Interval robotics

Chapter 1: Interval computation

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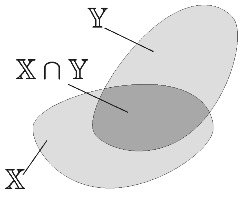
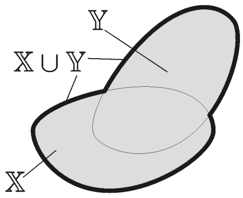
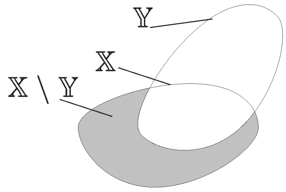
1 Notions on set theory

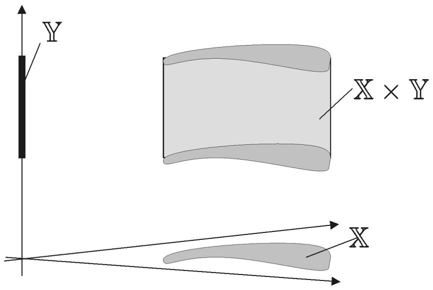
We define

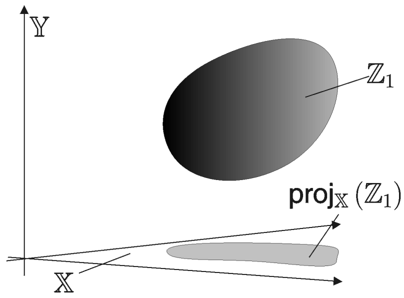
$$
\begin{aligned}
\mathbb{X} \cap \mathbb{Y} &\stackrel{\text{def}}{=} \{x \mid x \in \mathbb{X} \text{ and } x \in \mathbb{Y}\} \\
\mathbb{X} \cup \mathbb{Y} &\stackrel{\text{def}}{=} \{x \mid x \in \mathbb{X} \text{ or } x \in \mathbb{Y}\} \\
\mathbb{X} \setminus \mathbb{Y} &\stackrel{\text{def}}{=} \{x \mid x \in \mathbb{X} \text{ and } x \notin \mathbb{Y}\} \\
\mathbb{X} \times \mathbb{Y} &\stackrel{\text{def}}{=} \{(x, y) \mid x \in \mathbb{X} \text{ and } y \in \mathbb{Y}\} \\
\text{proj}_{\mathbb{X}}(\mathbb{Z}) &\stackrel{\text{def}}{=} \{x \in \mathbb{X} \mid \exists y \in \mathbb{Y},\ (x, y) \in \mathbb{Z}\}.
\end{aligned}
$$


$\mathbb{Y}$
$\mathbb{X} \cap \mathbb{Y}$
$\mathbb{X}$


$\mathbb{Y}$
$\mathbb{X} \cup \mathbb{Y}$
$\mathbb{X}$


$\mathbb{Y}$
$\mathbb{X}$
$\mathbb{X} \setminus \mathbb{Y}$


$\mathbb{Y}$
$\mathbb{X} \times \mathbb{Y}$
$\mathbb{X}$


$\mathbb{Y}$
$\mathbb{Z}_1$
$\mathsf{proj}_{\mathbb{X}}(\mathbb{Z}_1)$
$\mathbb{X}$

Exercise: If $\mathbb{X} = \{a, b, c, d\}$ and $\mathbb{Y} = \{b, c, x, y\}$, then

$$\begin{aligned}
\mathbb{X} \cap \mathbb{Y} &= \text{?} \\
\mathbb{X} \cup \mathbb{Y} &= \text{?} \\
\mathbb{X} \setminus \mathbb{Y} &= \text{?} \\
\mathbb{X} \times \mathbb{Y} &= \text{?}
\end{aligned}$$

