array}{l} t_3(\mathbf{x}, y) \Leftrightarrow (f(\mathbf{x}) - y \leq 0) \\ t_4(\mathbf{x}, y) \Leftrightarrow (f(\mathbf{x}) - y < 0) \, . \end{array}$$

Example. Consider the optimization problem where $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$. The problem is

$$\mathbb{P} \ = \min_{\mathbf{x}\in[\mathbf{x}]} \mathbf{f}\left(\mathbf{x}\right)$$

The set

$$\mathbb{P} = \{\mathbf{y} \mid \exists \mathbf{x} \in [\mathbf{x}]\,, \mathbf{f}(\mathbf{x}) \leq \mathbf{y}\} \cap \{\mathbf{y}, \forall \mathbf{x} \in [\mathbf{x}]\,, \neg\left(\mathbf{f}(\mathbf{x}) < \mathbf{y}\right)\}$$

is called the *Pareto set*. Here, $\mathbf{a} < \mathbf{b}$ means that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$.

The decomposition is

$$\begin{array}{lll} \mathbb{P} = \{\mathbf{y} \mid t_1(\mathbf{y}) \wedge t_2(\mathbf{y})\} & \text{where} & \left\{ \begin{array}{l} t_1(\mathbf{y}) \Leftrightarrow (\exists \mathbf{x} \in [\mathbf{x}] \\ t_2(\mathbf{y}) \Leftrightarrow (\forall \mathbf{x} \in [\mathbf{x}] \end{array} \right. \\ t_1(\mathbf{y}) \Leftrightarrow (\exists \mathbf{x} \in [\mathbf{x}]\,, t_3(\mathbf{x},\mathbf{y})) & \text{where} & t_3(\mathbf{x},\mathbf{y}) \Leftrightarrow (\mathbf{f}(\mathbf{x}) \leq \mathbf{y} \\ t_2(\mathbf{y}) = \neg t_4(\mathbf{y}) & \text{where} & t_4(\mathbf{y}) \Leftrightarrow (\exists \mathbf{x} \in [\mathbf{x}]\,, \mathbf{f}(\\ t_4(\mathbf{y}) \Leftrightarrow (\exists \mathbf{x} \in [\mathbf{x}]\,, t_5(\mathbf{x},\mathbf{y})) & \text{where} & t_5(\mathbf{x},\mathbf{y}) \Leftrightarrow (\mathbf{f}(\mathbf{x}) < \mathbf{y} \end{array}$$

Example. Consider

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x}, \mathbf{y}) = 0\}.$$

The set $\{(\mathbf{x}, \mathbf{y}) \in [\mathbf{x}] \times [\mathbf{y}], f(\mathbf{x}, \mathbf{y}) = 0\}$ has an empty volume and the inclusion test associated with $f(\mathbf{x}, \mathbf{y}) = 0$ will never return 1.

If f is continuous

$$(\exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x}, \mathbf{y}) = 0) \Leftrightarrow (\exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x}, \mathbf{y}) \geq 0) \wedge (\exists \mathbf{y} \in [\mathbf{y}$$

Thus

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x}, \mathbf{y}) \geq 0\} \cap \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}],$$

The decomposition is thus

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid t_1(\mathbf{x}) \wedge t_2(\mathbf{x})\} \quad \text{where} \quad \left\{ \begin{array}{l} t_1(\mathbf{x}) \Leftrightarrow (\exists \mathbf{y} \in [\mathbf{y} \\ t_2(\mathbf{x}) \Leftrightarrow (\exists \mathbf{y} \in [\mathbf{y} \end{array} \right.$$

$$t_1(\mathbf{x}) \Leftrightarrow \exists \mathbf{y} \in [\mathbf{y}], t_3(\mathbf{x}, \mathbf{y}) \quad \text{where} \quad t_3(\mathbf{x}, \mathbf{y}) \Leftrightarrow (f(\mathbf{x}, \mathbf{y})$$

$$t_2(\mathbf{x}) \Leftrightarrow \exists \mathbf{y} \in [\mathbf{y}], t_4(\mathbf{x}, \mathbf{y}) \quad \text{where} \quad t_4(\mathbf{x}, \mathbf{y}) \Leftrightarrow (f(\mathbf{x}, \mathbf{y})$$

7 Bounded-error estimation

Model : $\phi(\mathbf{p}, t) = p_1 e^{-p_2 t}$.

