LINEAR MATRIX INEQUALITIES FOR INTERVAL CONSTRAINT PROPAGATION

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I. MOTIVATIONS

Model : $y_{\mathsf{m}}(\mathbf{p}, t) = p_1 e^{-p_2 t}$.
Parameters : p_1, p_2.
Sampling times : $t_1, t_2, \dots, t_m$
Data bars : $[y_1^-, y_1^+], [y_2^-, y_2^+], \dots, [y_m^-, y_m^+]$

Feasible set for the parameters

$$\begin{aligned} \mathbb{S} = \{\mathbf{p} \in \mathbb{R}^2,\ & y_{\mathsf{m}}(\mathbf{p}, t_1) \in [y_1^-, y_1^+], \\ & \dots \\ & y_{\mathsf{m}}(\mathbf{p}, t_m) \in [y_m^-, y_m^+]\}. \end{aligned}$$

Presentation of the software **Setdemo**

As a conclusion,

- interval methods are fast when the number n of variables is small (here 2).

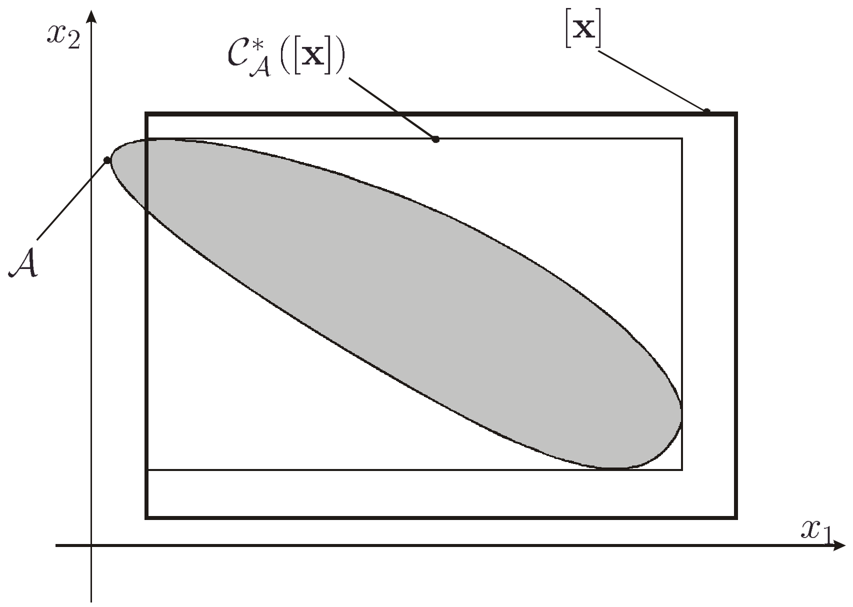
- when n is large, constraint propagation can be used to limit the number of bisections

II. PROJECTORS

A *constraint* $\mathcal{A}(x_1, \ldots, x_n)$ is a subset of $\mathbb{R}^n$.
A *projector* associated with $\mathcal{A}$ is a function

$$\mathcal{C}^*_{\mathcal{A}} : [\mathbf{x}] \rightarrow [[\mathbf{x}] \cap \mathcal{A}] \,.$$

A constraint $\mathcal{A}$ is *projectable* if a polynomial projector is available for $\mathcal{A}$.



Presentation of the solver **Proj2d**

Among projectable constraints, we have

- $acyclic\ constraints$ such as

$$\begin{aligned} x_1^2 + x_3 \sin(x_2) &\geq 0 \\ x_2 + \exp(x_4) &= 1 \end{aligned}$$

which can be projected using interval arithmetic,

- $linear\ constraints$ such as

$$\begin{aligned} 6x_1 - 4x_2 + 6x_4 + 2 &\geq 0 \\ 9x_1 + x_3 + 6x_4 - 5 &= 0 \end{aligned}$$

which can be projected using linear programming,

- *linear matrix inequalities* (LMI) such as

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & -4 \end{pmatrix} + x_1 \begin{pmatrix} -9 & 2 & 2 \\ 2 & 4 & 5 \\ 2 & 5 & 2 \end{pmatrix} + \ldots + x_n \begin{pmatrix} 1 & 7 & 3 \\ 7 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix} \succeq 0$$

which can be projected using convex optimization techniques.