Exercise: If $\mathbb{X} = \{a, b, c, d\}$ and $\mathbb{Y} = \{b, c, x, y\}$, then

$$\begin{aligned}
\mathbb{X} \cap \mathbb{Y} &= \{b, c\} \\
\mathbb{X} \cup \mathbb{Y} &= \{a, b, c, d, x, y\} \\
\mathbb{X} \setminus \mathbb{Y} &= \{a, d\} \\
\mathbb{X} \times \mathbb{Y} &= \{(a, b), (a, c), (a, x), (a, y), \\
&\quad \ldots, (d, b), (d, c), (d, x), (d, y)\}
\end{aligned}$$

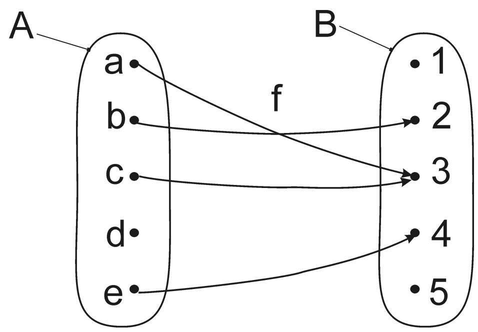
The *direct image* of $\mathbb{X}$ by f is

$$f\left(\mathbb{X}\right) \triangleq \{f(x) \mid x \in \mathbb{X}\}.$$

The *reciprocal image* of $\mathbb{Y}$ by f is

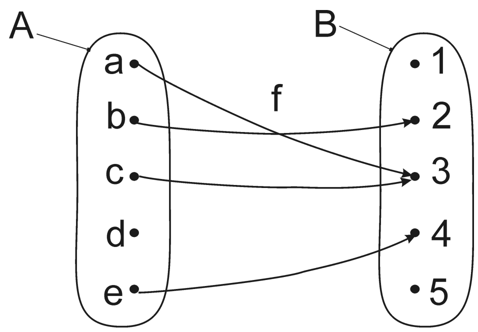
$$f^{-1}\left(\mathbb{Y}\right) \triangleq \{x \in \mathbb{X} \mid f(x) \in \mathbb{Y}\}.$$

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= ?. \\ f^{-1}(B) &= ?. \\ f^{-1}(f(A)) &= ? \\ f^{-1}(f(\{b,c\})) &= ?. \end{aligned}$$

Exercise: If f is defined as follows



$$
\begin{aligned}
f(A) &= \{2,3,4\} = \mathsf{Im}(f). \\
f^{-1}(B) &= \{a,b,c,e\} = \mathsf{dom}(f). \\
f^{-1}(f(A)) &= \{a,b,c,e\} \subset A \\
f^{-1}(f(\{b,c\})) &= \{a,b,c\}.
\end{aligned}
$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= ? \\ f^{-1}([4,9]) &= ?. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= [4,9] \\ f^{-1}([4,9]) &= [-3,-2] \cup [2,3]. \end{aligned}$$

This is consistent with the property

$$f^{-1}(f(\mathbb{Y})) \supset \mathbb{Y}.$$

2 Interval arithmetic

If $\diamond \in \{+, -, \cdot, /, \max, \min\}$

$$[x] \diamond [y] = [\{x \diamond y \mid x \in [x], y \in [y]\}] \,.$$

For instance,

$$\begin{aligned} [-1, 3] + [2, 5] &= [?, ?], \\ [-1, 3] \cdot [2, 5] &= [?, ?], \\ [-2, 6]/[2, 5] &= [?, ?]. \end{aligned}$$

If $\diamond \in \{+, -, \cdot, /, \mathrm{max}, \mathrm{min}\}$

$$[x] \diamond [y] = [\{x \diamond y \mid x \in [x], y \in [y]\}]\,.$$

For instance,

$$\begin{aligned} [-1,3] + [2,5] &= [1,8], \\ [-1,3].[2,5] &= [-5,15], \\ [-2,6]/[2,5] &= [-1,3]. \end{aligned}$$

$$\begin{aligned}
[x^-,x^+]+[y^-,y^+] = \quad & [x^- + y^-, x^+ + y^+],\\
[x^-,x^+].[y^-,y^+] = \quad & [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+,\\
& x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+],
\end{aligned}$$