Prior feasible box for the parameters : $[\mathbf{p}] \subset \mathbb{R}^2$

Measurement times : $t_1, t_2, \ldots, t_m$

Data bars : $[y_1^-, y_1^+], [y_2^-, y_2^+], \ldots, [y_m^-, y_m^+]$

$\mathbb{S} = \{\mathbf{p} \in [\mathbf{p}], \phi(\mathbf{p}, t_1) \in [y_1^-, y_1^+], \ldots, \phi(\mathbf{p}, t_m) \in [y_m^-, y_m^+]\}$

If

$$\boldsymbol{\phi}(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p}, t_1) \\ \\ \phi(\mathbf{p}, t_m) \end{pmatrix}$$

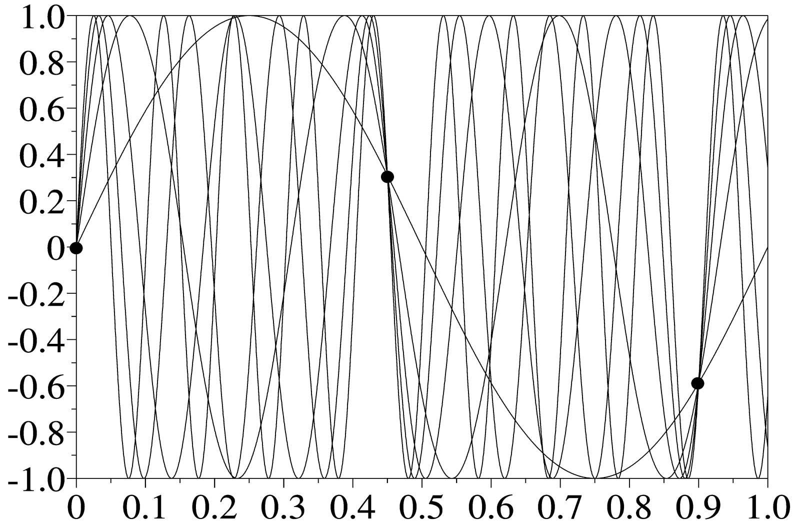
and

$$[\mathbf{y}] = [y_1^-, y_1^+] \times \cdots \times [y_m^-, y_m^+]$$

then

$$\mathbb{S} = [\mathbf{p}] \cap \boldsymbol{\phi}^{-1}([\mathbf{y}]).$$

If now $\phi(\mathbf{p}, t) = p_1 \sin(2\pi p_2 t)$ and $t_k = k\delta, \ldots$. $\mathbb{S}$ contains an infinite number of connected components.



8 Robustification against outliers

Define a *relaxing function* for the box $[\mathbf{y}] = [y_1] \times \cdots \times [y_n]$

$$\lambda(\mathbf{y}) = \pi_{[y_1]}(y_1) + \cdots + \pi_{[y_n]}(y_n)$$

where

$$\pi_{[a,b]}(x) \begin{cases} = 1 & \text{if} \quad x \in [a,b] \\ = 0 & \text{if} \quad x \notin [a,b]. \end{cases}$$

Allow up to q of the n output variables y_i to escape their prior feasible intervals. The posterior feasible set becomes

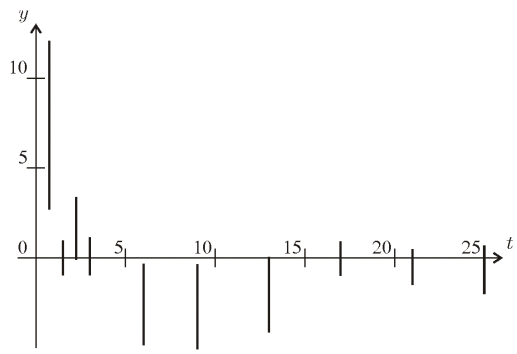
$$\hat{\mathbb{P}}_q = \{\mathbf{p} \in [\mathbf{p}] \mid \pi_{[y_1]}(\phi_1(\mathbf{p})) + \cdots + \pi_{[y_n]}(\phi_n(\mathbf{p})) \geqslant n - q\}.$$

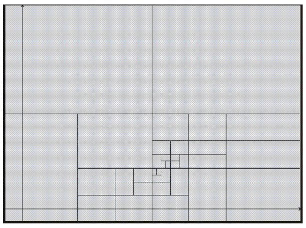
This is a set inversion problem. The set $\hat{\mathbb{P}}_q$ can thus be characterized by Sivia.

