Thick separators

L. Jaulin, B. Desrochers
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Thick sets

A *thin set* is a subset of $\mathbb{R}^n$.
A *thick set* $[\![\mathbb{X}]\!]$ of $\mathbb{R}^n$ is an interval of $(\mathscr{P}(\mathbb{R}^n), \subset)$.

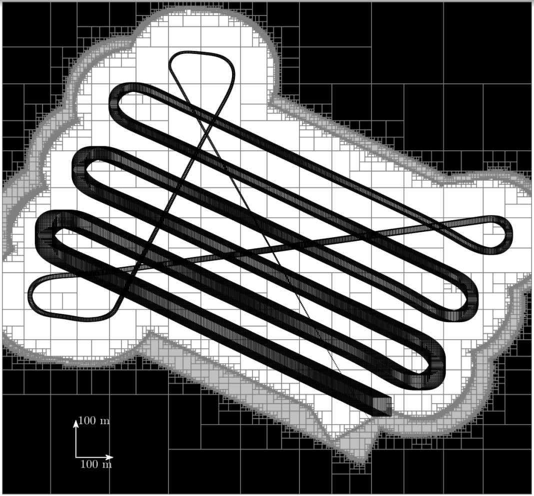
$$[\![\mathbb{X}]\!] = \qquad [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!] = \{\mathbb{X} \in \mathscr{P}(\mathbb{R}^n) | \mathbb{X}^{\subset} \subset \mathbb{X} \subset \mathbb{X}^{\supset}\}.$$

A thickset $[\![\mathbb{X}]\!]$ is also the partition $\{\mathbb{X}^{in}, \mathbb{X}^{?}, \mathbb{X}^{out}\}$, where

$$\begin{array}{rcl} \mathbb{X}^{in} & = & \mathbb{X}^{\subset} \\ \mathbb{X}^{?} & = & \mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset} \\ \mathbb{X}^{out} & = & \overline{\mathbb{Z}^{\supset}}. \end{array}$$

The subset $\mathbb{X}^{?}$ is called the *penumbra*.

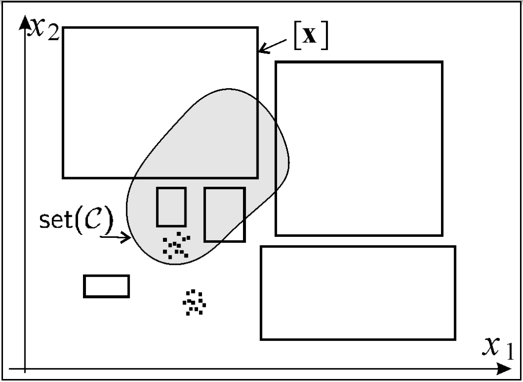
Redermor DGA-TN, Brest

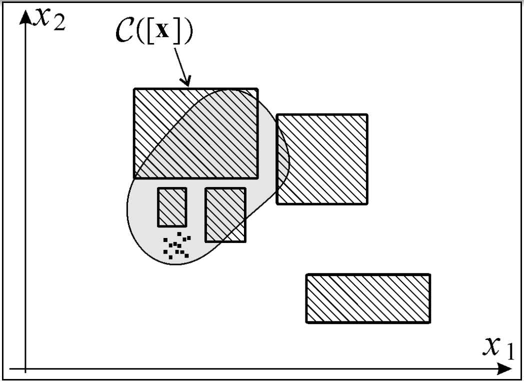


Penumbra

Contractors

$$\mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \qquad \text{(contractance)}$$
$$[\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}]) \qquad \text{(monotonicity)}$$


x_2
[x]
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

Properties

Inclusion

$$\mathscr{C}_1 \subset \mathscr{C}_2 \Leftrightarrow \forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathscr{C}_1([\mathbf{x}]) \subset \mathscr{C}_2([\mathbf{x}]).$$

A set $\mathbb{S}$ is *consistent* with $\mathscr{C}$ (we write $\mathbb{S} \sim \mathscr{C}$) if

$$\mathscr{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}.$$

$\mathscr{C}$ is *minimal* if

$$\left.\begin{array}{c} \mathbb{S} \sim \mathscr{C} \\ \mathbb{S} \sim \mathscr{C}_1 \end{array}\right\} \Rightarrow \mathscr{C} \subset \mathscr{C}_1.$$

