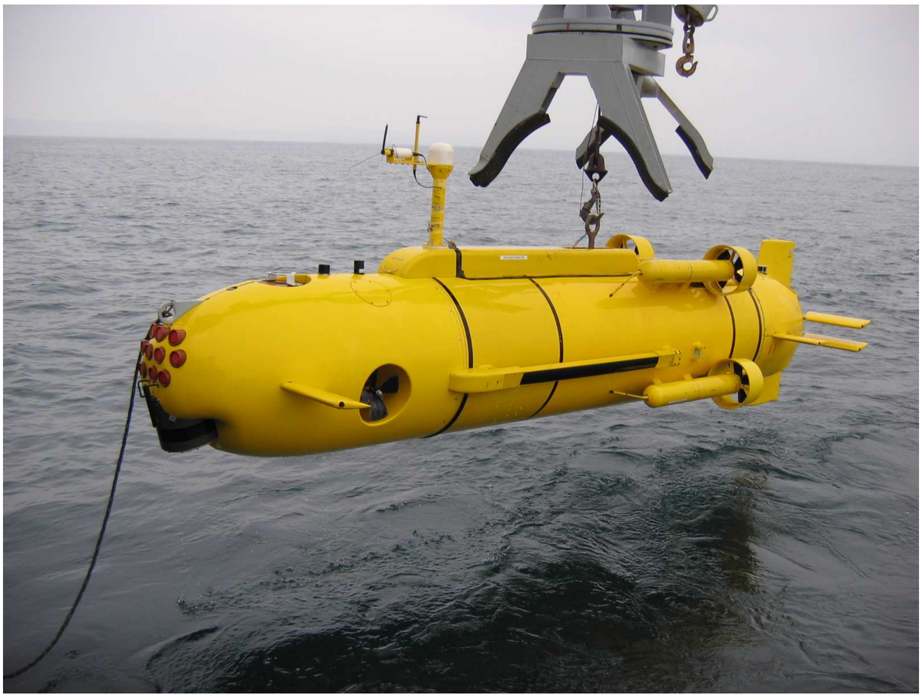
Interval robotics

Chapter 6: SLAM

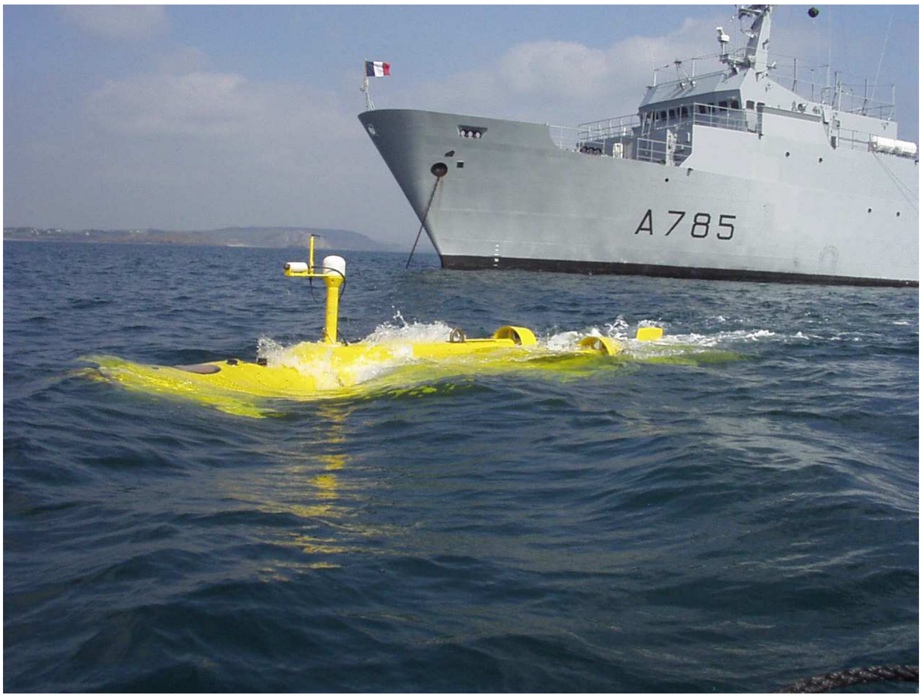
Luc Jaulin,
ENSTA-Bretagne, Brest, France

1 Basic SLAM

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}) & \text{(evolution equation)} \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{u}) & \text{(observation equation)} \\ \mathbf{z}_i &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{m}_i) & \text{(mark equation)} \end{cases}$$



Redermor, GESMA
(Groupe d'Etude Sous-Marine de l'Atlantique)

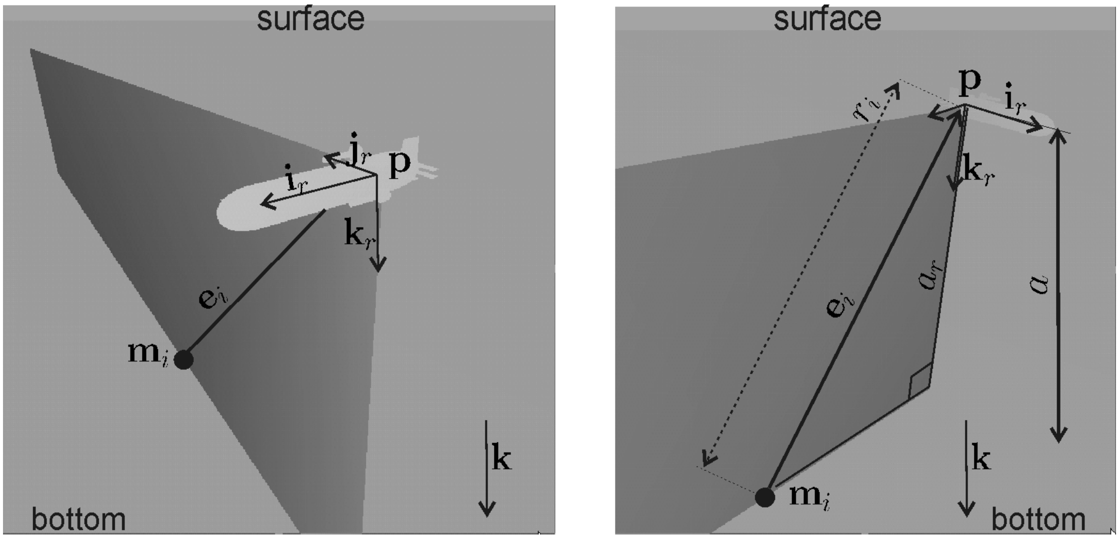

A785

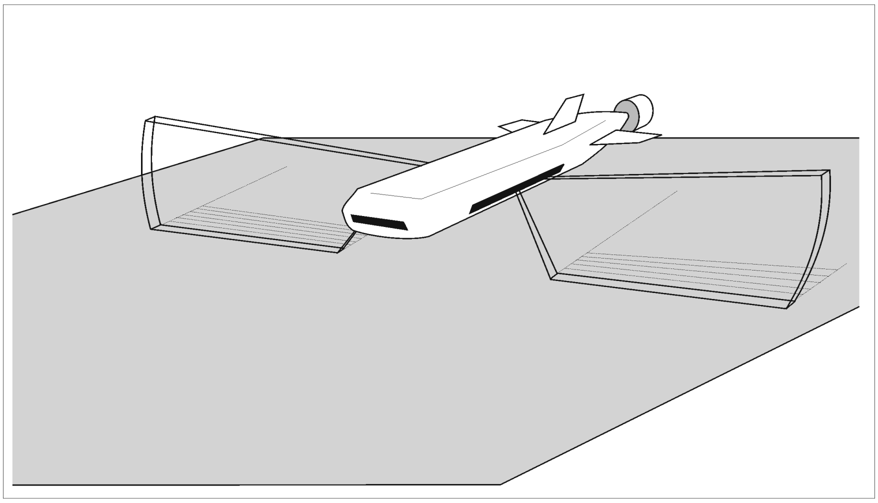
1.1 Sensors

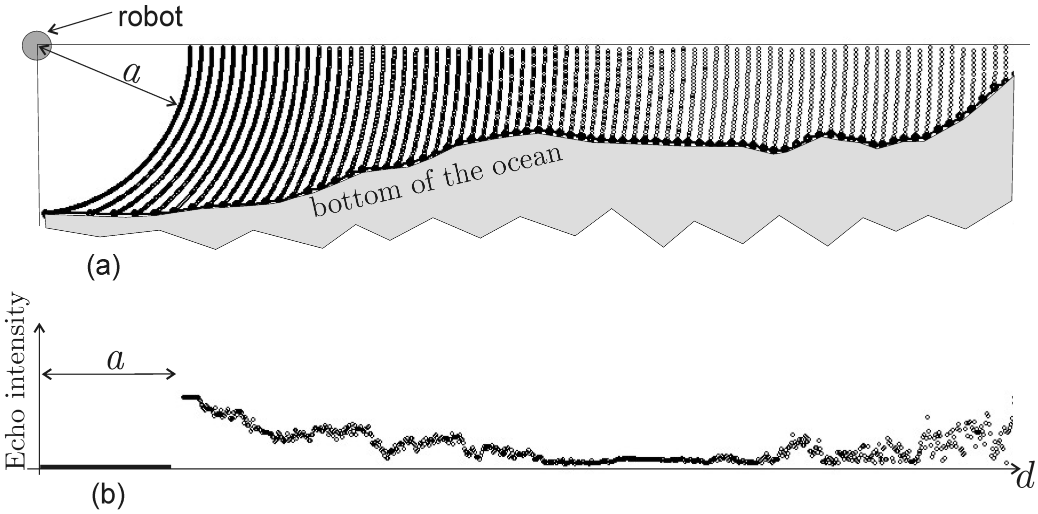
GPS (Global positioning system), only at the surface.

