Inner and Outer Approximations of Projections of Equalities

Alexandre Goldsztejn and Luc Jaulin,
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1 Problem

Consider a function

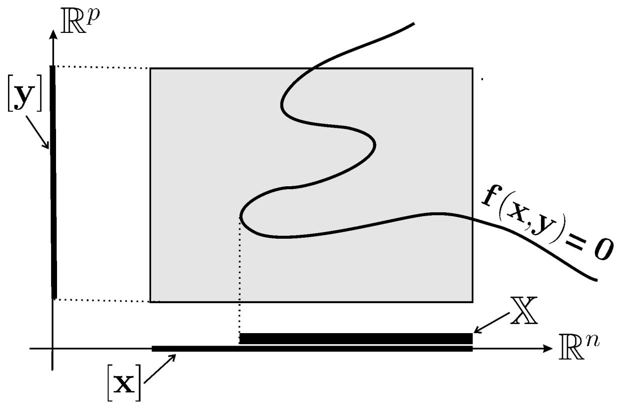
$$\mathbf{f}:\left\{\begin{array}{lll}\mathbb{R}^n\times\mathbb{R}^p & \rightarrow & \mathbb{R}^m\\ (\mathbf{x},\mathbf{y}) & \mapsto & \mathbf{f}(\mathbf{x},\mathbf{y})\end{array}\right.$$

and two boxes $[\mathbf{x}]\subset\mathbb{R}^n$ and $[\mathbf{y}]\subset\mathbb{R}^p$. Characterize the set

$$\mathbb{X}=\{\mathbf{x}\in[\mathbf{x}],\exists\mathbf{y}\in[\mathbf{y}],\mathbf{f}(\mathbf{x},\mathbf{y})=\mathbf{0}\},$$

i.e., find two subpavings $\mathbb{X}^-,\mathbb{X}^+$ such that

$$\mathbb{X}^-\subset\mathbb{X}\subset\mathbb{X}^+.$$



Remark: Finding $\mathbb{X}^+$ can be considered as already solved. This is not the case for $\mathbb{X}^-$.

2 Examples

2.1 Image set problem

(Robust control, state estimation, multi-objective problems)

Characterize the set

$$\mathbb{Y} = \mathbf{f}([\mathbf{x}]),$$

where $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^p$,$n \geq p$ and $[\mathbf{x}]$ is a box of $\mathbb{R}^n$. We have

$$\mathbb{Y} = \{\mathbf{y} \in \mathbb{R}^p, \exists \mathbf{x} \in [\mathbf{x}], \mathbf{f}(\mathbf{x}) - \mathbf{y} = \mathbf{0}\}.$$

2.2 Control

Consider the system

$$\begin{cases} \frac{d\mathbf{x}}{dt}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) \\ \mathbf{y}(t) & = & \mathbf{g}(\mathbf{x}(t),\mathbf{u}(t)). \end{cases}$$

The set of all feasible steady outputs is

$$\mathbb{Y} = \{\mathbf{y} \in \mathbb{R}^p, \exists \mathbf{u} \in [\mathbf{u}], \exists \mathbf{x} \in [\mathbf{x}],\ \mathbf{h}(\mathbf{y},\mathbf{u},\mathbf{x}) = \mathbf{0} \quad \}.$$

where

$$\mathbf{h}(\mathbf{y},\mathbf{u},\mathbf{x}) = \begin{pmatrix} \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \mathbf{y} - \mathbf{g}(\mathbf{x},\mathbf{u}) \end{pmatrix}$$