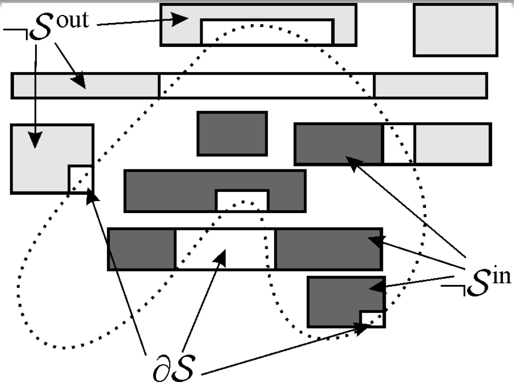
Separators

A *separator* $\mathscr{S}$ is pair of contractors $\left\{\mathscr{S}^{\text{in}},\mathscr{S}^{\text{out}}\right\}$ such that

$$\mathscr{S}^{\text{in}}([\mathbf{x}])\cup\mathscr{S}^{\text{out}}([\mathbf{x}])=[\mathbf{x}] \qquad \text{(complementarity).}$$

A set $\mathbb{S}$ is *consistent* with $\mathscr{S}$ (we write $\mathbb{S} \sim \mathscr{S}$), if

$$\mathbb{S} \sim \mathscr{S}^{\text{out}} \text{ and } \overline{\mathbb{S}} \sim \mathscr{S}^{\text{in}}.$$


$\neg\mathcal{S}^{\text{out}}$
$\neg\mathcal{S}^{\text{in}}$
$\partial\mathcal{S}$

Properties

Inclusion

$$\mathscr{S}_1 \subset \mathscr{S}_2 \Leftrightarrow \mathscr{S}_1^{\text{in}} \subset \mathscr{S}_2^{\text{in}} \text{ and } \mathscr{S}_1^{\text{out}} \subset \mathscr{S}_2^{\text{out}}.$$

Here $\subset$ means *more accurate.*

$\mathscr{S}$ is *minimal* if

$$\mathscr{S}_1 \subset \mathscr{S} \Rightarrow \mathscr{S}_1 = \mathscr{S}.$$

i.e., if $\mathscr{S}^{\text{in}}$ and $\mathscr{S}^{\text{out}}$ are both minimal.

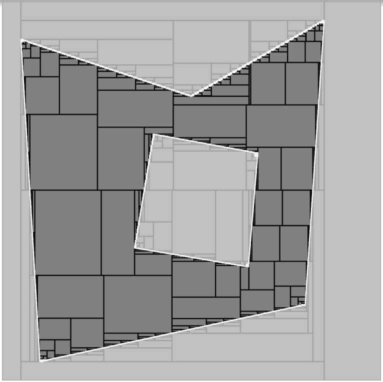
Algebra

If $\mathscr{S}_i = \left\{\mathscr{S}_i^{\text{in}}, \mathscr{S}_i^{\text{out}}\right\}, i \geq 1$, are separators, we define

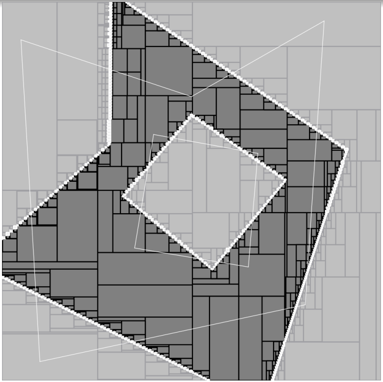
$$
\begin{array}{rcll}
\mathscr{S}_1 \cap \mathscr{S}_2 & = & \left\{\mathscr{S}_1^{\text{in}} \cup \mathscr{S}_2^{\text{in}}, \mathscr{S}_1^{\text{out}} \cap \mathscr{S}_2^{\text{out}}\right\} & \text{(intersection)} \\
\mathscr{S}_1 \cup \mathscr{S}_2 & = & \left\{\mathscr{S}_1^{\text{in}} \cap \mathscr{S}_2^{\text{in}}, \mathscr{S}_1^{\text{out}} \cup \mathscr{S}_2^{\text{out}}\right\} & \text{(union)} \\
\mathscr{S}_1 \backslash \mathscr{S}_2 & = & \mathscr{S}_1 \cap \overline{\mathscr{S}_2}. & \text{(difference)}
\end{array}
$$