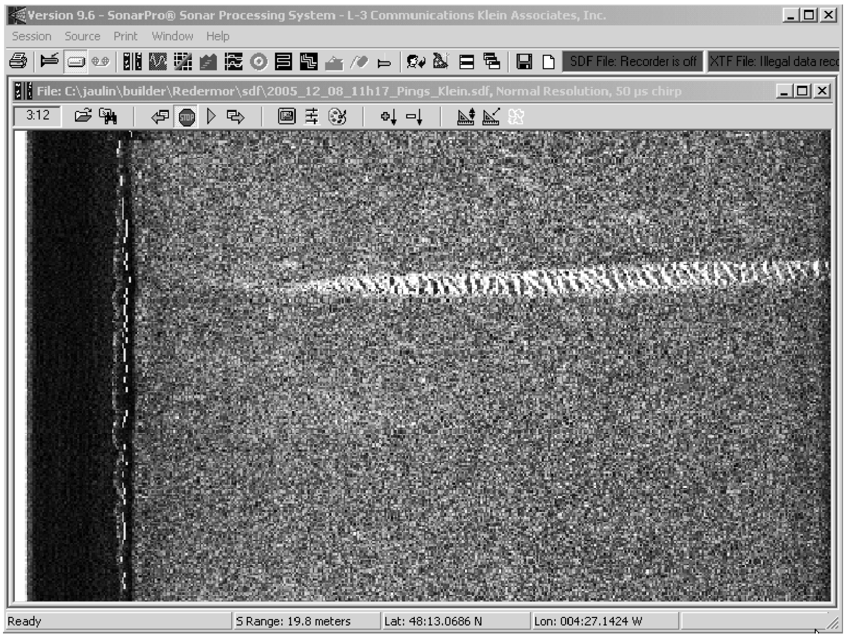
$$t_0 = 6000 \text{ s}, \quad \ell^0 = (-4.4582279^\circ, 48.2129206^\circ) \pm 2.5m$$
$$t_f = 12000 \text{ s}, \quad \ell^f = (-4.4546607^\circ, 48.2191297^\circ) \pm 2.5m$$

Sonar (KLEIN 5400 side scan sonar).

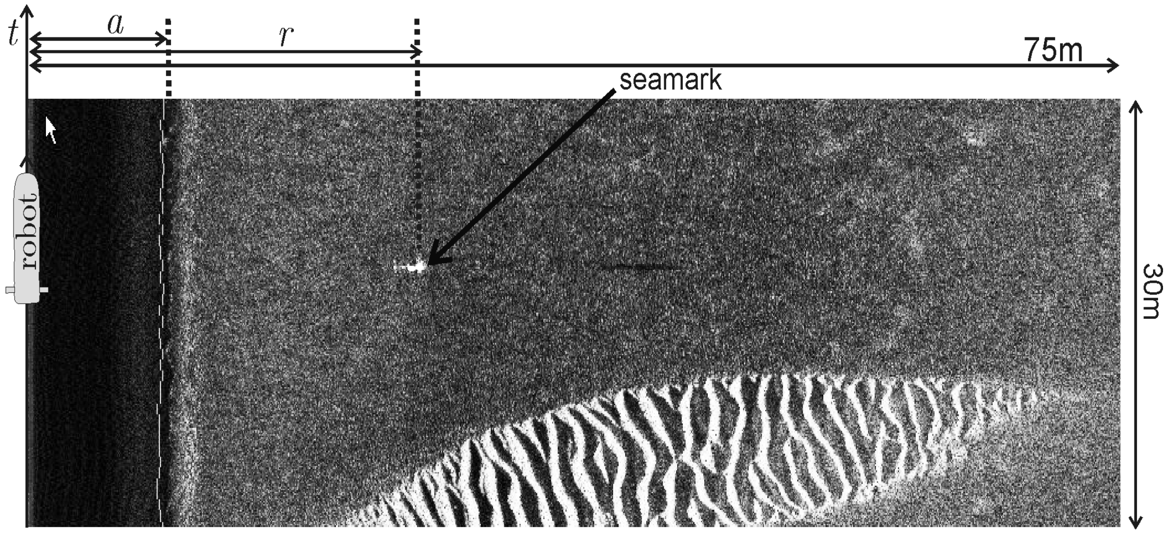





robot
a
bottom of the ocean
(a)
Echo intensity
a
d
(b)



Screenshot of SonarPro



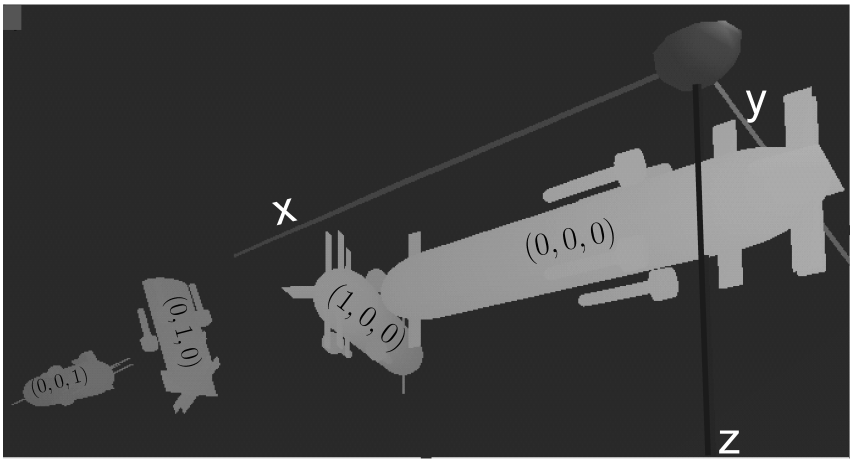
Mine detection with SonarPro

Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \tilde{\mathbf{v}}_r + 0.004 * [-1, 1] \, .\tilde{\mathbf{v}}_r + 0.004 * [-1, 1]$$

Inertial central (Octans III from IXSEA).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.\,[-1,1] \\ 1.75 \times 10^{-4}.\,[-1,1] \\ 5.27 \times 10^{-3}.\,[-1,1] \end{pmatrix}.$$



Six mines have been detected.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	7054	7092	7374	7748	9038	9688	10024	10817	11172	11232	11279	11688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

Exercise. Draw the association graph associated with the detections.

1.2 Constraints

$$t \in \{6000.0, 6000.1, 6000.2, \ldots, 11999.4\},$$

$$i \in \{0, 1, \ldots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) * \frac{\pi}{180}\right) & 0 \end{pmatrix} \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix}$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix},$$

$$\mathbf{R}_{\varphi}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{pmatrix},$$

$$\mathbf{R}(t) = \mathbf{R}_{\psi}(t)\mathbf{R}_{\theta}(t)\mathbf{R}_{\varphi}(t),$$

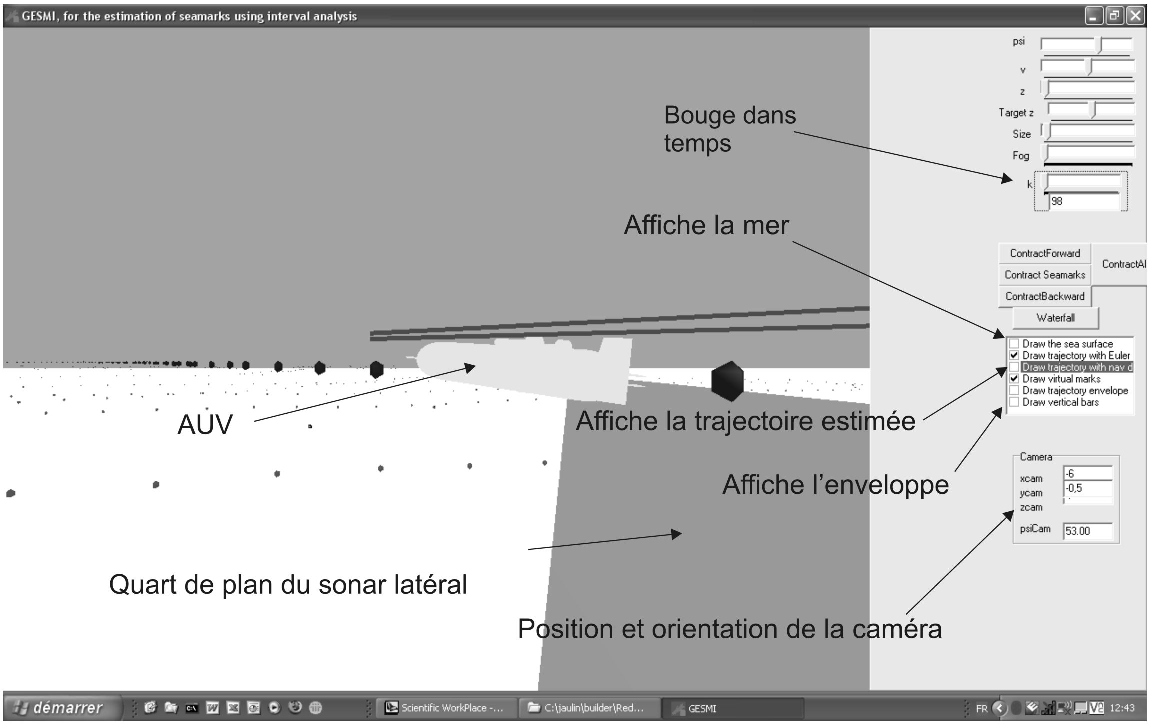
$$\dot{\mathbf{p}}(t) = \mathbf{R}(t).\mathbf{v}_r(t),$$