2.3 Robotics

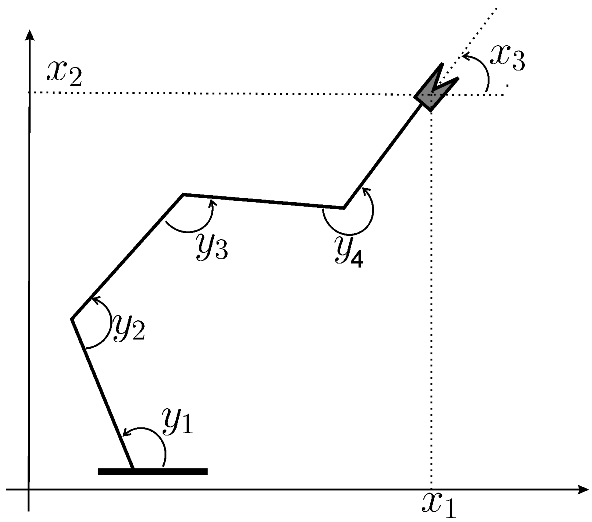
Geometric model of a robot :

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$$

where $\mathbf{y}$ is the configuration vector of the robot and $\mathbf{x}$ is the position vector of the tool.

The workspace is

$$\mathbb{W} = \{\mathbf{x} \mid \exists \mathbf{y} \in [\mathbf{y}],\ \mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0}\}\,.$$



3 Already solved

3.1 Set inversion

When

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0} \Leftrightarrow \mathbf{h}(\mathbf{x}) = \mathbf{y}$$

then

$$\begin{aligned}
\mathbb{X} &= \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0}\} \\
&= \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{h}(\mathbf{x}) = \mathbf{y}\} \\
&= \{\mathbf{x} \in [\mathbf{x}], \mathbf{h}(\mathbf{x}) \in [\mathbf{y}]\} \\
&= [\mathbf{x}] \cap \mathbf{h}^{-1}([\mathbf{y}]).
\end{aligned}$$

The set $\mathbb{X}$ can thus be defined by inequalities.

3.2 Image set

When

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0} \Leftrightarrow \mathbf{h}(\mathbf{y}) = \mathbf{x}$$

then

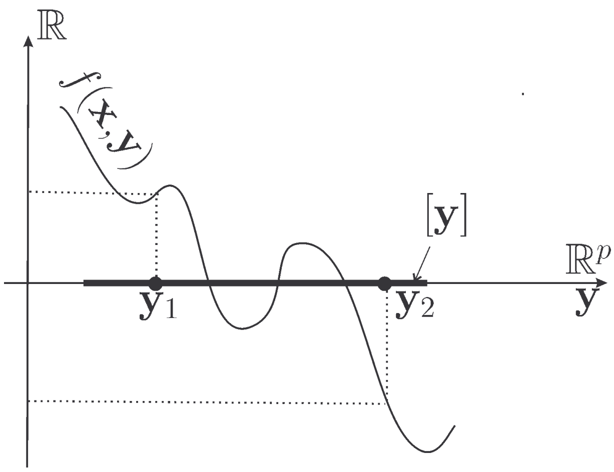
$$\begin{aligned} \mathbb{X} &= \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0}\} \\ &= \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{h}(\mathbf{y}) = \mathbf{x}\} \\ &= [\mathbf{x}] \cap \mathbf{h}([\mathbf{y}]). \end{aligned}$$

When dim$(\mathbf{x})$ =dim$(\mathbf{y})$ the problem has recently been solved.

3.3 Only one equality

If $f \in \mathbb{R}$ is continuous,

$$\exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) = 0 \Leftrightarrow \begin{cases} \exists \mathbf{y}_1 \in [\mathbf{y}], & f(\mathbf{x},\mathbf{y}_1) \geq 0 \\ \exists \mathbf{y}_2 \in [\mathbf{y}], & f(\mathbf{x},\mathbf{y}_2) \leq 0 \end{cases},$$



Example

$$\begin{aligned}\mathbb{X} &= \Big\{(x_1, x_2), \exists y \in [-2, 2], x_1^2 + x_2^2 + y^2 - 1 = 0\Big\} \\ &= \left\{(x_1, x_2), \left\{\begin{array}{ll} \exists y_1 \in [-2, 2] & x_1^2 + x_2^2 + y_1^2 - 1 \leq 0 \\ \exists y_2 \in [-2, 2] & x_1^2 + x_2^2 + y_2^2 - 1 \geq 0 \end{array}\right.\right\}\end{aligned}$$