As an illustration, consider the model

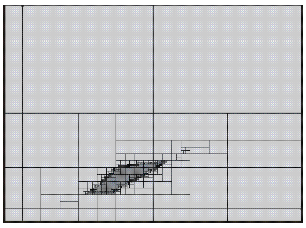
$$\phi(\mathbf{p}, t) = 20\,\mathsf{exp}(-p_1 t) - 8\,\mathsf{exp}(-p_2 t)$$

with the data bars represented on the figure below

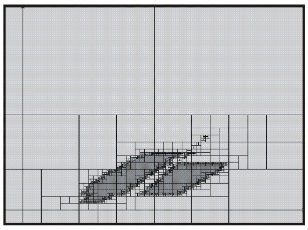




(a)



(b)



(c)

(a) no outlier assumed; (b) one outlier assumed; (c) two outliers assumed;

9 Robust stability

The *stability domain* $\mathbb{S}_{\mathrm{p}}$ of the polynomial

$$P(s,\mathbf{p}) = s^n + a_{n-1}(\mathbf{p})s^{n-1} + \ldots + a_1(\mathbf{p})s + a_0(\mathbf{p})$$

is the set of all $\mathbf{p}$ such that $P(s,\mathbf{p})$ is stable.

If $P(s, \mathbf{p})$ is given by

$$s^3+(p_1+p_2+2)s^2+(p_1+p_2+2)s+2p_1p_2+6p_1+6p_2+2.25,$$

Its Routh table is given by

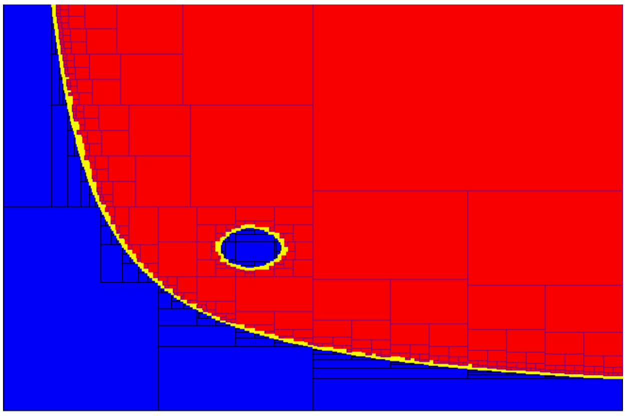
1	p_1+p_2+2
p_1+p_2+2	$2p_1p_2+6p_1+6p_2+2.25$
$\frac{(p_1-1)^2+(p_2-1)^2-0.25}{p_1+p_2+2}$	0
$2(p_1+3)(p_2+3)-15.75$	0

Its stability domain is thus defined by

$$\mathbb{S}_\mathsf{p} \triangleq \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{r}(\mathbf{p}) > \mathbf{0}\} = \mathbf{r}^{-1}\left(]0, +\infty[^{\times n}\right).$$

where

$$\mathbf{r}(\mathbf{p}) = \left(\begin{array}{c} p_1 + p_2 + 2 \\ (p_1 - 1)^2 + (p_2 - 1)^2 - 0.25 \\ 2(p_1 + 3)(p_2 + 3) - 15.75 \end{array} \right).$$



Stability domain $\mathbb{S}_p$ generated by Proj2d

10 Application to global optimization

Consider the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \text{ s.t. } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}.$$

Its epigraph is defined by

$$\mathbb{S} = \{(\mathbf{x}, a) \in \mathbb{R}^n \times \mathbb{R} \mid a \geq f(\mathbf{x}) \text{ and } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}\}$$

Define the ith *profile* of the epigraph

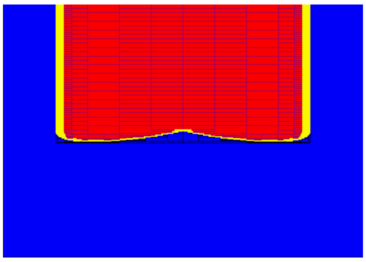
$$\mathbb{S}_i = \{(x_i, a) \in \mathbb{R} \times \mathbb{R} \mid \exists\, (x_1, \ldots, x_{i-1}, x_i, \ldots, x_n) \mid a \geq f$$

Example.

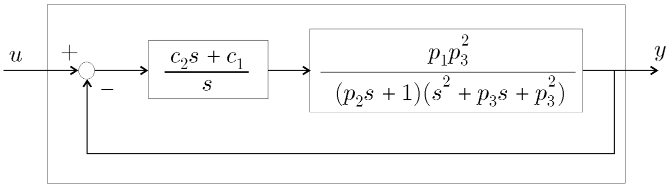
Consider, for instance, the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sin x_1 x_2 \text{ s.t. } x_1^2 + x_2^2 \in [1,2].$$

The sets $\mathbb{S}_1$ (and also $\mathbb{S}_2$) are obtained by Proj2d.