III. DECOMPOSITION

Decompose the CSP to be solved into a conjunction of projectable constraints.

Example: The CSP

$$\mathcal{A} : \begin{cases} y_1 = x_1 + 2x_2 + x_3 \\ y_2 = x_2 + x_3 - \frac{x_1}{\sqrt{x_1^2 + x_2^2}^3} \end{cases}$$

can be decomposed into the 3 following projectable constraints:

$$\mathcal{A}_1 : \begin{cases} y_1 = x_1 + 2x_2 + x_3 \\ y_2 = x_2 + x_3 - z_1 \end{cases}$$

$$\mathcal{A}_2 : z_1 = x_1 / z_2$$

$$\mathcal{A}_3 : z_2 = \sqrt{x_1^2 + x_2^2}^3$$

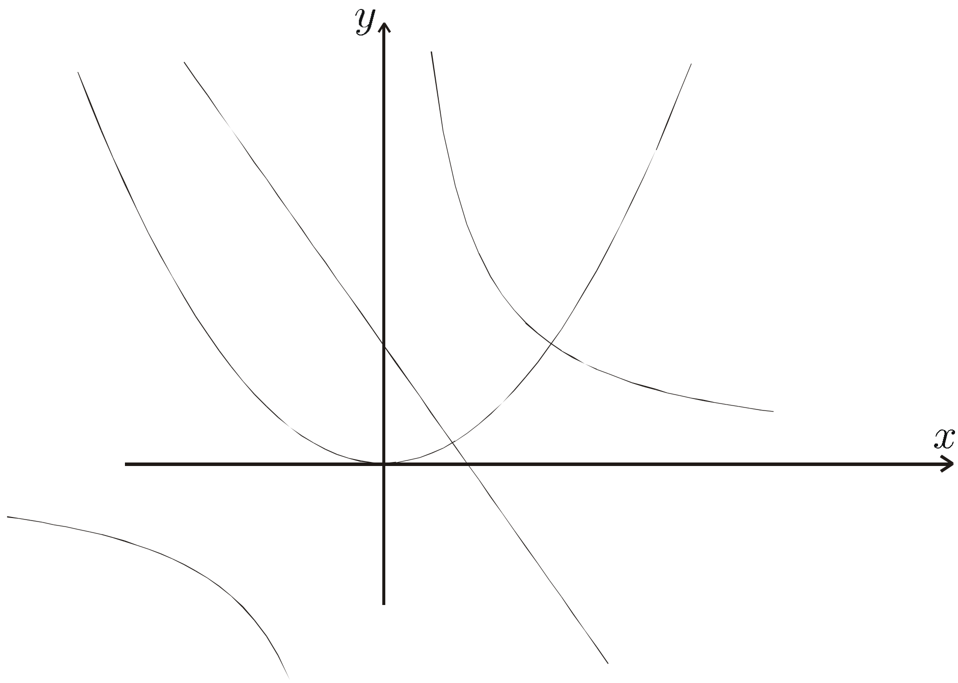
IV. PROPAGATION

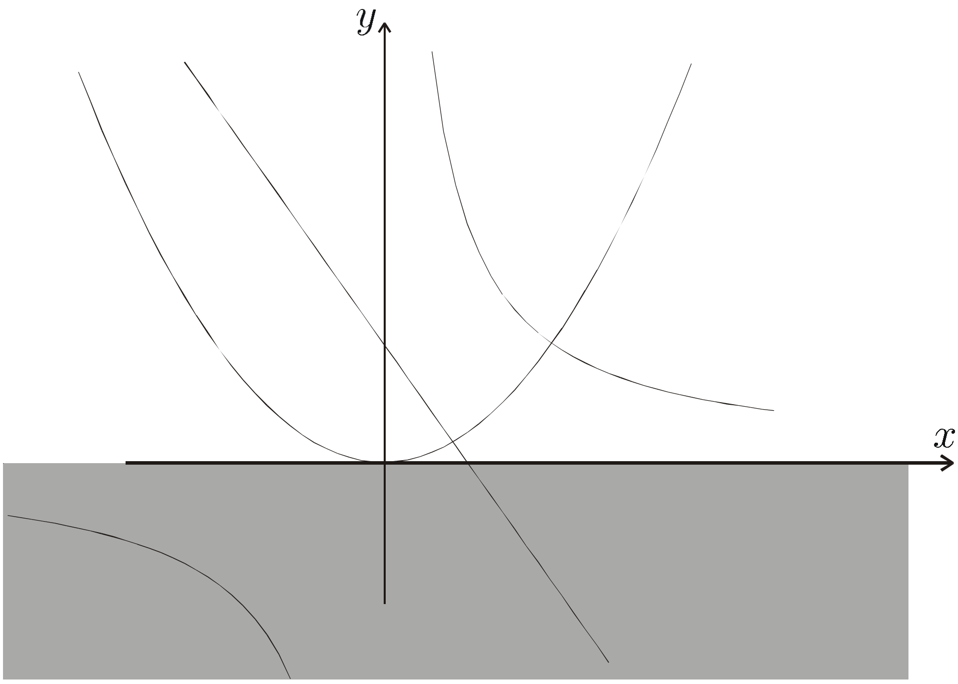
Consider the three following constraints