If $f \in \{$cos, sin,sqr, sqrt, log, exp, $\ldots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

For instance,

$$\begin{aligned}
\mathsf{sin}([0, \pi]) &= ?, \\
\mathsf{sqr}([-1, 3]) &= [-1, 3]^2 = ?, \\
\mathsf{abs}([-7, 1]) &= ?, \\
\mathsf{sqrt}([-10, 4]) &= \sqrt{[-10, 4]} = ?, \\
\mathsf{log}([-2, -1]) &= ?.
\end{aligned}$$

If $f \in \{$cos, sin,sqr, sqrt, log, exp, $\ldots\}$

$$f\left([x]\right) = \left[\{f(x) \mid x \in [x]\}\right].$$

For instance,

$$\begin{array}{rcl}
\sin\left([0,\pi]\right) & = & [0,1], \\
\mathrm{sqr}\left([-1,3]\right) & = & [-1,3]^2 = [0,9], \\
\mathrm{abs}\left([-7,1]\right) & = & [0,7], \\
\mathrm{sqrt}\left([-10,4]\right) & = & \sqrt{[-10,4]} = [0,2], \\
\log\left([-2,-1]\right) & = & \emptyset.
\end{array}$$

3 Boxes

A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

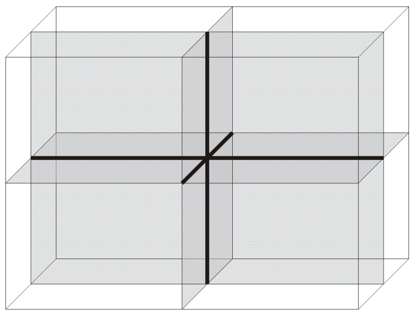
$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

The *width* $w([\mathbf{x}])$ of a box $[\mathbf{x}]$ is the length of its largest side. For instance

$$w([1,2] \times [-1,3]) = 4$$

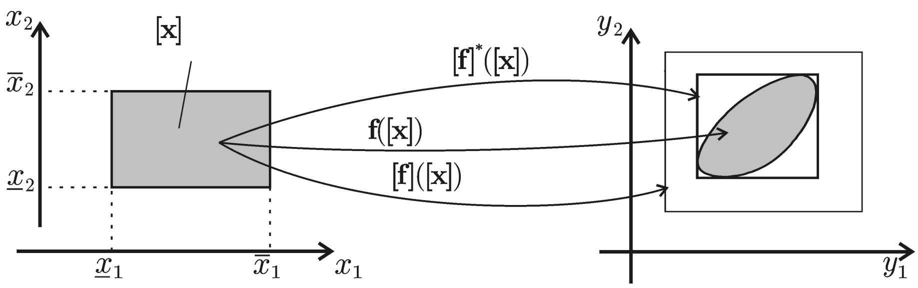
The *principal plane* of $[\mathbf{x}]$ is the symmetric plane $[\mathbf{x}]$ perpendicular to its largest side.



4 Inclusion function

The interval function $[\mathbf{f}]$ from $\mathbb{IR}^n$ to $\mathbb{IR}^m$, is an *inclusion function* of $\mathbf{f}$ if