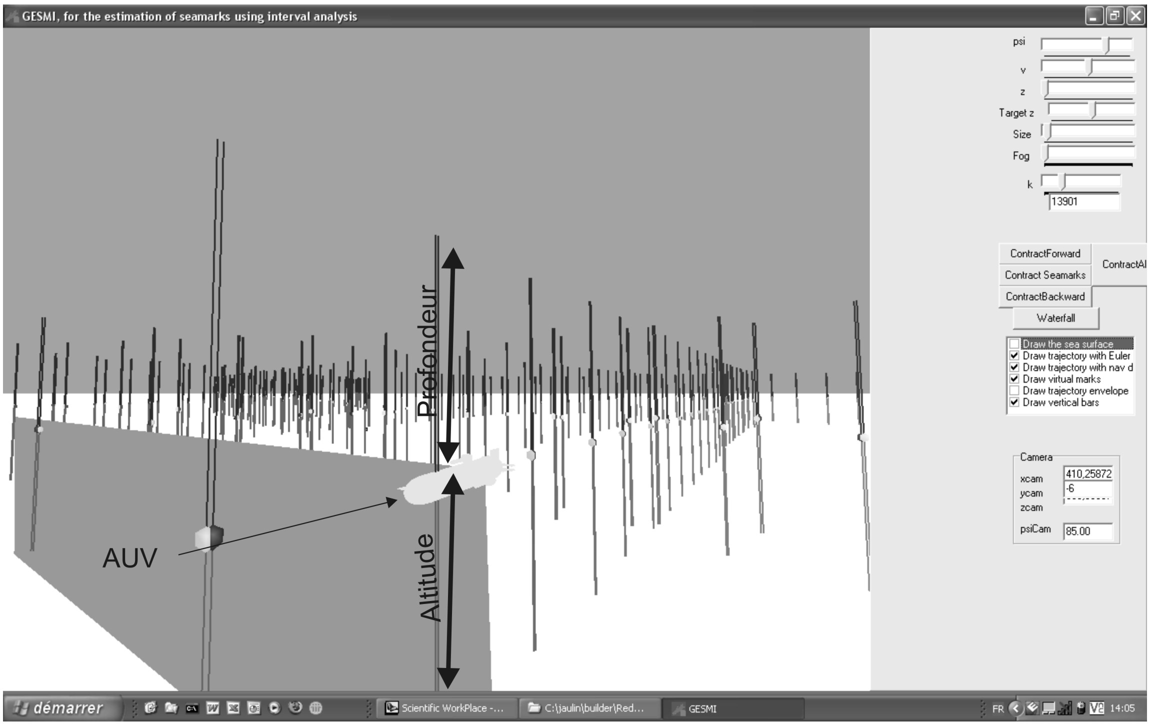
$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| \ = r(i),$$

$$\mathbf{R}^{\mathsf{T}}(\tau(i))\,(\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^{\times 2},$$

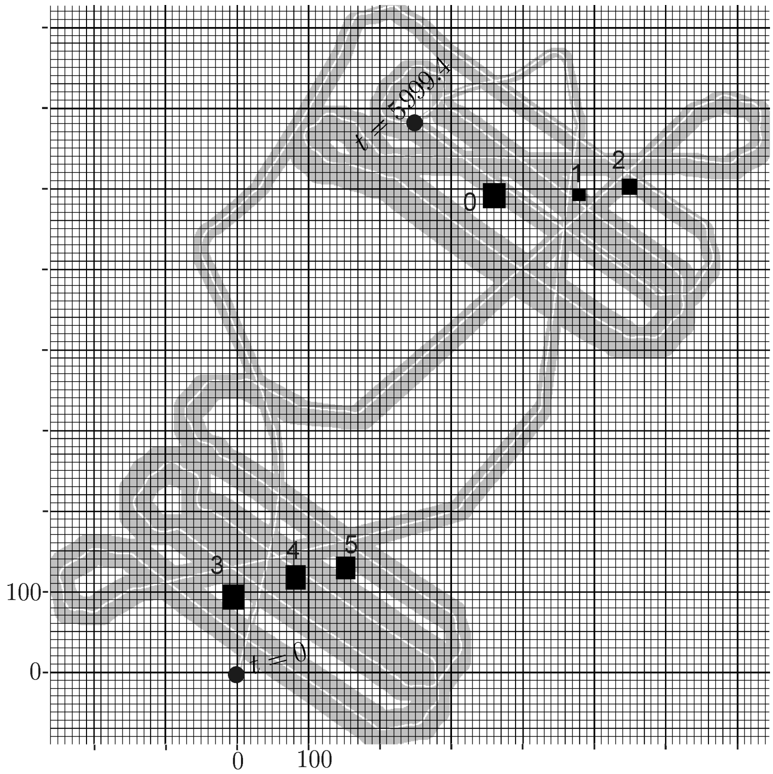
$$m_z(\sigma(i)) - p_z(\tau(i)) - a(\tau(i)) \in [-0.5, 0.5]$$

1.3 GESMI

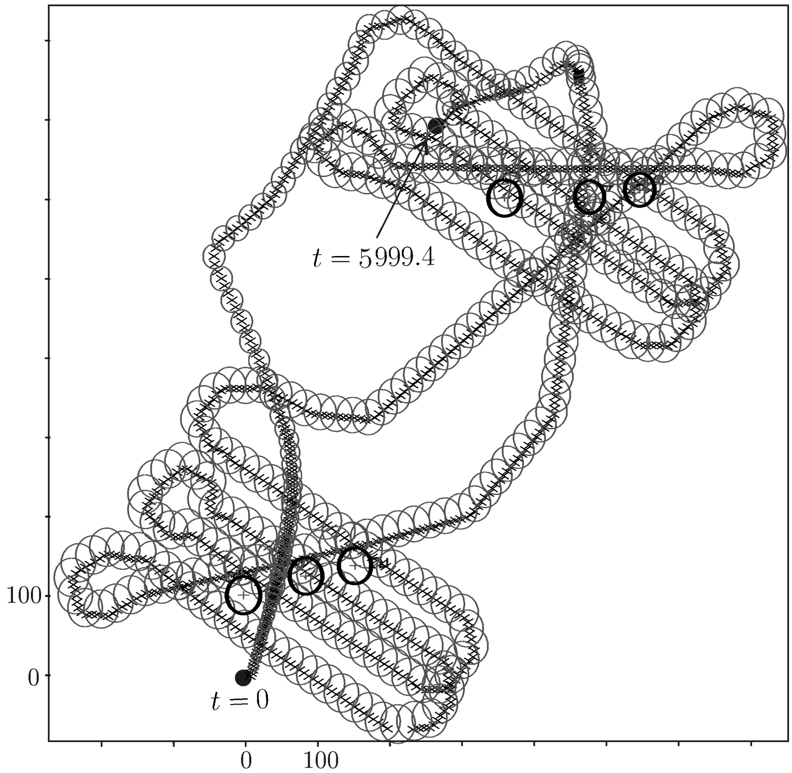

GESMI, for the estimation of seamarks using interval analysis
Bouge dans temps
Affiche la mer
AUV
Affiche la trajectoire estimée
Affiche l'enveloppe
Quart de plan du sonar latéral
Position et orientation de la caméra
psi
v
z
Target z
Size
Fog
k
98
ContractForward
Contract Seamarks
ContractBackward
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
Camera
xcam
ycam
zcam
psiCam
-6
-0,5
53.00
démarrer
GESMI
12:43



GESMI, for the estimation of seamarks using interval analysis
Profondeur
AUV
Altitude
psi
v
z
Target z
Size
Fog
k
13901
ContractForward
Contract Seamarks
ContractBackward
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
Camera
xcam
ycam
zcam
psiCam
410,25872
-6
85.00
démarrer
GESMI
14:05



t = 5999.4
0
1
2
3
4
5
t = 0
100
0
0
100



Extended Kalman smoother

2 Intervals in lattices

2.1 Lattices

A *lattice* $(\mathcal{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds.

The least upper bound of x and y is called the *join*: $x \vee y$.

The greatest lower bound is called the *meet*: $x \wedge y$.

The Cartesian product of two lattices $(\mathcal{E}_1, \leq_1)$ and $(\mathcal{E}_2, \leq_2)$ is a lattice $(\mathcal{E}, \leq)$ with

$$(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow ((a_1 \leq_1 b_1) \text{ and } (a_2 \leq_2 b_2)).$$

Exercise. $\mathcal{L} = ((\mathbb{B}, \mathbb{R}), \leq)$ is a lattice.

$$\begin{array}{lcl}
(\text{false}, 5) \vee (\text{true}, 2) & = & ? \\
(\text{false}, 5) \wedge (\text{true}, 2) & = & ? \\
\top(\mathcal{L}) & = & ? \\
\bot(\mathcal{L}) & = & ?
\end{array}$$

Example. The set $(\mathbb{R}^n, \leq)$ is a lattice with

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow \forall i \in \{\mathbf{1}, \ldots, n\}, x_i \leq y_i.$$

Example.