Theorem. If $\mathbb{S}_i$ are subsets of $\mathbb{R}^n$, we have

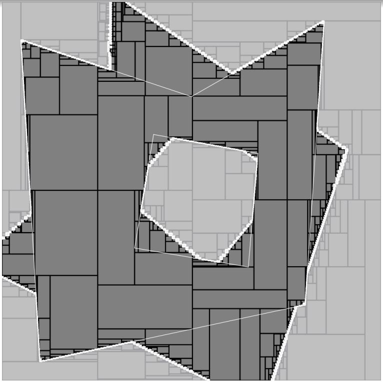
$$\begin{array}{cccc}
\text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathscr{S}_1 \cap \mathscr{S}_2 \\
\text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathscr{S}_1 \cup \mathscr{S}_2 \\
\text{(iii)} & \overline{\mathbb{S}}_i & \sim & \overline{\mathscr{S}_i} \\
\text{(iv)} & \mathbb{S}_i & \sim & \mathscr{S}_i^k,\ k \geq 0 \\
\text{(vi)} & \mathbb{S}_1 \backslash \mathbb{S}_2 & \sim & \mathscr{S}_1 \backslash \mathscr{S}_2.
\end{array}$$



Set $\mathbb{M}$



$$\mathrm{Rot}(\mathbb{M})$$



$\mathrm{Rot}(\mathbb{M}) \cup \mathbb{M}$

Thick separators

A *thick separator* $[\![\mathscr{S}]\!]$ for $[\![\mathbb{X}]\!]$ is a 3-uple of contractors $\{\mathscr{S}^{in}, \mathscr{S}^{?}, \mathscr{S}^{out}\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$

$$\begin{array}{c} \mathscr{S}^{in}([\mathbf{x}]) \cap \mathbb{X}^{in} = [\mathbf{x}] \cap \mathbb{X}^{in} \\ \mathscr{S}^{?}([\mathbf{x}]) \cap \mathbb{X}^{?} = [\mathbf{x}] \cap \mathbb{X}^{?} \\ \mathscr{S}^{out}([\mathbf{x}]) \cap \mathbb{X}^{out} = [\mathbf{x}] \cap \mathbb{X}^{out} \end{array}$$

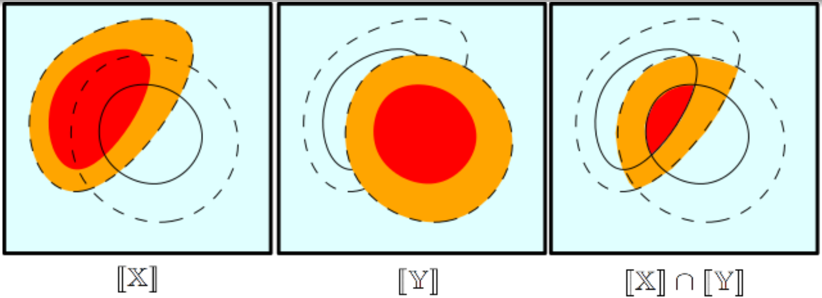
Algebra

Intersection. Consider two thick separators $[\![\mathscr{S}_{\mathbb{X}}]\!] = \{\mathscr{S}_{\mathbb{X}}^{in}, \mathscr{S}_{\mathbb{X}}^{?}, \mathscr{S}_{\mathbb{X}}^{out}\}$ and $[\![\mathscr{S}_{\mathbb{Y}}]\!] = \{\mathscr{S}_{\mathbb{Y}}^{in}, \mathscr{S}_{\mathbb{Y}}^{?}, \mathscr{S}_{\mathbb{Y}}^{out}\}$. A thick separator $[\![\mathscr{S}_{\mathbb{Z}}]\!] = \{\mathscr{S}_{\mathbb{Z}}^{in}, \mathscr{S}_{\mathbb{Z}}^{?}, \mathscr{S}_{\mathbb{Z}}^{out}\}$ for

$$[\![\mathbb{Z}]\!] = [\![\mathbb{Z}^{\subset}, \mathbb{Z}^{\supset}]\!] = [\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!]$$