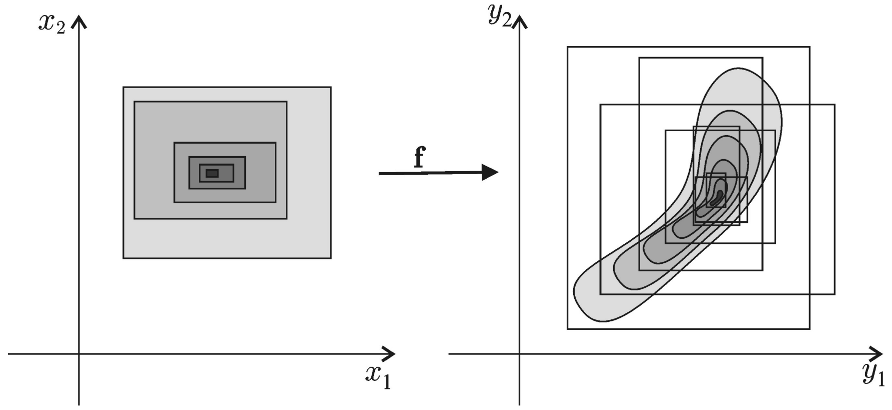
$$\forall\,[\mathbf{x}] \in \mathbb{IR}^n, \quad \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



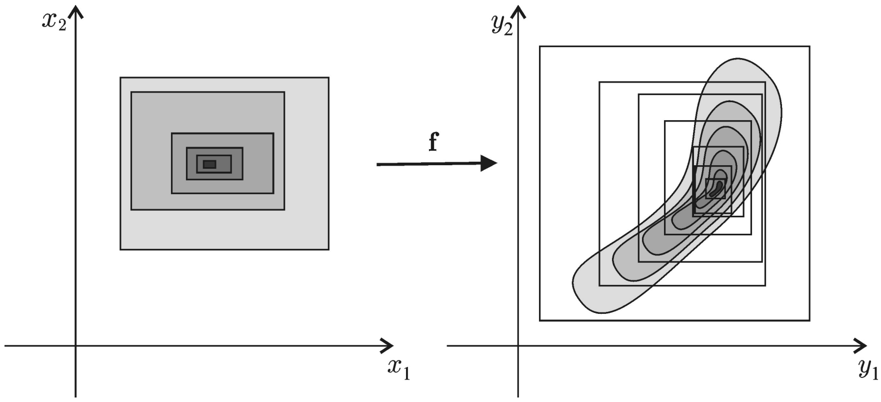
Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

The inclusion function $[\mathbf{f}]$ is

monotonic	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([\mathbf{f}]([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{y}]))$
minimal	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [\mathbf{f}]([\mathbf{x}]) = [\mathbf{f}([\mathbf{x}])]$
thin	if	$w([\mathbf{x}]) = 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) = 0$
convergent	if	$w([\mathbf{x}]) \to 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) \to 0.$



Convergent but non-monotonic



Convergent and monotonic

The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = ?.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

If $[x] = [-3, 4]$, we have

$$\begin{aligned} [f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\ &= [0,16] + [-6,8] + 4 \\ &= [-2,28]. \end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{ccc} \mathbb{IR}^2 & \rightarrow & \mathbb{IR}^3 \\ ([x_1], [x_2]) & \rightarrow & \left([x_1] * [x_2], [x_1]^2, [x_1] - [x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by the algorithm

Algorithm $\mathbf{f}$(in: $\mathbf{x} = (x_1, x_2, x_3)$, out: $\mathbf{y} = (y_1, y_2)$)
$z := x_1$;
for $k := 0$ to 100
$z := x_2(z + kx_3)$;
next;
$y_1 := z$;
$y_2 :=$ sin(zx_1);

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in: $[\mathbf{x}]$, out: $[\mathbf{y}]$)
$[z] := [x_1]$;
for $k := 0$ to 100
$\quad [z] := [x_2] * ([z] + k * [x_3])$;
next;
$[y_1] := [z]$;
$[y_2] := \sin([z] * [x_1])$;

Is $[\mathbf{f}]$ convergent? thin? monotonic?