The powerset $\mathcal{P}(\mathbb{E})$ of all subsets of $\mathbb{E}$ is a lattice with respect to the inclusion $\subset$.

What is the meet ? What is the join ?

Example

The set $\mathcal{F}$ of all functions from $\mathbb{R}$ to $\mathbb{R}^n$ is a lattice with

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall t \in \mathbb{R},\ \mathbf{f}(t) \leq \mathbf{g}(t)$$

An interval of $\mathcal{F}$ is called a *tube*.

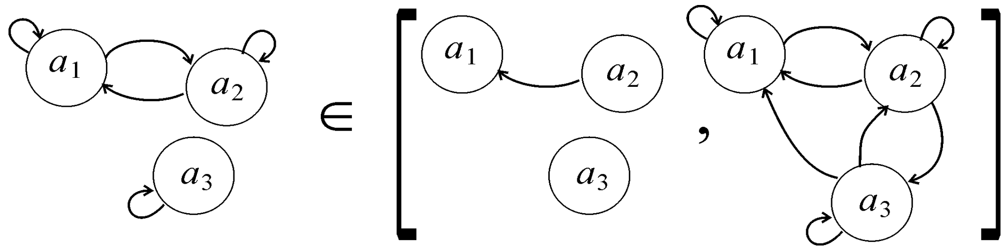
Intervals. A *closed interval* (or *interval* for short) $[x]$ of a lattice $\mathcal{E}$ is a subset of $\mathcal{E}$ which satisfies

$$[x] = \{x \in \mathcal{E} \mid \wedge[x] \leq x \leq \vee[x]\}\,.$$

The set $\mathbb{I}\mathcal{L}$ of all intervals of a lattice $\mathcal{L}$ is also a lattice with respect to $\subset$.

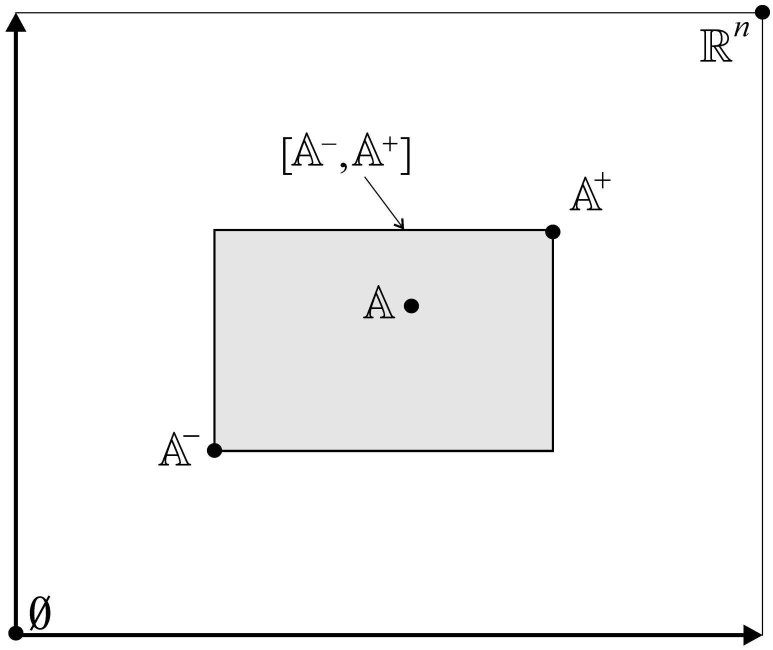
Exercice. Draw the Hasse diagram of the set of Boolean interval $\mathbb{I}\mathbb{B}$.

Graph intervals

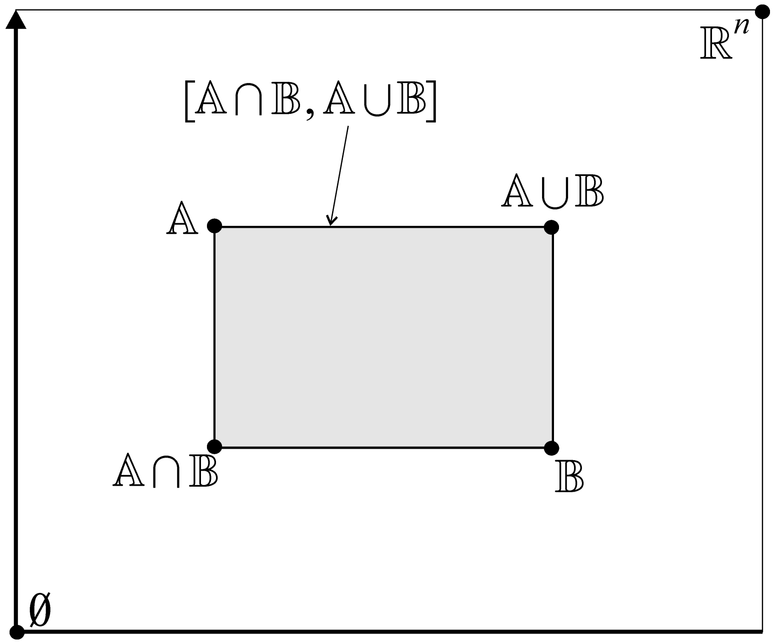


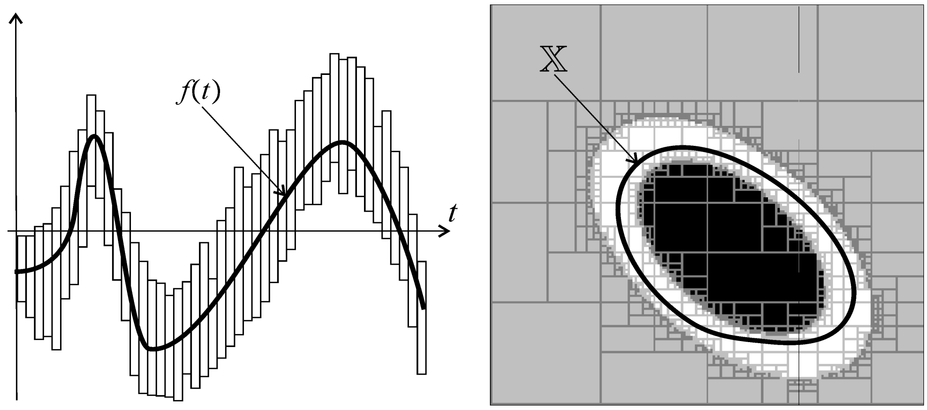
Both $\emptyset$ and $\mathcal{E}$ are intervals of $\mathcal{E}$.

Exercise. Contract the graph interval with respect to the constraint "$\mathcal{G}$ is an equivalence relation".



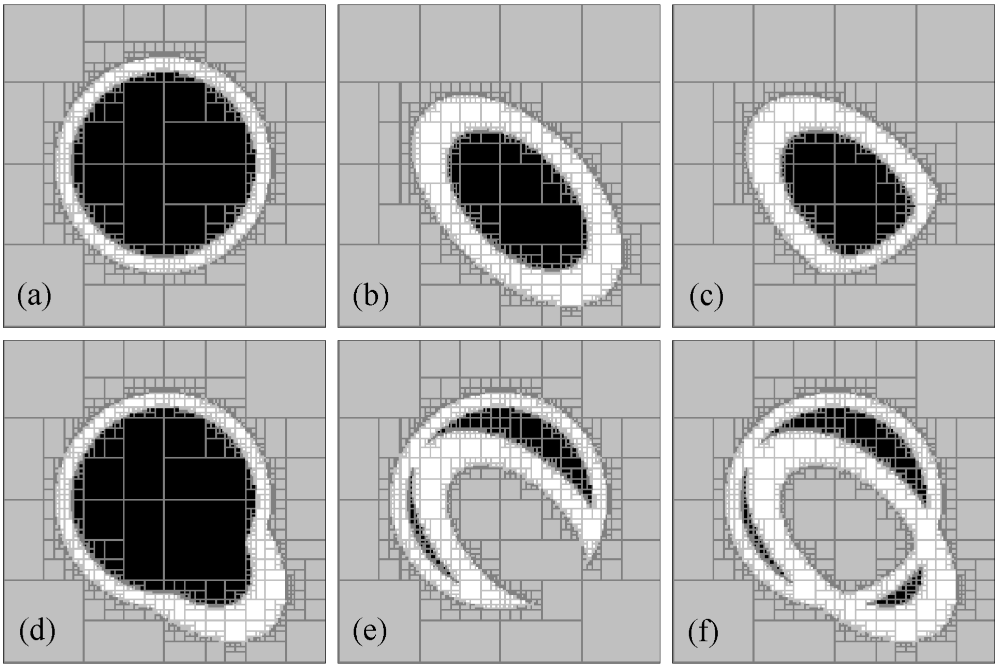
Interval in the lattice $(\mathcal{P}(\mathbb{R}^n), \subset)$


$\mathbb{R}^n$
$[\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B}]$
$\mathbb{A}$
$\mathbb{A}\cup\mathbb{B}$
$\mathbb{A}\cap\mathbb{B}$
$\mathbb{B}$
$\emptyset$