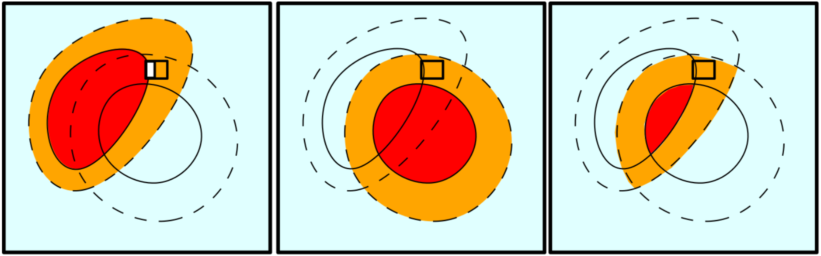
is

$$\left\{\mathscr{S}_{\mathbb{X}}^{in} \cap \mathscr{S}_{\mathbb{Y}}^{in}, \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{in}\right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{?}\right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{in} \cap \mathscr{S}_{\mathbb{Y}}^{?}\right), \mathscr{S}_{\mathbb{X}}^{out} \sqcup \mathscr{S}_{\mathbb{Y}}^{out}\right\}.$$



Intersection of two thick sets

Illustration. Take one box [x] .



$[\![\mathbb{X}]\!]$ $\quad$ $[\![\mathbb{Y}]\!]$ $\quad$ $[\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!]$

We have

$$[\![\mathscr{S}_{\mathbb{X}}]\!]([\mathsf{x}]) = \left\{\mathscr{S}_{\mathbb{X}}^{in}, \mathscr{S}_{\mathbb{X}}^{?}, \mathscr{S}_{\mathbb{X}}^{out}\right\}([\mathsf{x}]) = \{[\mathsf{a}], [\mathsf{x}], \emptyset\}$$

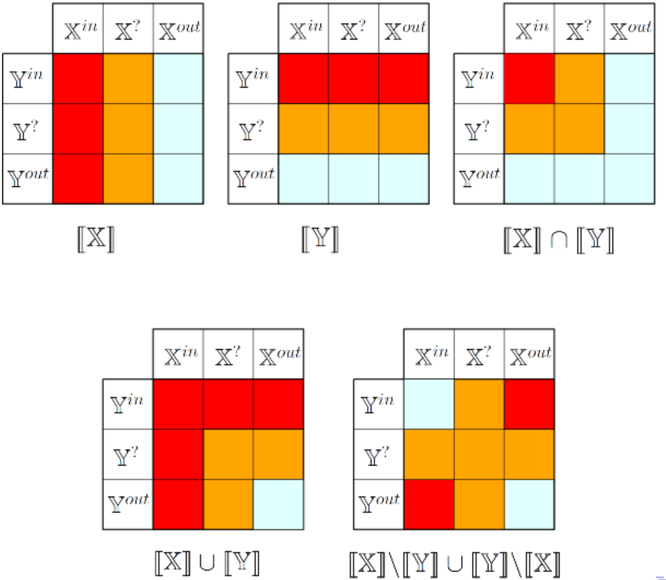
where [a] the white box. Moreover,

$$[\![\mathscr{S}_{\mathbb{Y}}]\!]([\mathsf{x}]) = \left\{\mathscr{S}_{\mathbb{Y}}^{in}, \mathscr{S}_{\mathbb{Y}}^{?}, \mathscr{S}_{\mathbb{Y}}^{out}\right\}([\mathsf{x}]) = \{\emptyset, [\mathsf{x}], \emptyset\}.$$