5 Centred inclusion functions

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable over $[\mathbf{x}]$, and if $\mathbf{m} = \text{mid}([\mathbf{x}])$. The mean-value theorem implies

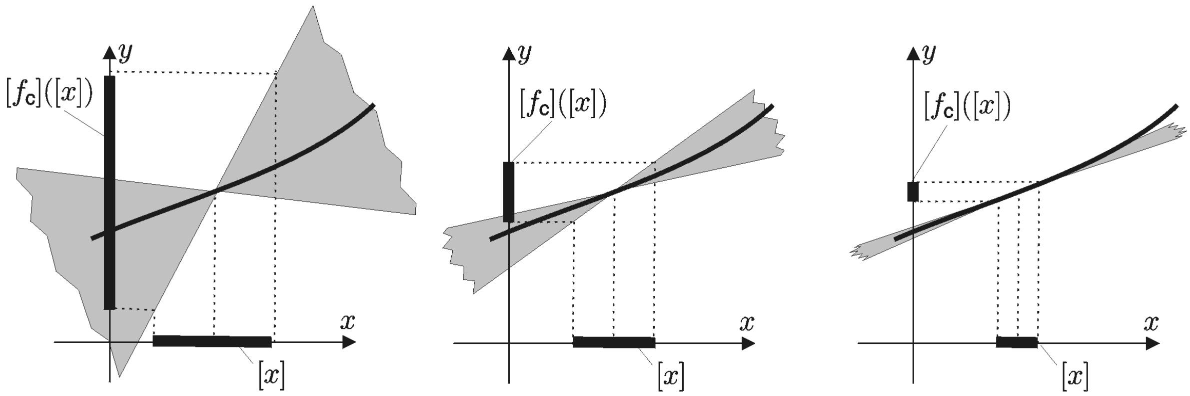
$$\forall \mathbf{x} \in [\mathbf{x}], \exists \mathbf{z} \in [\mathbf{x}] \mid f(\mathbf{x}) = f(\mathbf{m}) + \frac{df}{d\mathbf{x}}(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{m}).$$

Thus,

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \in f(\mathbf{m}) + \frac{df}{d\mathbf{x}}([\mathbf{x}]) \cdot (\mathbf{x} - \mathbf{m}),$$

Therefore, an inclusion function for $\mathbf{f}$ is

$$[f_{\mathsf{c}}]([\mathbf{x}]) \overset{\text{def}}{=} f(\mathbf{m}) + \left[\frac{df}{d\mathbf{x}}\right]([\mathbf{x}]) \cdot ([\mathbf{x}] - \mathbf{m}).$$



y
$[f_c]([x])$
x
$[x]$
y
$[f_c]([x])$
x
$[x]$
y
$[f_c]([x])$
x
$[x]$

6 Boolean intervals

Boolean intervals

A *Boolean number* is an element of

$$\mathbb{B} \triangleq \{\textit{false}, \textit{true}\} = \{0, 1\}\,.$$

If we define the relation $\leq$ as

$$0 \leq 0, 0 \leq 1, 1 \leq 1,$$

then, the set $(\mathbb{B}, \leq)$ is a lattice for which intervals can be defined.

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{?, ?, ?, ?\},$$

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{\emptyset, 0, 1, [0, 1]\},$$

Boolean interval arithmetic

$$\begin{aligned}
[a] \vee [b] &= \{a \vee b \mid a \in [a], b \in [b]\}, \\
[a] \wedge [b] &= \{a \wedge b \mid a \in [a], b \in [b]\}, \\
\neg [a] &= \{\neg a \mid a \in [a]\}.
\end{aligned}$$

Exercise: Compute

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = ?$$

Exercise: Compute

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = 1 \wedge [0,1] = [0,1].$$

7 Inclusion tests

A *test* is a function t from $\mathbb{R}^n$ to $\mathbb{B}$. An *inclusion test* $[t]$ is an inclusion function for t. Thus

$$\begin{array}{lll} ([t]([\mathbf{x}]) = 1) & \Rightarrow & (\forall \mathbf{x} \in [\mathbf{x}], \ t(\mathbf{x}) = 1), \\ ([t]([\mathbf{x}]) = 0) & \Rightarrow & (\forall \mathbf{x} \in [\mathbf{x}], \ t(\mathbf{x}) = 0). \end{array}$$