An interval function (or tube) and a set interval

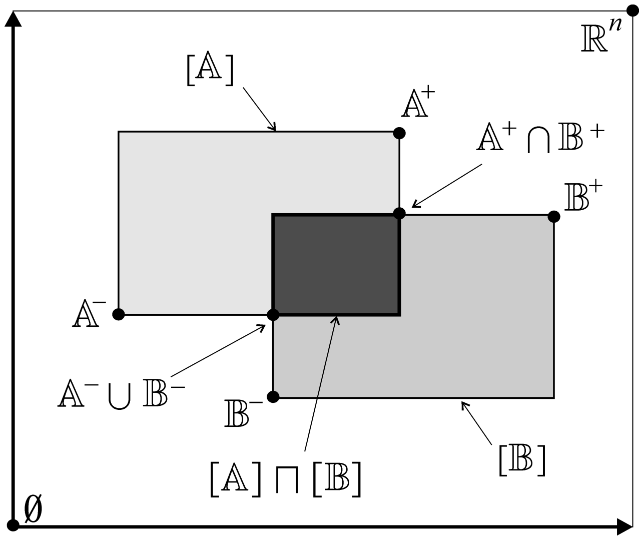
2.2 Interval arithmetic



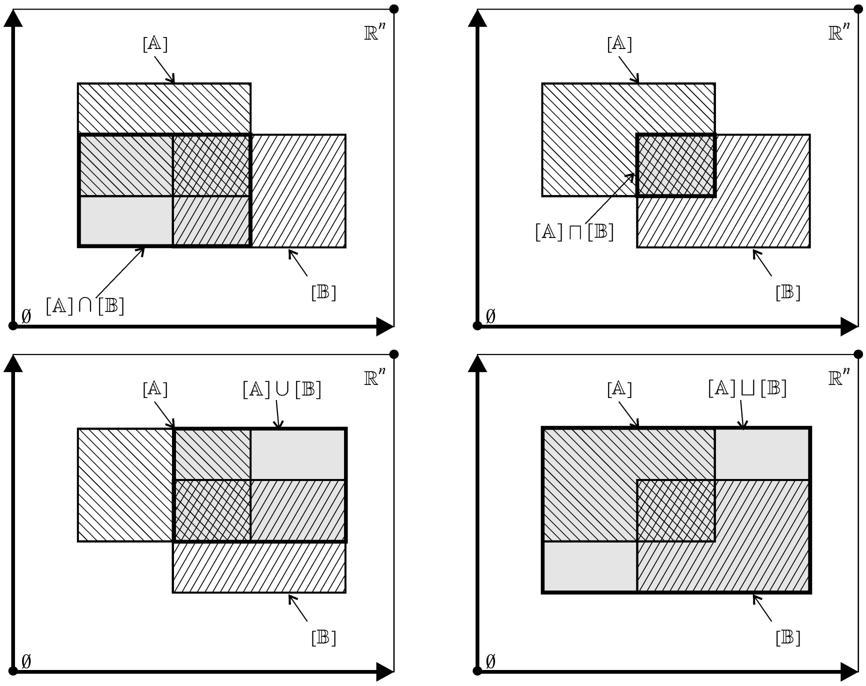
$$[\mathbb{A}],[\mathbb{B}],\ [\mathbb{A}]\cap[\mathbb{B}],\ [\mathbb{A}]\cup[\mathbb{B}],$$
$$[\mathbb{A}]\setminus[\mathbb{B}],\ ([\mathbb{A}]\cup[\mathbb{B}])\setminus([\mathbb{A}]\cap[\mathbb{B}]).$$

Intersection.

$$\begin{aligned} [\mathbb{A}] \sqcap [\mathbb{B}] &= \{\mathbb{X}, \mathbb{X} \in [\mathbb{A}] \text{ and } \mathbb{X} \in [\mathbb{B}]\} \\ &= \left[\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+\right]. \end{aligned}$$



$\mathbb{R}^n$
$[\mathbb{A}]$
$\mathbb{A}^+$
$\mathbb{A}^+ \cap \mathbb{B}^+$
$\mathbb{B}^+$
$\mathbb{A}^-$
$\mathbb{A}^- \cup \mathbb{B}^-$
$\mathbb{B}^-$
$[\mathbb{B}]$
$[\mathbb{A}] \sqcap [\mathbb{B}]$
$\emptyset$



[𝔸]
ℝⁿ
[𝔸] ∩ [𝔹]
[𝔹]
∅
[𝔸]
ℝⁿ
[𝔸] ⊓ [𝔹]
[𝔹]
∅
[𝔸]
[𝔸] ∪ [𝔹]
ℝⁿ
[𝔹]
∅
[𝔸]
[𝔸] ⊔ [𝔹]
ℝⁿ
[𝔹]
∅

2.3 Contractors in lattices

A CSP is composed of a set of variables $\{x_1, \ldots, x_n\}$,
of constraints $\{c_1, \ldots, c_m\}$ and of domains $\{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$.

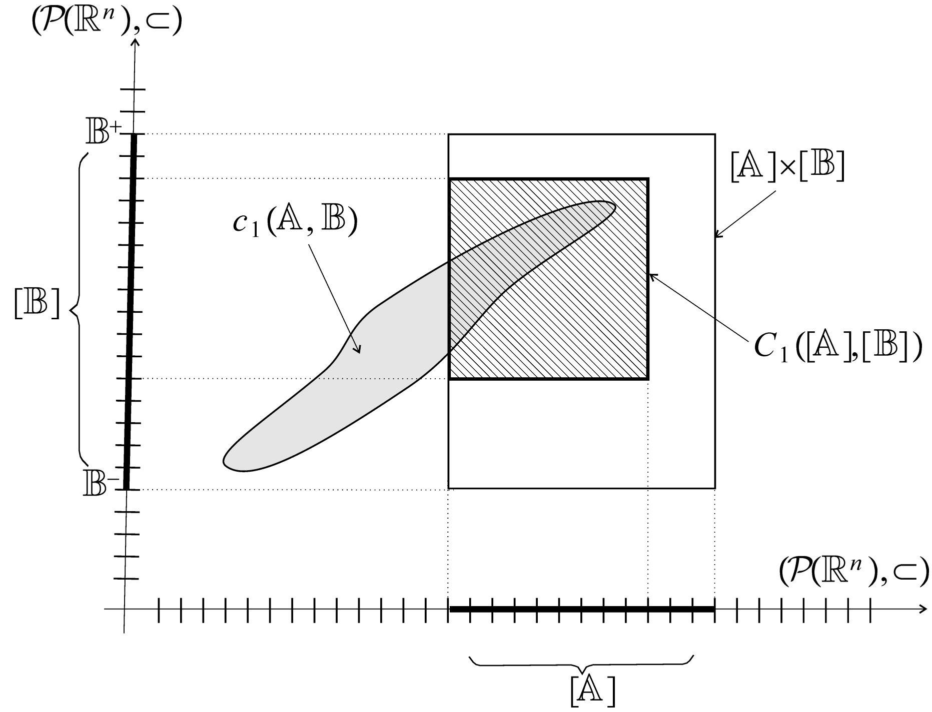
The domains $\mathbb{X}_i$ should belong to a lattice $(\mathcal{L}_i, \subset)$.

For SLAM, the domains are

(i) intervals of $\mathbb{R}^n$ to represent the location of the marks,

(ii) tubes to represent the unknown trajectory and

(iii) intervals of subsets of $\mathbb{R}^n$ to represent the free space.


$(\mathcal{P}(\mathbb{R}^n), \subset)$
$\mathbb{B}^+$
$[\mathbb{A}] \times [\mathbb{B}]$
$c_1(\mathbb{A}, \mathbb{B})$
$[\mathbb{B}]$
$C_1([\mathbb{A}],[\mathbb{B}])$
$\mathbb{B}^-$
$(\mathcal{P}(\mathbb{R}^n), \subset)$
$[\mathbb{A}]$

Example

$$\left\{ \begin{array}{c} \mathbb{A} \subset \mathbb{B} \\ \mathbb{A} \in [\mathbb{A}] \, , \mathbb{B} \in [\mathbb{B}] \, . \end{array} \right.$$

Since

$$\mathbb{A} \subset \mathbb{B} \Leftrightarrow \mathbb{A} = \mathbb{A} \cap \mathbb{B} \Leftrightarrow \mathbb{B} = \mathbb{A} \cup \mathbb{B}.$$

the optimal contractor is

$$\left\{ \begin{array}{ll} \text{(i)} & [\mathbb{A}] := [\mathbb{A}] \sqcap ([\mathbb{A}] \cap [\mathbb{B}]) \\ \text{(ii)} & [\mathbb{B}] := [\mathbb{B}] \sqcap ([\mathbb{A}] \cup [\mathbb{B}]) \end{array} \right.$$

Tarski theorem.