$$\begin{aligned} \mathcal{A}_1 &: y = x^2, \\ \mathcal{A}_2 &: xy = 1, \\ \mathcal{A}_3 &: y = -2x + 1. \end{aligned}$$

To each variable x, y, we give the domain $]-\infty, \infty[$.

$\underline{Constraint\ propagation}$ projects all constraints upto the equilibrium.


y
x



y
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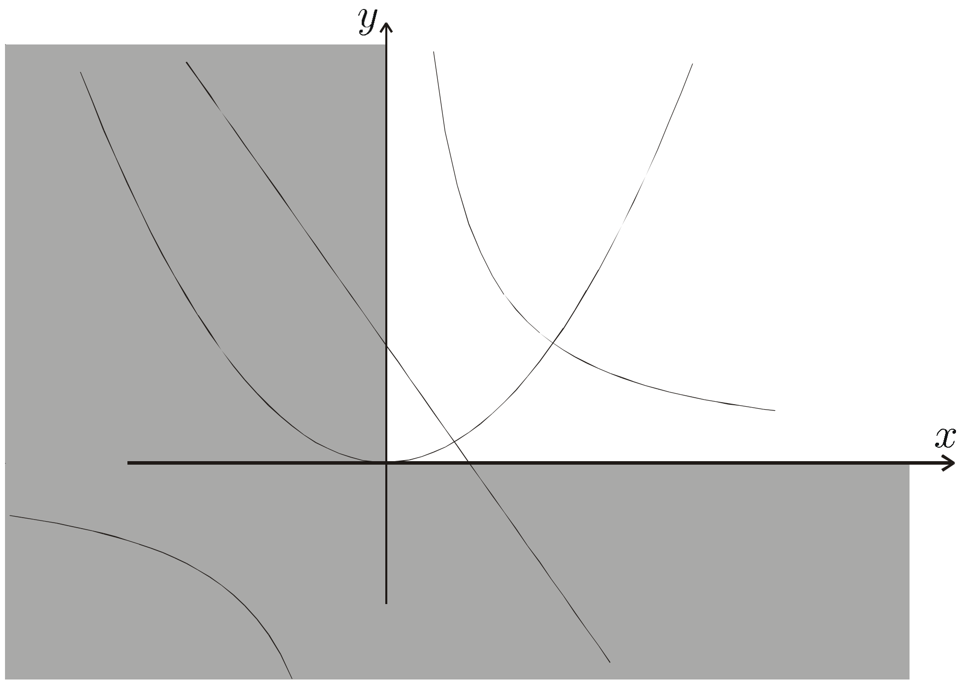
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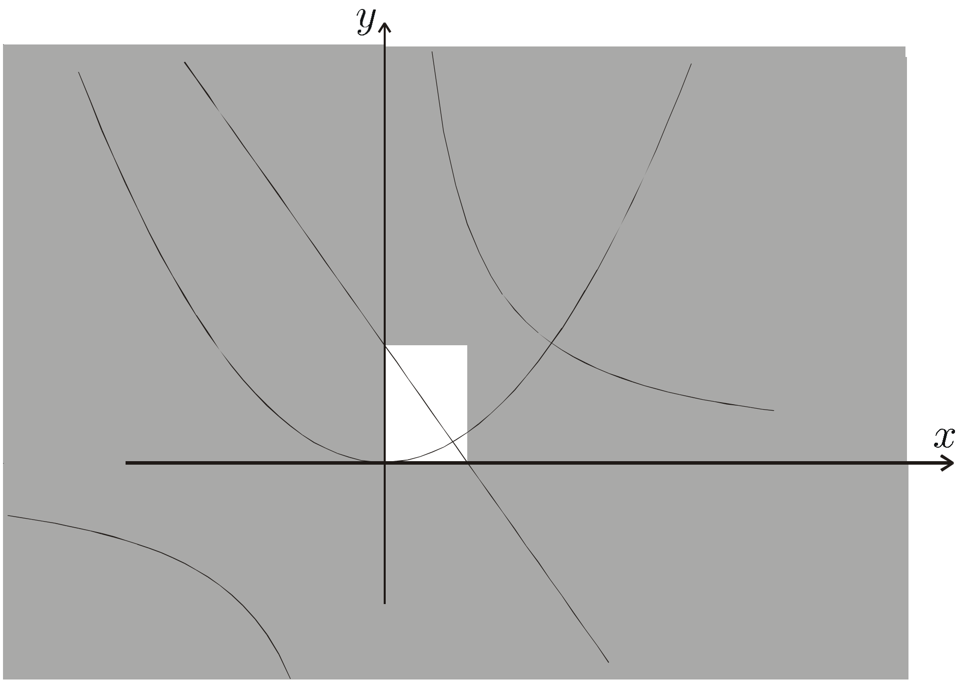
y
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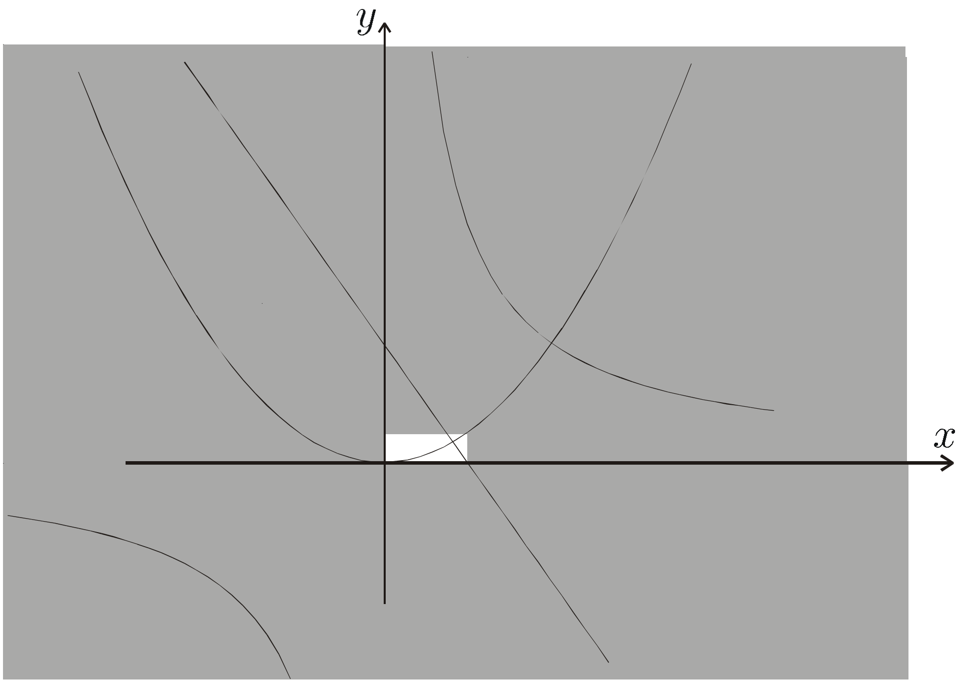