$$
\begin{array}{rcl}
[\![\mathscr{S}_{\mathbb{Z}}]\!] & = & \left\{\mathscr{S}_{\mathbb{Z}}^{in}, \mathscr{S}_{\mathbb{Z}}^{?}, \mathscr{S}_{\mathbb{Z}}^{out}\right\}([\mathbf{x}]) \\
& = & \{ \quad \mathscr{S}_{\mathbb{X}}^{in} \cap \mathscr{S}_{\mathbb{Y}}^{in}([\mathbf{x}]), \\
& = & \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{in}\right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{?}\right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{in} \cap \mathscr{S}_{\mathbb{Y}}^{?}\right)([\mathbf{x}]), \\
& = & \mathscr{S}_{\mathbb{X}}^{out} \sqcup \mathscr{S}_{\mathbb{Y}}^{out}([\mathbf{x}]) \quad \} \\
& = & \{ \quad [\mathbf{a}] \cap \emptyset, ([\mathbf{x}] \cap \emptyset) \sqcup ([\mathbf{x}] \cap [\mathbf{x}]) \sqcup ([\mathbf{a}] \cap [\mathbf{x}]), \emptyset \sqcup \emptyset \quad \} \\
& = & \{\emptyset, [\mathbf{x}], \emptyset\}
\end{array}
$$

We conclude that $[\mathbf{x}] \subset \mathbb{Z}^{in}$.

Using Karnaugh maps


$\mathbb{X}^{in}$ $\mathbb{X}^{?}$ $\mathbb{X}^{out}$
$\mathbb{Y}^{in}$ $\mathbb{Y}^{?}$ $\mathbb{Y}^{out}$
$[\![\mathbb{X}]\!]$
$[\![\mathbb{Y}]\!]$
$[\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!]$
$[\![\mathbb{X}]\!] \cup [\![\mathbb{Y}]\!]$
$[\![\mathbb{X}]\!] \backslash [\![\mathbb{Y}]\!] \cup [\![\mathbb{Y}]\!] \backslash [\![\mathbb{X}]\!]$

Union. For

$$[\![\mathbb{Z}]\!] = [\![\mathbb{X}]\!] \cup [\![\mathbb{Y}]\!],$$

we read from the Karnaugh map

$$\begin{array}{rcc} \mathbb{Z}^{in} & = & \mathbb{X}^{in} \cup \mathbb{Y}^{in} \\ \mathbb{Z}^{?} & = & \left(\mathbb{X}^{?} \cap \mathbb{Y}^{out}\right) \cup \left(\mathbb{X}^{?} \cap \mathbb{Y}^{?}\right) \cup \left(\mathbb{X}^{out} \cap \mathbb{Y}^{?}\right) \\ \mathbb{Z}^{out} & = & \mathbb{X}^{out} \cap \mathbb{Y}^{out}. \end{array}$$

A thick separator $[\![\mathscr{S}_{\mathbb{Z}}]\!] = \left\{\mathscr{S}_{\mathbb{Z}}^{in}, \mathscr{S}_{\mathbb{Z}}^{?}, \mathscr{S}_{\mathbb{Z}}^{out}\right\}$ for $[\![\mathbb{Z}]\!]$ is

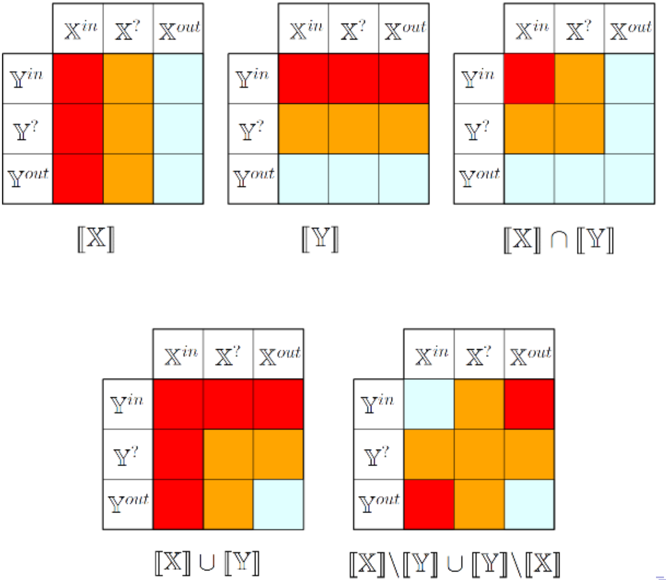
$$\left\{\mathscr{S}_{\mathbb{X}}^{in} \sqcup \mathscr{S}_{\mathbb{Y}}^{in}, \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{out}\right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{?}\right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{out} \cap \mathscr{S}_{\mathbb{Y}}^{?}\right), \mathscr{S}_{\mathbb{X}}^{out} \cap \mathscr{S}_{\mathbb{Y}}^{out}\right\}$$