If $(\mathcal{L}, \leq)$ is a lattice and $f : \mathcal{L} \rightarrow \mathcal{L}$ is monotonic (i.e., $a \leq b \Rightarrow f(a) \leq f(b)$), then $x_{k+1} = f(x_k)$, converges to the greatest x_∞ such that

$$\begin{cases} x_\infty = f(x_\infty) & \text{(fixed point)} \\ x_\infty \leq x_0 & \end{cases}$$

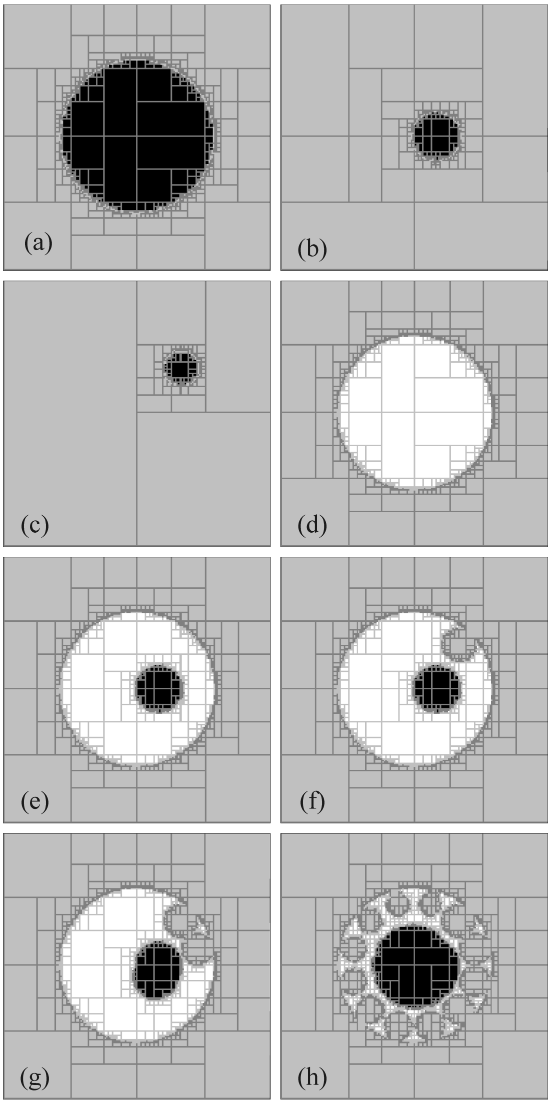
2.4 Propagation

Consider the following CSP

$$\begin{cases} \text{(i)} & \mathbb{X} \subset \mathbb{A} \\ \text{(ii)} & \mathbb{B} \subset \mathbb{X} \\ \text{(iii)} & \mathbb{X} \cap \mathbb{C} = \emptyset \\ \text{(iv)} & f(\mathbb{X}) = \mathbb{X}, \end{cases}$$

where $\mathbb{X} \subset \mathbb{R}^2$, f is a rotation of $-\frac{\pi}{6}$, and

$$\begin{cases} \mathbb{A} & = & \left\{(x_1, x_2), x_1^2 + x_2^2 \leq 3\right\} \\ \mathbb{B} & = & \left\{(x_1, x_2), (x_1 - 0.5)^2 + x_2^2 \leq 0.3\right\} \\ \mathbb{C} & = & \left\{(x_1, x_2), (x_1 - 1)^2 + (x_2 - 1)^2 \leq 0.15\right\} \end{cases}$$



(a) $[\mathbb{A}]$

(b) $[\mathbb{B}]$

(c) $[\mathbb{C}]$

(d) $\mathbb{X} \subset \mathbb{A}$

(e) $\mathbb{B} \subset \mathbb{X}$

(f) $\mathbb{X} \cap \mathbb{C} = \emptyset$

(g) $f\,(\mathbb{X}) = \mathbb{X}$

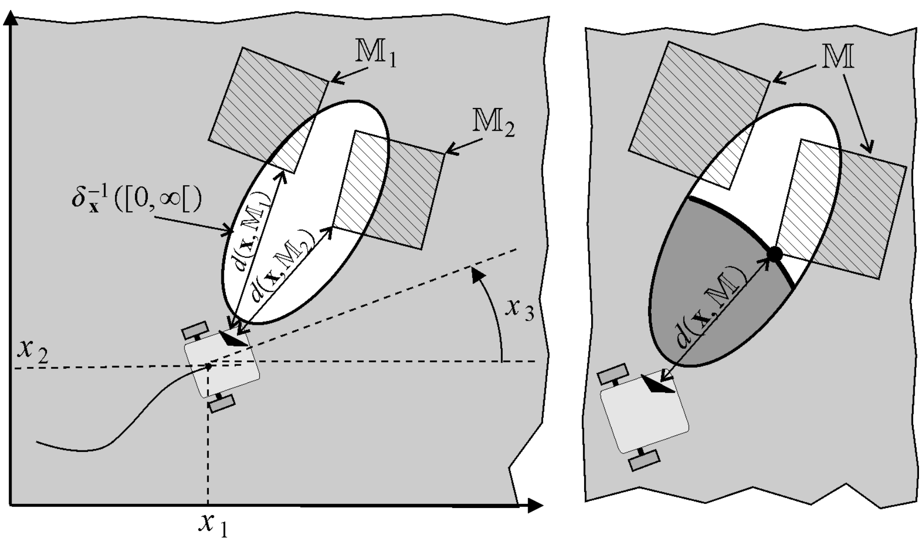
(h) $(f\,(\mathbb{X}) = \mathbb{X})^{\infty}$

3 Range-only SLAM with occupancy maps

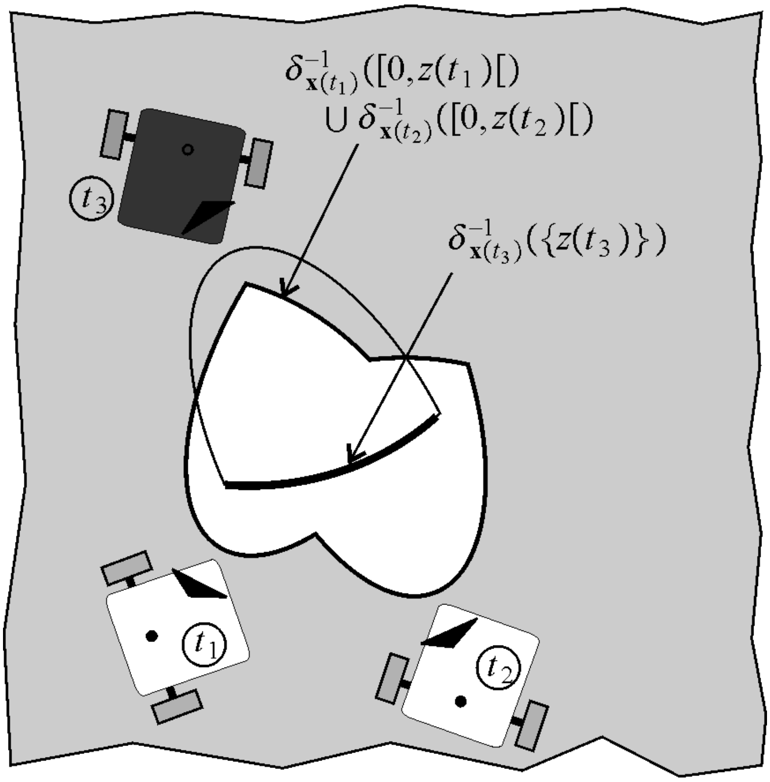
$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & \text{(evolution equation)} \\ z(t) & = & d(\mathbf{x}(t), \mathbb{M}) & \text{(map equation)} \end{cases}$$

where $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbb{M} \in \mathcal{C}(\mathbb{R}^q)$ is the occupancy map.

Unknown: the map $\mathbb{M}$ and the trajectory $\mathbf{x}$.



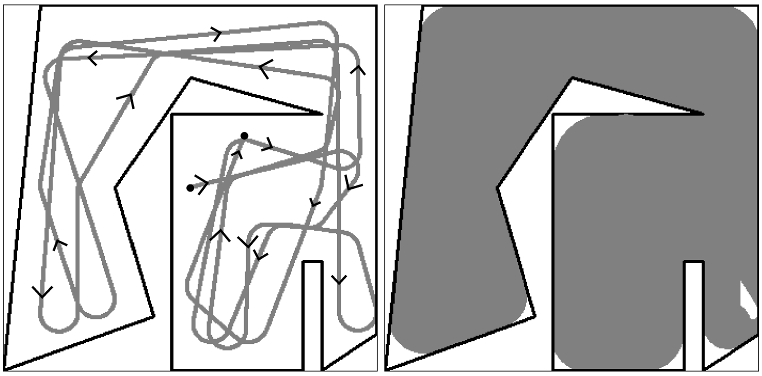
Impact, covering and dug zones



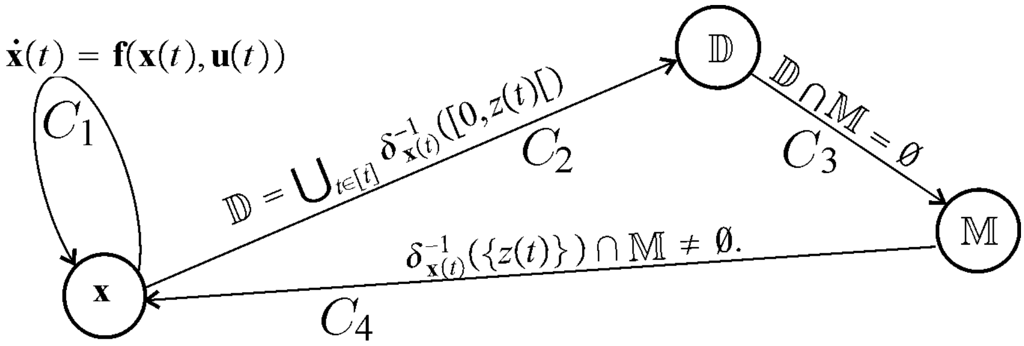
$\delta^{-1}_{\mathbf{x}(t_1)}([0,z(t_1)[)$
$\cup\, \delta^{-1}_{\mathbf{x}(t_2)}([0,z(t_2)[)$
$\delta^{-1}_{\mathbf{x}(t_3)}(\{z(t_3)\})$
t_3
t_1
t_2