11 Application to robust control



with $\mathbf{p} \in [\mathbf{p}] = [0.9, 1.1]^{\times 3}$ and $\mathbf{c} \in [\mathbf{c}] = [0, 1]^2$.

$$\mathbf{\Sigma}(\mathbf{p},\mathbf{c}) \text{ is stable } \Leftrightarrow r\left(\mathbf{c},\mathbf{p}\right) > 0.$$

Define

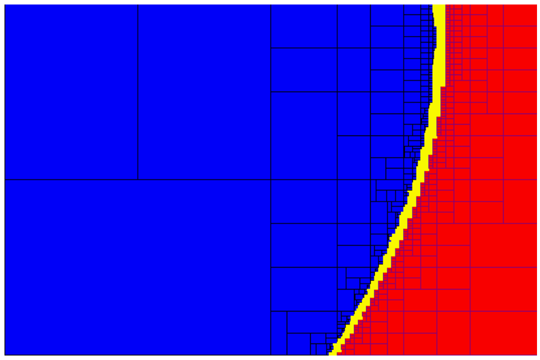
$$\mathbb{T}_{\mathsf{c}} = \{\mathbf{c} \in [\mathbf{c}] \mid \forall \mathbf{p} \in [\mathbf{p}], r(\mathbf{c},\mathbf{p}) > 0\}$$

The transfer function of $\mathbf{\Sigma}(\mathbf{p}, \mathbf{c})$ is

$$H(s) = \frac{\left(c_2 s + c_1\right) p_1 p_3^2}{p_2 s^4 + \left(p_2 p_3 + 1\right) s^3 + \left(p_2 p_3^2 + p_3\right) s^2 + \left(p_3^2 + c_2 p_1}$$

The first column of the corresponding Routh table is

$$\begin{pmatrix} p_2 \\ p_2 p_3 + 1 \\ p_2 p_3^2 + p_3 - \frac{p_2\left(p_3^2 + c_2 p_1 p_3^2\right)}{p_2 p_3 + 1} \\ p_3^2 + c_2 p_1 p_3^2 - \frac{\left(p_2 p_3 + 1\right)^2 \left(c_1 p_1 p_3^2\right)}{\left(p_2 p_3^2 + p_3\right)\left(p_2 p_3 + 1\right) - p_2\left(p_3^2 + c_2 p_1 p_3^2\right)} \\ c_1 p_1 p_3^2 \end{pmatrix}$$

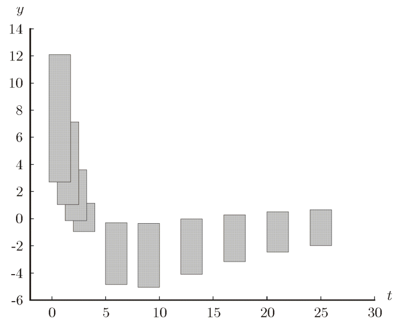


12 Application to bounded-error estimation with uncertain independent variables

Model:

$$\phi(\mathbf{p}, t) = 20\,\mathsf{exp}(-p_1 t) - 8\,\mathsf{exp}(-p_2 t)$$

Data



i	$\check{t}_i$	$[\check{t}_i]$	$[\check{y}_i]$
1	0.75	$[-0.25, 1.75]$	$[2.7, 12.1]$
2	1.5	$[0.5, 2.5]$	$[1.04, 7.14]$
3	2.25	$[1.25, 3.25]$	$[-0.13, 3.61]$
4	3	$[2, 4]$	$[-0.95, 1.15]$
5	6	$[5, 7]$	$[-4.85, -0.29]$
6	9	$[8, 10]$	$[-5.06, -0.36]$
7	13	$[12, 14]$	$[-4.1, -0.04]$
8	17	$[16, 18]$	$[-3.16, 0.3]$
9	21	$[20, 22]$	$[-2.5, 0.51]$
10	25	$[24, 26]$	$[-2, 0.67]$

The posterior feasible set is

$$\mathbb{S}_{\mathsf{p}} = \{\mathbf{p} \in [\mathbf{p}] \mid \exists t_1 \in [t_1], \ldots, \exists t_{10} \in [t_{10}], \phi(\mathbf{p}, t_1) \in [y_1], \ldots$$