Illustration with Proj2d

This trick does not works for more than one equality

$$\exists \mathbf{y}, (f_1(\mathbf{y}) = 0 \wedge f_2(\mathbf{y}) = 0) \quad \not\Leftrightarrow \quad \left\{ \begin{array}{l} (\exists \mathbf{y}, f_1(\mathbf{y}) = 0) \\ (\exists \mathbf{y}, f_2(\mathbf{y}) = 0) \end{array} \right.$$

4 Feasibility test

Consider one point $\mathbf{x} \in [\mathbf{x}]$, check if

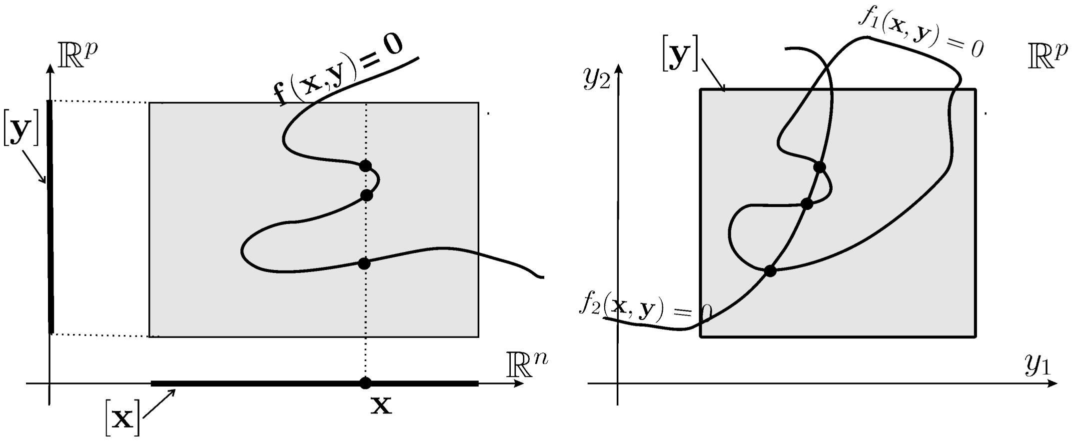
$$\mathbf{x} \in \mathbb{X} \overset{\text{def}}{=} \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\} .$$

Note that the set

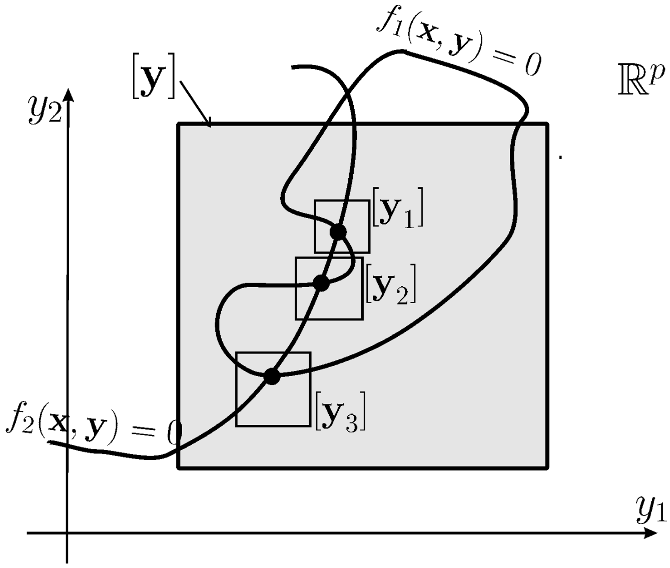
$$\{\mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$$

may contain a continuum of elements.

Step 1 : Find some approximate solutions $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots\}$ for $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.



Step 2 : Inflate the approximate solutions to get small boxes $[\mathbf{y}_1], [\mathbf{y}_2], [\mathbf{y}_3], \ldots$



Step 3 : Apply the new feasibility test to each

$$[\mathbf{y}] \in \{[\mathbf{y}_1], [\mathbf{y}_2], [\mathbf{y}_3], \dots\}$$

Proposition : Consider $\mathbf{g}(\mathbf{y}) : \mathbb{R}^p \longrightarrow \mathbb{R}^m$, a differentiable function and $[\bar{\mathbf{y}}]$ a box of $\mathbb{R}^m$.