Range-only SLAM equations

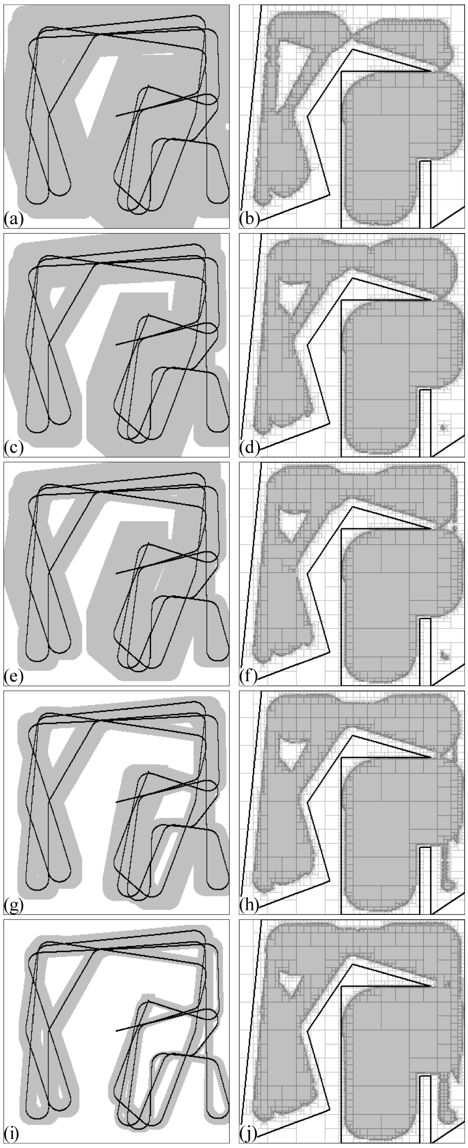
$$\begin{cases} \dot{x}_1(t) & = & u_1(t)\cos(u_2(t)) \\ \dot{x}_2(t) & = & u_1(t)\sin(u_2(t)) \\ z(t) & = & d(\mathbf{x}(t), \mathbb{M}). \end{cases}$$

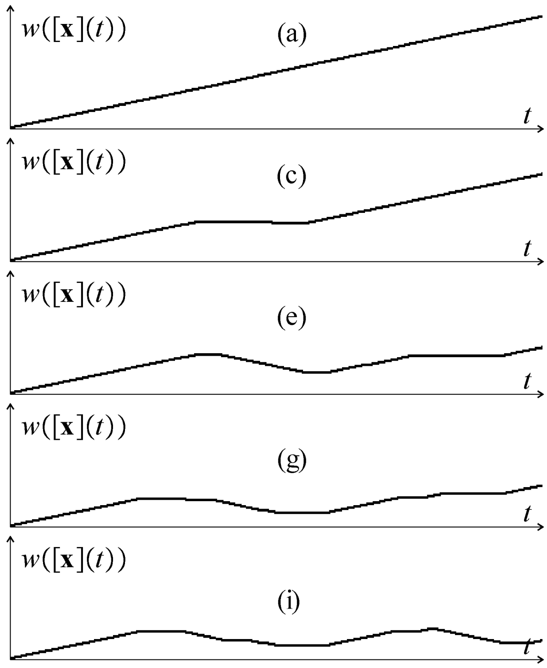


Actual trajectory and dug space



$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$
C_1
$\mathbf{x}$
$\mathbb{D} = \bigcup_{t\in[t]} \delta^{-1}_{\mathbf{x}(t)}([0, z(t)[)$
C_2
$\mathbb{D}$
$\mathbb{D} \cap \mathbb{M} = \emptyset$
C_3
$\mathbb{M}$
$\delta^{-1}_{\mathbf{x}(t)}(\{z(t)\}) \cap \mathbb{M} \neq \emptyset.$
C_4


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)



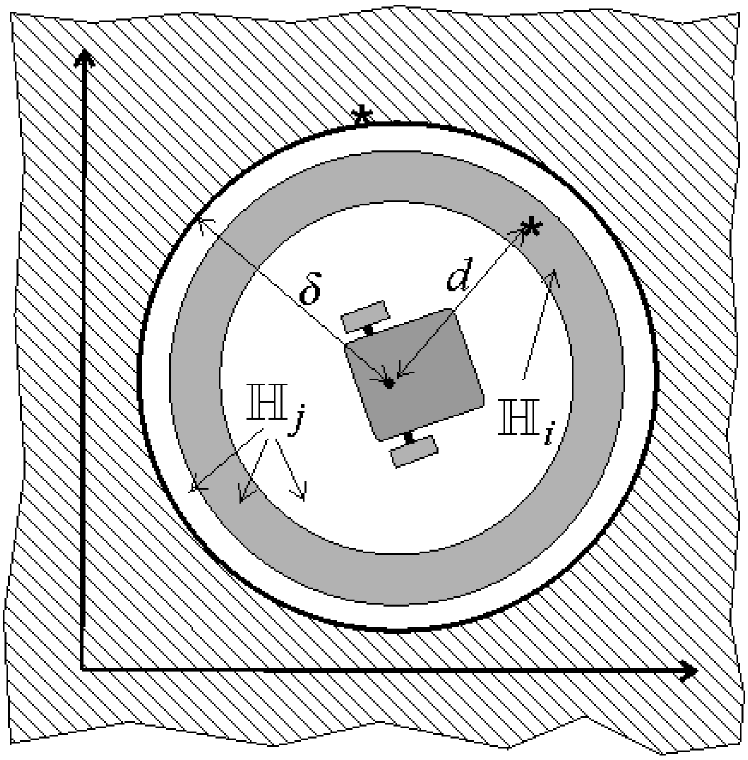
Width of the tubes $[\mathbf{x}](t)$

4 Range only SLAM with undistinguishable marks

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) & \text{(evolution equation)} \\ (t_i, \mathcal{H}_i(\mathbf{x})) & \text{(sector list)} \end{cases}$$

Example. A robot is located at (x_1, x_2). If at time t_3 the robot detects one single mark at a distance $d \in [4, 5]$m,

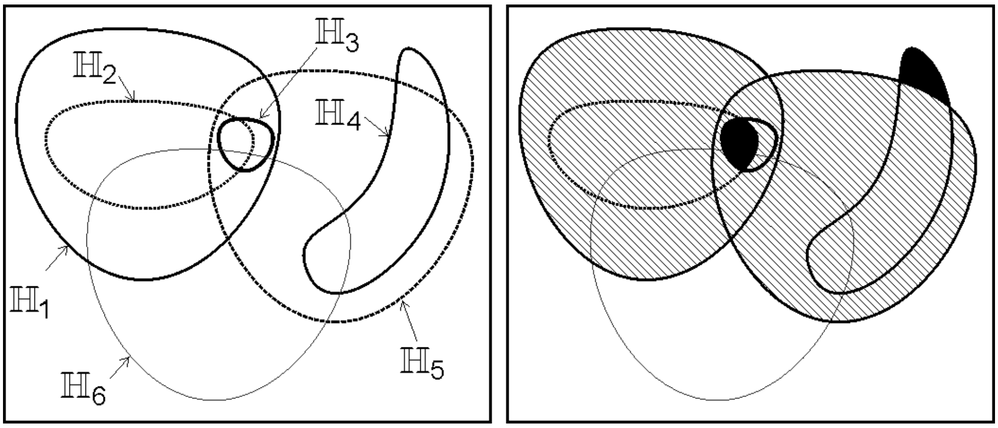
$$\mathcal{H}_3(\mathbf{x}) : \left\{ \mathbf{a} \in \mathbb{R}^2 | \, (x_1 - a_1)^2 + (x_2 - a_2)^2 \in [16, 25] \right\}.$$



The robot has detected the mark inside the ring

Theorem. Consider a set of marks $\mathcal{M} \subset \mathbb{R}^q$. Define the free space as $\mathbb{F} = \{\mathbf{p} \in \mathbb{R}^q \mid \mathbf{p} \notin \mathcal{M}\}$. Consider m sectors $\mathbb{H}_1, \ldots, \mathbb{H}_m$, each of them containing exactly one mark and define $\mathbf{a}(i) = \mathcal{M} \cap \mathbb{H}_i$. We have

(i) $\mathbb{H}_i \subset \mathbb{H}_j \Rightarrow \mathbf{a}(i) = \mathbf{a}(j)$
(ii) $\mathbb{H}_i \cap \mathbb{H}_j = \emptyset \Rightarrow \mathbf{a}(i) \neq \mathbf{a}(j)$
(iii) $\mathbb{H}_i \subset \mathbb{H}_j \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}$.



Each of the two black zones contains a single mark and that no mark exists in the hatched area.