XOR. For

$$[\![\mathbb{Z}]\!] = [\![\mathbb{X}]\!] \oplus [\![\mathbb{Y}]\!] = [\![\mathbb{X}]\!] \backslash [\![\mathbb{Y}]\!] \cup [\![\mathbb{Y}]\!] \backslash [\![\mathbb{X}]\!],$$

we read

$$\begin{array}{rcl} \mathbb{Z}^{in} & = & \left(\mathbb{X}^{in} \cap \mathbb{Y}^{out}\right) \cup \left(\mathbb{X}^{out} \cap \mathbb{Y}^{in}\right) \\ \mathbb{Z}^{?} & = & \mathbb{X}^{?} \cup \mathbb{Y}^{?} \\ \mathbb{Z}^{out} & = & \left(\mathbb{X}^{in} \cap \mathbb{Y}^{in}\right) \cup \left(\mathbb{X}^{out} \cap \mathbb{Y}^{out}\right). \end{array}$$


$\mathbb{X}^{in}$ $\mathbb{X}^{?}$ $\mathbb{X}^{out}$
$\mathbb{Y}^{in}$ $\mathbb{Y}^{?}$ $\mathbb{Y}^{out}$
$[\![\mathbb{X}]\!]$
$[\![\mathbb{Y}]\!]$
$[\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!]$
$[\![\mathbb{X}]\!] \cup [\![\mathbb{Y}]\!]$
$[\![\mathbb{X}]\!] \backslash [\![\mathbb{Y}]\!] \cup [\![\mathbb{Y}]\!] \backslash [\![\mathbb{X}]\!]$

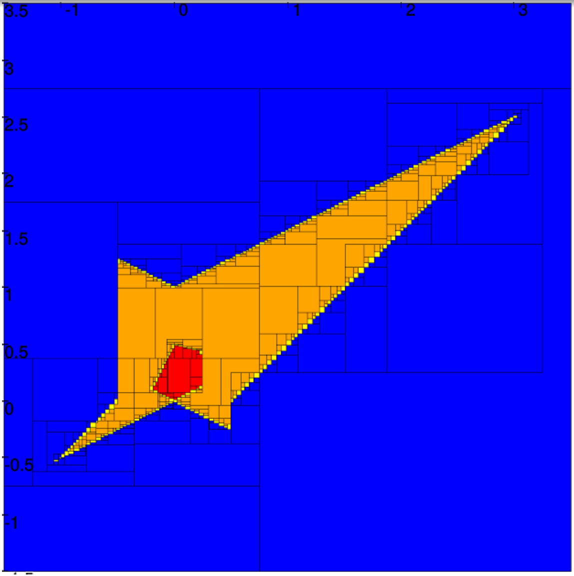
Therefore a thick separator for the thick set $[\![\mathbb{Z}]\!] = [\![\mathbb{X}]\!] \oplus [\![\mathbb{Y}]\!]$ is

$$\left\{ \mathscr{S}_{\mathbb{X}}^{in} \sqcup \mathscr{S}_{\mathbb{Y}}^{in}, \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{out} \right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{?} \cap \mathscr{S}_{\mathbb{Y}}^{?} \right) \sqcup \left(\mathscr{S}_{\mathbb{X}}^{out} \cap \mathscr{S}_{\mathbb{Y}}^{?} \right), \mathscr{S}_{\mathbb{X}}^{out} \cap \mathscr{S}_{\mathbb{Y}}^{out} \right\}.$$

Test-case

Example from [Kreinovich, Shary, 2016]:

$$\begin{cases} [2,4]\cdot x_1 + [-2,0]\cdot x_2 \in [-1,1] \\ [-1,1]\cdot x_1 + [2,4]\cdot x_2 \in [0,2] \end{cases}$$


-1
0
1
2
3
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1