y
x

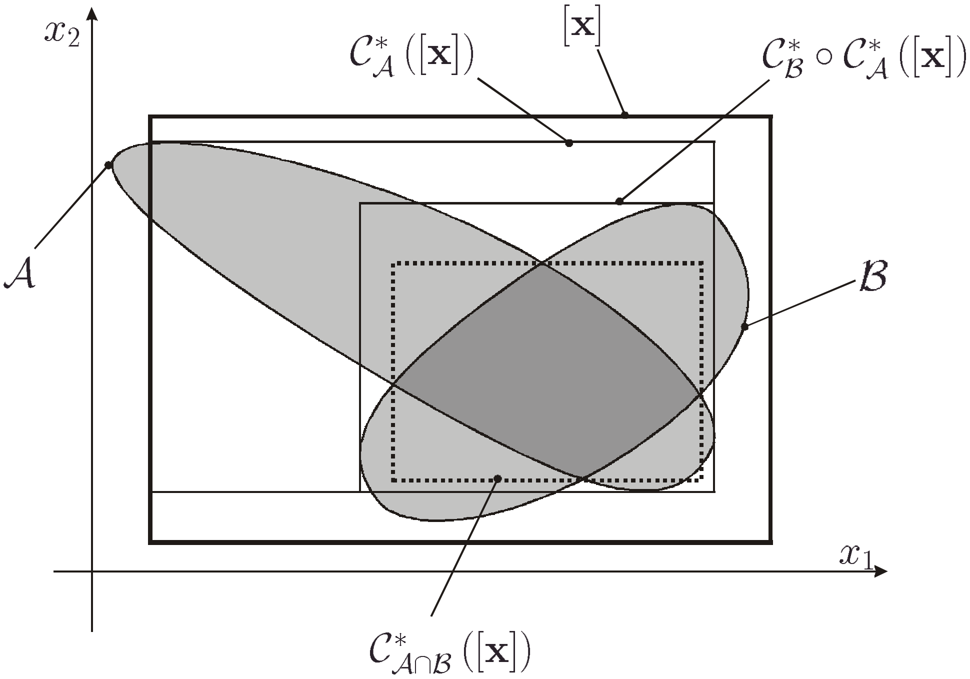
$$
\begin{aligned}
(C_1) \Rightarrow\ & y \in\ ]-\infty,\infty[^2 =\ [0,\infty[\\
(C_2) \Rightarrow\ & x \in\ 1/[0,\infty[= [0,\infty[\\
(C_3) \Rightarrow\ & y \in\ [0,\infty[\cap\ ((-2)\,.[0,\infty[+1) \\
& \qquad = [0,\infty[\cap\ (]-\infty,1]) = [0,1] \\
& x \in\ [0,\infty[\cap\ (-[0,1]/2 + 1/2) \\
& \qquad = [0,1/2] \\
(C_1) \Rightarrow\ & y \in\ [0,1] \cap [0,1/2]^2 = [0,1/4] \\
(C_2) \Rightarrow\ & x \in\ [0,1/2]\ \cap\ 1/[0,1/4] = \emptyset \\
& y \in\ [0,1/4] \cap 1/\emptyset = \emptyset
\end{aligned}
$$

V. LOCAL CONSISTENCY

If $\mathcal{C}^*_{\mathcal{A}}$ and $\mathcal{C}^*_{\mathcal{B}}$ are two projectors for two $\mathcal{A}$ and $\mathcal{B}$, the contractor

$$\mathcal{C}^*_{\mathcal{A}} \circ \mathcal{C}^*_{\mathcal{B}} \circ \mathcal{C}^*_{\mathcal{A}} \circ \mathcal{C}^*_{\mathcal{B}} \circ \mathcal{C}^*_{\mathcal{A}} \circ \mathcal{C}^*_{\mathcal{B}} \circ \dots$$

is not necessarily a projector for $\mathcal{A} \cap \mathcal{B}$.



x_2
$[\mathbf{x}]$
$\mathcal{C}^*_{\mathcal{A}}([\mathbf{x}])$
$\mathcal{C}^*_{\mathcal{B}} \circ \mathcal{C}^*_{\mathcal{A}}([\mathbf{x}])$
$\mathcal{A}$
$\mathcal{B}$
x_1
$\mathcal{C}^*_{\mathcal{A}\cap\mathcal{B}}([\mathbf{x}])$

Problem: Decrease the pessimism due to local consistency.

Solution: Increase the class of projectable constraints to avoid unnecessary decompositions.