If the CSP

$$\begin{cases} \mathbf{y} \in [\bar{\mathbf{y}}] \\ \mathbf{z} \in [\mathbf{0}, \mathbf{g}(\text{mid}([\bar{\mathbf{y}}])] \\ \mathbf{g}(\mathbf{y}) = \mathbf{z} \\ a_{i,j} = \text{sign}(y_j - \bar{y}_j^-).\text{sign}(\bar{y}_j^+ - y_j).\frac{\partial g_i}{\partial y_j}(\mathbf{y}) \\ \text{rank}\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mp} \end{pmatrix} < m \end{cases}$$

is inconsistent, then

$$\exists \mathbf{y} \in [\bar{\mathbf{y}}], \mathbf{g}(\mathbf{y}) = \mathbf{0}.$$

Remark 1: One should not confuse the domain $[\mathbf{y}]$ for $\mathbf{y}$ with the box $[\bar{\mathbf{y}}]$. The box $[\bar{\mathbf{y}}]$ should no be contracted during a constraint propagation.

Remark 2 : The rank constraint can be transformed into inequalities. For instance

$$\text{rank}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} < 2 \Leftrightarrow \begin{cases} a_{11}.a_{22} - a_{12}.a_{21} & = & 0 \\ a_{11}.a_{23} - a_{13}.a_{21} & = & 0 \end{cases}$$

Our rank contractor uses the interval Gauss elimination.

Example : Proving that

$$\exists \mathbf{y} \in [1,3] \times [2,4], y_1^2 + y_2^2 - 4 = 0$$

amounts to prove that the following CSP is inconsistent

$$\begin{cases} 1 \leq y_1 \leq 3 \\ 2 \leq y_2 \leq 4 \\ 0 \leq z \leq 2^2 + 3^2 - 4 \\ y_1^2 + y_2^2 - 4 = z \\ a_{11} = \text{sign}(y_1 - 1).\text{sign}(3 - y_1).\,(2y_1) \\ a_{12} = \text{sign}(y_2 - 2).\text{sign}(4 - y_2).\,(2y_2) \\ a_{11} = 0, a_{12} = 0 \text{ (rank condition)} \end{cases}$$

Interval constraint propagation can prove the inconsistency in a very efficient way.

Summary : Given

$$\mathbb{X} \overset{\text{def}}{=} \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$$

At the moment, we are able

- to prove that a box $[\mathbf{x}]$ is outside $\mathbb{X}$,
- to prove that a point $\mathbf{x}$ is inside $\mathbb{X}$.

The next step explains how to prove that $[\mathbf{x}] \cap \partial\mathbb{X}$. Such a box is said to be *boundary free.*

5 Boundary test

Theorem : Consider $\mathbf{f}(\mathbf{x},\mathbf{y}) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ a differentiable function, $[\mathbf{x}]$ a box of $\mathbb{R}^n$ and $[\bar{\mathbf{y}}]$ a box of $\mathbb{R}^p$. If the CSP

$$\left\{\begin{array}{l} \mathbf{x} \in [\mathbf{x}], \mathbf{y} \in [\bar{\mathbf{y}}] \\ \mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0}, \\ a_{i,j} = \text{sign}(y_j - \bar{y}_j^-).\text{sign}(\bar{y}_j^+ - y_j).\frac{\partial f_i}{\partial y_j}(\mathbf{x},\mathbf{y}) \\ \text{rank}\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mp} \end{pmatrix} < m \end{array}\right.$$