(i) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$

(ii) $\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$

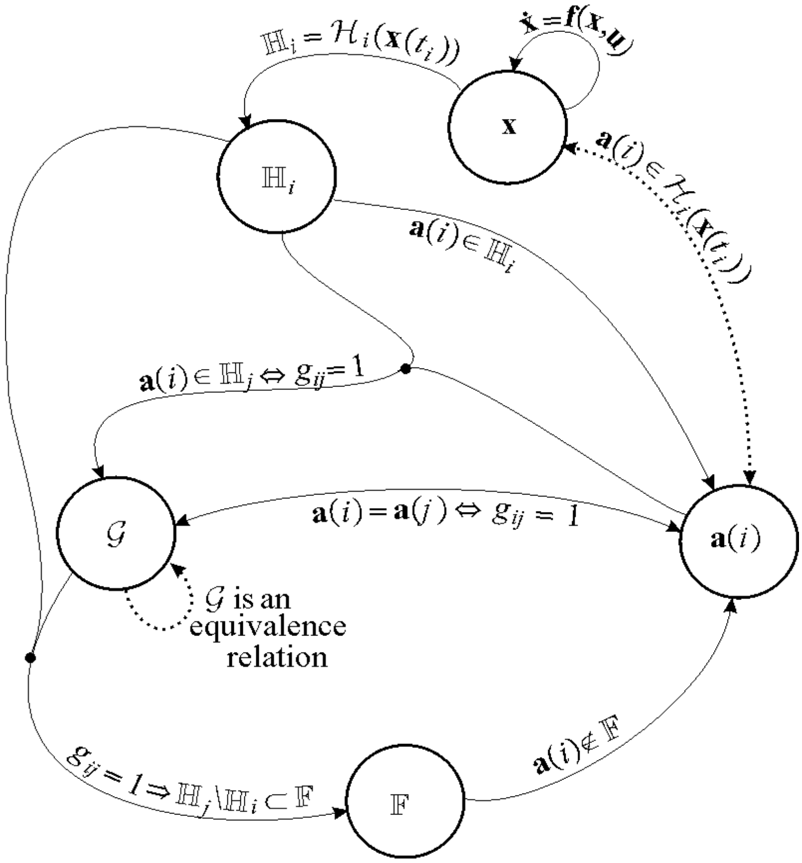
(iii) $\mathbf{a}(i) \in \mathbb{H}_i$

(iv) $\mathbf{a}(i) = \mathbf{a}(j) \Leftrightarrow g_{ij} = \mathbf{1}$

(v) $\mathbf{a}(i) \in \mathbb{H}_j \Leftrightarrow g_{ij} = \mathbf{1}$

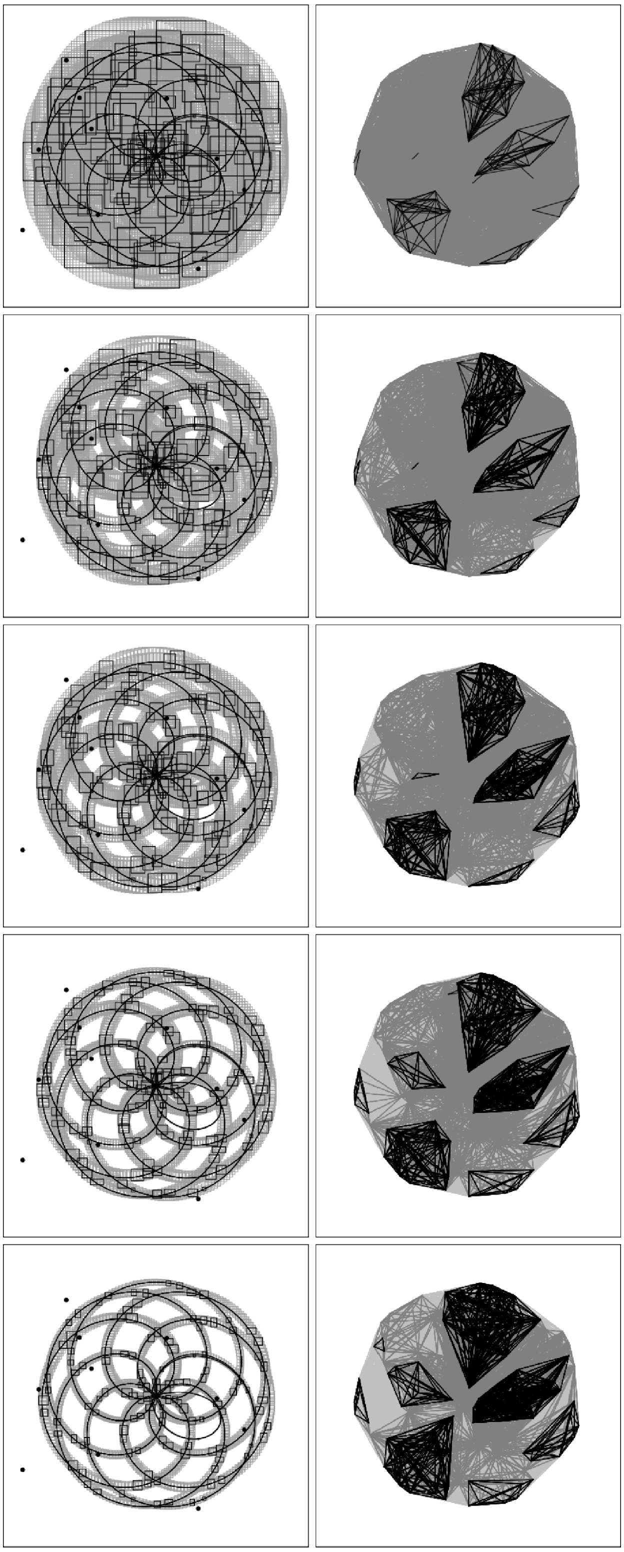
(vi) $g_{ij} = \mathbf{1} \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}$

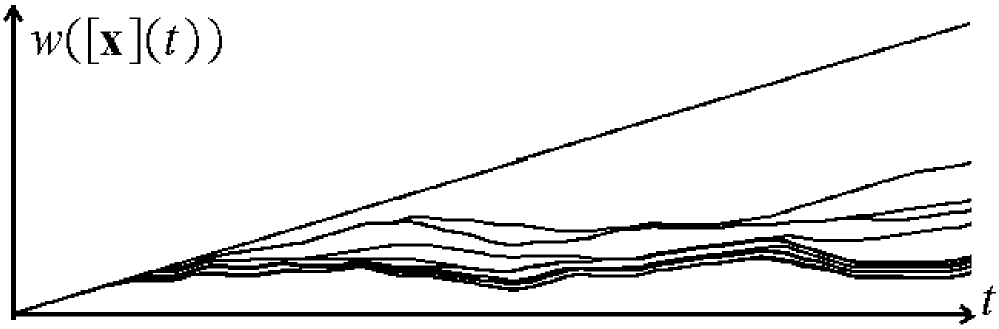
(vii) $\mathbf{a}(i) \notin \mathbb{F}$



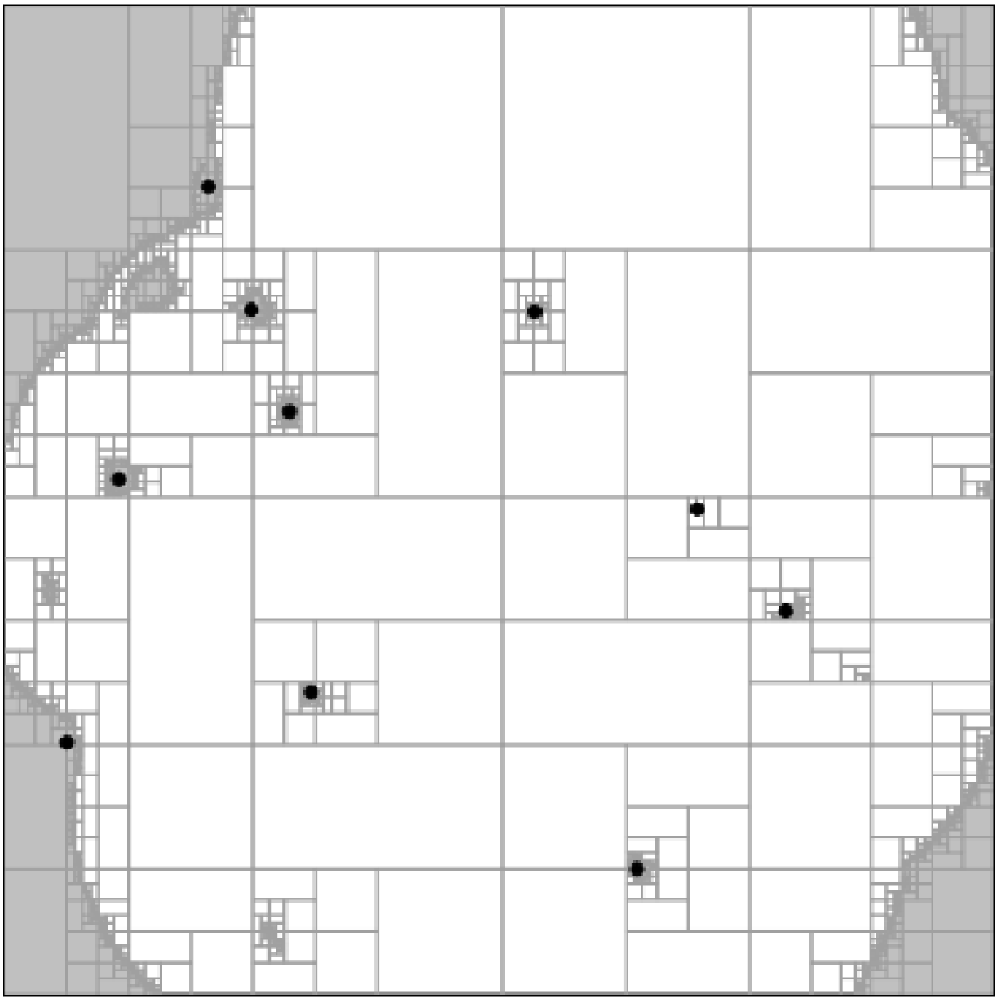
Contractor graph

4.1 Testcase





Width of the tubes $[\mathbf{x}](t)$



Free space $\mathbb{F}$.