VI. LINEAR MATRIX INEQUALITY

The constraint $C(x_1, x_2)$ given by

$$\begin{pmatrix} 3+4x_1+7x_2 & 4+5x_1-3x_2 & 6+2x_1+2x_2 \\ 4+5x_1-3x_2 & 2+5x_1+5x_2 & 9+15x_1+x_2 \\ 6+2x_1+2x_2 & 9+15x_1+x_2 & 1+10x_1+8x_2 \end{pmatrix} \succeq 0$$

or equivalently

$$\begin{pmatrix} 3 & 4 & 6 \\ 4 & 2 & 9 \\ 6 & 9 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 4 & 5 & 2 \\ 5 & 5 & 15 \\ 2 & 15 & 10 \end{pmatrix} + x_2 \begin{pmatrix} 7 & -3 & 2 \\ -3 & 5 & 1 \\ 3 & 1 & 8 \end{pmatrix} \succeq 0$$

is an LMI.

Linear constraints are LMIs

A set of linear inequalities is an LMI. For instance

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + b_1 & \geq 0 \\ a_{21}x_1 + a_{22}x_2 + b_2 & \geq 0 \end{cases}$$

is equivalent to the following LMI

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + b_1 & 0 \\ 0 & a_{21}x_1 + a_{22}x_2 + b_2 \end{pmatrix} \succeq 0,$$

i.e.,

$$\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} + x_1 \begin{pmatrix} a_{11} & 0 \\ 0 & a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} & 0 \\ 0 & a_{22} \end{pmatrix} \succeq 0.$$

Ellipsoids are LMI sets (Schur complement theorem).

$$3x_1^2 + 2x_2^2 - 2x_1x_2 \le 5 \Leftrightarrow \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 2 & 1 \\ x_2 & 1 & 3 \end{pmatrix} \succeq 0$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \succeq 0.$$

LMIs are involved in global optimization

Function :

$$f(\mathbf{x}) = 4x_1^2 - x_2 x_1^4 + x_1^6 + x_1^2 x_2^3 - 4x_2^2 + x_1 x_2^4$$

Gradient

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 6x_1 - 4x_2 x_1^3 + 6x_1^5 + 2x_1 x_2^3 + x_2^4 \\ -x_1^4 + 3x_1^2 x_2^2 - 8x_2 + 4x_1 x_2^3 \end{pmatrix}$$

Hessian matrix

$$\begin{pmatrix} 8 - 12x_1^2 x_2 + 30x_1^4 + 2x_2^3 & -4x_1^3 + 6x_1 x_2^2 + 4x_2^3 \\ -4x_1^3 + 6x_1 x_2^2 + 4x_2^3 & 6x_1^2 x_2 - 8 + 12x_1 x_2^2 \end{pmatrix}$$

Constraints

$$\begin{cases} 4x_1^2 - x_2x_1^4 + x_1^6 + x_1^2x_2^3 - 4x_2^2 + x_1x_2^4 \leq f^+ \\ 6x_1 - 4x_1^3x_2 + 6x_1^5 + 2x_1x_2^3 + x_2^4 = 0 \\ -x_1^4 + 3x_1^2x_2^2 - 8x_2 + 4x_1x_2^3 = 0 \\ \begin{pmatrix} 8 - 12x_1^2x_2 + 30x_1^4 + 2x_2^3 & -4x_1^3 + 6x_1x_2^2 + 4x_2^3 \\ -4x_1^3 + 6x_1x_2^2 + 4x_2^3 & 6x_1^2x_2 - 8 + 12x_1x_2^2 \end{pmatrix} \succeq 0 \end{cases}$$

Decomposition : The previous set of constraints can be decomposed into the following set of projectable constraints

$$\mathcal{A}_1 : \left\{ \begin{array}{c} a_1 = x_1^2,\ a_2 = x_1^3,\ a_3 = x_1^4,\ a_4 = x_1^5,\ a_5 = x_1^6, \\ b_1 = x_2^2,\ b_2 = x_2^3,\ b_3 = x_2^4, \end{array} \right.$$

$$\mathcal{A}_2 : c_1 = a_3 x_2, \quad c_2 = a_1 b_2, \quad c_4 = a_2 x_2, \quad c_6 = a_1 b_1,$$