An *inclusion test* $[t_{\mathbb{A}}]$ for a set $\mathbb{A}$ of $\mathbb{R}^n$ is an inclusion test for the test $(\mathbf{x} \in \mathbb{A})$. We have

$$\begin{array}{lllll} [t_{\mathbb{A}}]([\mathbf{x}]) = 1 & \Rightarrow & (\forall \mathbf{x} \in [\mathbf{x}],\ t_{\mathbb{A}}(\mathbf{x}) = 1) & \Leftrightarrow & ([\mathbf{x}] \subset \mathbb{A}), \\ [t_{\mathbb{A}}]([\mathbf{x}]) = 0 & \Rightarrow & (\forall \mathbf{x} \in [\mathbf{x}],\ t_{\mathbb{A}}(\mathbf{x}) = 0) & \Leftrightarrow & ([\mathbf{x}] \cap \mathbb{A} = \end{array}$$

$[t]$ is *monotonic*	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([t]([\mathbf{x}]) \subset [t]([\mathbf{y}]))$
$[t]$ is *minimal*	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [t]([\mathbf{x}]) = t([\mathbf{x}])$
$[t]$ is *thin*	if	$\forall \mathbf{x} \in \mathbb{R}^n,\ [t](\mathbf{x}) \neq [0,1]$.

If $\mathbb{A}$ and $\mathbb{B}$ are two sets, we have

$$\begin{aligned} t_{\mathbb{A}\cap\mathbb{B}} &= t_{\mathbb{A}} \wedge t_{\mathbb{B}} \\ t_{\mathbb{A}\cup\mathbb{B}} &= t_{\mathbb{A}} \vee t_{\mathbb{B}} \\ t_{\neg\mathbb{A}} &= \neg t_{\mathbb{A}} = 1 - t_{\mathbb{A}}. \end{aligned}$$

Thus

$$\begin{aligned}
\left[t_{\mathbb{A}\cap\mathbb{B}}\right]([\mathbf{x}]) &\triangleq ([t_{\mathbb{A}}] \wedge [t_{\mathbb{B}}])([\mathbf{x}]) = [t_{\mathbb{A}}]([\mathbf{x}]) \wedge [t_{\mathbb{B}}]([\mathbf{x}]),\\
\left[t_{\mathbb{A}\cup\mathbb{B}}\right]([\mathbf{x}]) &\triangleq ([t_{\mathbb{A}}] \vee [t_{\mathbb{B}}])([\mathbf{x}]) = [t_{\mathbb{A}}]([\mathbf{x}]) \vee [t_{\mathbb{B}}]([\mathbf{x}]),\\
\left[t_{\neg\mathbb{A}}\right]([\mathbf{x}]) &\triangleq \neg [t_{\mathbb{A}}]([\mathbf{x}]) = \mathbf{1} - [t_{\mathbb{A}}]([\mathbf{x}]).
\end{aligned}$$

Exercise: Consider the test

$$t : \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \{0,1\} \\ (x_1, x_2)^\mathsf{T} & \mapsto & (x_1 + x_2^2 \leqslant 5). \end{array}$$

The minimal inclusion test $[t]$ associated with t is

$$[t]([\mathbf{x}]) = \begin{cases} 1 & \text{if } ? \\ 0 & \text{if } ? \\ [0,1] & \text{if } ? \end{cases}$$

Exercise: Consider the test

$$t : \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \{0,1\} \\ (x_1, x_2)^\mathsf{T} & \mapsto & (x_1 + x_2^2 \leqslant 5). \end{array}$$

The minimal inclusion test $[t]$ associated with t is

$$[t]\left([\mathbf{x}]\right) = \begin{cases} 1 & \text{if } [x_1] + [x_2]^2 \in \,]-\infty, 5], \\ 0 & \text{if } [x_1] + [x_2]^2 \in \,]5, \infty[\\ [0,1] & \text{otherwise,} \end{cases}$$