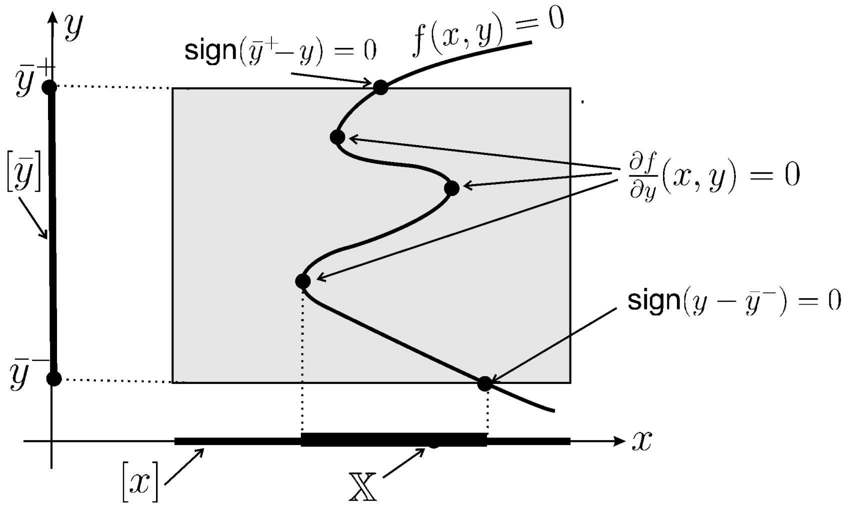
is inconsistent, then

$$[\mathbf{x}] \cap \partial\mathbb{X} = \emptyset.$$

Example : If $n = p = m = 1$, the theorem becomes

$$\begin{cases} x \in [x], y \in [\bar{y}] \\ f(x,y) = 0, \\ a = \text{sign}(y - \bar{y}^-).\text{sign}(\bar{y}^+ - y).\frac{\partial f}{\partial y}(x,y) \\ a = 0 \end{cases}$$

is inconsistent, then $[\mathbf{x}] \cap \partial\mathbb{X} = \emptyset$.



Summary : Given

$$\mathbb{X} \stackrel{\text{def}}{=} \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}.$$

We are able

- to prove that a box $[\mathbf{x}]$ is outside $\mathbb{X}$,

- to prove that a point $\mathbf{x}$ is inside $\mathbb{X}$.

- to prove that $[\mathbf{x}] \cap \partial\mathbb{X} = \emptyset$

A bisection algorithm to characterize $\mathbb{X}$ can thus be developed.

6 Polar diagram of a sailboat

The state equations of a sailboat are

$$
\left\{
\begin{array}{rcl}
\dot{x} & = & v \cos\theta, \\
\dot{y} & = & v \sin\theta - \beta V, \\
\dot{\theta} & = & \omega, \\
\dot{\delta}_s & = & u_1, \\
\dot{\delta}_r & = & u_2, \\
\dot{v} & = & \frac{f_s \sin\delta_s - f_r \sin\delta_r - \alpha_f v}{m}, \\
\dot{\omega} & = & \frac{(\ell - r_s \cos\delta_s) f_s - r_r \cos\delta_r f_r - \alpha_\theta \omega}{J}, \\
f_s & = & \alpha_s \left(V \cos(\theta + \delta_s) - v \sin\delta_s \right), \\
f_r & = & \alpha_r v \sin\delta_r.
\end{array}
\right.
$$

The set of feasible chosen input vectors is

$$
\begin{aligned}
\mathbb{W} = \{ \quad (\theta, v) \mid \quad & \exists(\delta_r, \delta_s) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]^{\times 2} \\
& (V \cos(\theta + \delta_s) - v \sin \delta_s) \sin \delta_s \\
& -\tfrac{\alpha_r}{\alpha_s} v \sin^2 \delta_r - \tfrac{\alpha_f}{\alpha_s} v && = 0 \\
& (1 - \cos \delta_s) \,. \\
& (V \cos(\theta + \delta_s) - v \sin \delta_s) \\
& -r_r \tfrac{\alpha_r}{\alpha_s} v \cos \delta_r \sin \delta_r && = 0 \quad \}
\end{aligned}
$$

With $V = 10, r_r = 2, \alpha_s = 500, \alpha_r = 300, \alpha_f = 60$, we get