$$\mathcal{A}_3 : c_8 = x_1 b_1, \quad c_3 = x_1 b_3, \quad c_7 = a_1 x_2, \quad c_5 = x_1 b_2,$$

$$\mathcal{A}_4 : \left\{ \begin{array}{c} 4a_1 - c_1 + a_5 + c_2 - 4b_1 + c_3 \leq f^+ \\ 6x_1 - 4c_4 + 6a_4 + 2c_5 + b_3 = 0 \\ -a_3 + 3c_6 - 8x_2 + 4c_5 = 0 \end{array} \right.$$

$$\mathcal{A}_5 : \begin{pmatrix} 8 - 12c_7 + 30a_3 + 2b_2 & -4a_2 + 6c_8 + 4b_2 \\ -4a_2 + 6c_8 + 4b_2 & 6c_7 - 8 + 12c_8 \end{pmatrix} \succeq 0$$

Better : The constraint given by

$$\begin{cases} a_1 \geq x_1^2, \quad b_1 \geq x_2^2, \\ \mathcal{A}_4 : \begin{cases} 4a_1 - c_1 + a_5 + c_2 - 4b_1 + c_3 \leq f^+ \\ 6x_1 - 4c_4 + 6a_4 + 2c_5 + b_3 = 0 \\ -a_3 + 3c_6 - 8x_2 + 4c_5 = 0 \end{cases} \\ \mathcal{A}_5 : \begin{pmatrix} 8 - 12c_7 + 30a_3 + 2b_2 & -4a_2 + 6c_8 + 4b_2 \\ -4a_2 + 6c_8 + 4b_2 & 6c_7 - 8 + 12c_8 \end{pmatrix} \succeq 0 \end{cases}$$

is convex (better, it is an LMI). Its projector can be built using convex techniques.

VI. PROJECTION OF LMIs

A matrix $\mathbf{A}$ of $\mathcal{S}^n$ is $positive\ semidefinite$ (PSD), denoted by $\mathbf{A} \succeq 0$, if

$$\forall \mathbf{z} \in \mathbb{R}^n, \mathbf{z}^\mathsf{T} \mathbf{A} \mathbf{z} \geq 0.$$

Theorem: The set of PSD matrices is convex.

Proof:

$$\begin{aligned}
\mathcal{S}_+^n &= \left\{\mathbf{A} \in \mathcal{S}^n | \forall \mathbf{z} \in \mathbb{R}^n, \mathbf{z}^\mathsf{T}\mathbf{A}\mathbf{z} \geq 0\right\} \\
&= \bigcap_{\mathbf{z}\in\mathbb{R}^n} \left\{\mathbf{A} \in \mathcal{S}^n \mid \mathbf{z}^\mathsf{T}\mathbf{A}\mathbf{z} \geq 0\right\} \\
&= \bigcap_{\mathbf{z}\in\mathbb{R}^n} \left\{\mathbf{A} \in \mathcal{S}^n \mid \sum_{i,j} z_i z_j a_{ij} \geq 0\right\}.
\end{aligned}$$

For instance, the constraint $\mathcal{C}\left(x_1, x_2\right)$ given by

$$\begin{pmatrix} 3+x_1+7x_2 & 4+5x_1-x_2 & 6+2x_1+x_2 \\ 4+5x_1-x_2 & 2+5x_1+5x_2 & 9+5x_1+x_2 \\ 6+2x_1+x_2 & 9+5x_1+x_2 & 4+x_1+8x_2 \end{pmatrix} \succeq 0$$

is convex, because it is the reciprocal image of the convex set $\mathcal{S}_+^3$ by the $affine$ function:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} 3+x_1+7x_2 & 4+5x_1-x_2 & 6+2x_1+x_2 \\ 4+5x_1-x_2 & 2+5x_1+5x_2 & 9+5x_1+x_2 \\ 6+2x_1+x_2 & 9+5x_1+x_2 & 4+x_1+8x_2 \end{pmatrix}$$

The projection of an LMI with m variables requires $2m$ LMI optimizations.

Each of them can be performed by SeDuMi which has a worst-case complexity of

$$O\left(\left(n^{3.5}m + n^{2.5}m^2\right)\log\left(\frac{1}{\varepsilon}\right)\right).$$