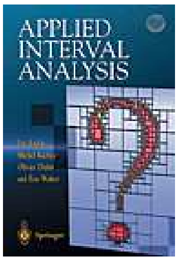
Linear and nonlinear control with intervals



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1 What is control theory ?

Consider one system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

where $\mathbf{x}$ is the state and $\mathbf{u}$ is the control.

Control problem: Find a controller

$$\mathbf{u} = \mathbf{r}(\mathbf{x}, \mathbf{w}),$$

where $\mathbf{w}$ is setpoint, such that the closed loop system behaves as desired.

2 What is interval analysis ?

Problem. Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\operatorname{abs}([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned}[f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ &\quad + \sin [x_1] \cdot \sin [x_2] + 2.\end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

3 Stability of linear systems

Linear system

$$a_n y^{(n)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1 \dot{u} + b_0 u.$$

The system is *stable* iff all roots of

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0.$$

are in $\mathbb{C}^-$.

Routh table

a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\dots$	0	0
a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\dots$	0	0
b_1	b_2	b_3			0	0
c_1	c_2	c_3			0	0
$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$

with

$$
\begin{array}{ccc}
b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} & b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} & \dots \\
c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} & c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} & \dots \\
\dots & \vdots &
\end{array}
$$

The roots of $P(s)$ are in $\mathbb{C}^-$ if all entries on the left column have the same sign.

Example. One motorbike where θ is the handlebar angle and ϕ is the roll:

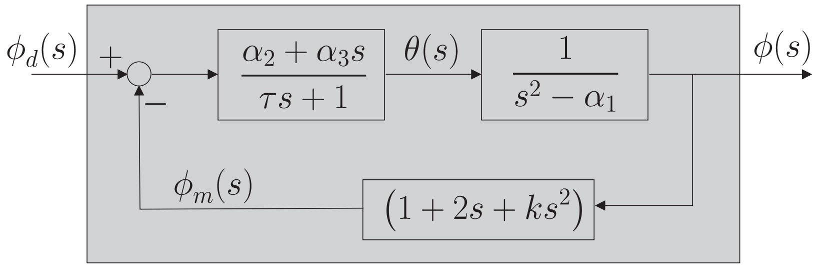
$$\phi(s) = \frac{1}{s^2 - \alpha_1}\theta(s)$$

Controller

$$\theta(s) = \frac{\alpha_2 + \alpha_3 s}{\tau s + 1} \left(\phi_d(s) - \phi_m(s)\right)$$

Sensor:

$$\phi_m(s) = \left(1 + 2s + ks^2\right) \phi(s)$$


$\phi_d(s)$
+
−
$\frac{\alpha_2+\alpha_3 s}{\tau s+1}$
$\theta(s)$
$\frac{1}{s^2-\alpha_1}$
$\phi(s)$
$\phi_m(s)$
$\left(1+2s+ks^2\right)$

We have

$$\phi(s) = \frac{\alpha_2 + \alpha_3 s}{\left(s^2 - \alpha_1\right)(\tau s + 1) + (\alpha_2 + \alpha_3 s)\left(1 + 2s + ks^2\right)}\phi_d(s)$$

The characteristic polynomial is

$$\begin{aligned}
& \left(s^2 - \alpha_1\right)(\tau s + 1) + (\alpha_2 + \alpha_3 s)\left(1 + 2s + ks^2\right) \\
= \; & a_3 s^3 + a_2 s^2 + a_1 s + a_0
\end{aligned}$$

with

$$\begin{aligned}
a_3 &= \tau + \alpha_3 k \\
a_2 &= \alpha_2 k + 2\alpha_3 + 1 \\
a_1 &= \alpha_3 - \alpha_1 \tau + 2\alpha_2 \\
a_0 &= -\alpha_1 + \alpha_2.
\end{aligned}$$

The Routh table is:

a_3	a_1	
a_2	a_0	(1)
$\frac{a_2a_1 - a_3a_0}{a_2}$	0	
a_0	0	

The system is stable if $a_3, a_2, \frac{a_2a_1-a_3a_0}{a_2}$, a_0 have the same sign.

For

$$\begin{aligned}\alpha_1 &\in [8.8, 9.2]\,, \alpha_2 \in [2.8, 3.2]\,, \alpha_3 \in [0.8, 1.2]\,,\\ \tau &\in [1.8, 2.2]\,, k \in [-3.2, -2.8]\end{aligned}$$

we get the robust stability of the closed loop system.

4 Stability domains

The stability domain $\mathbb{S}_{\mathsf{p}}$ of

$$P(s,\mathbf{p}) = s^n + a_{n-1}(\mathbf{p})s^{n-1} + \ldots + a_1(\mathbf{p})s + a_0(\mathbf{p})$$

is the set of all $\mathbf{p}$ such that $P(s,\mathbf{p})$ is stable.

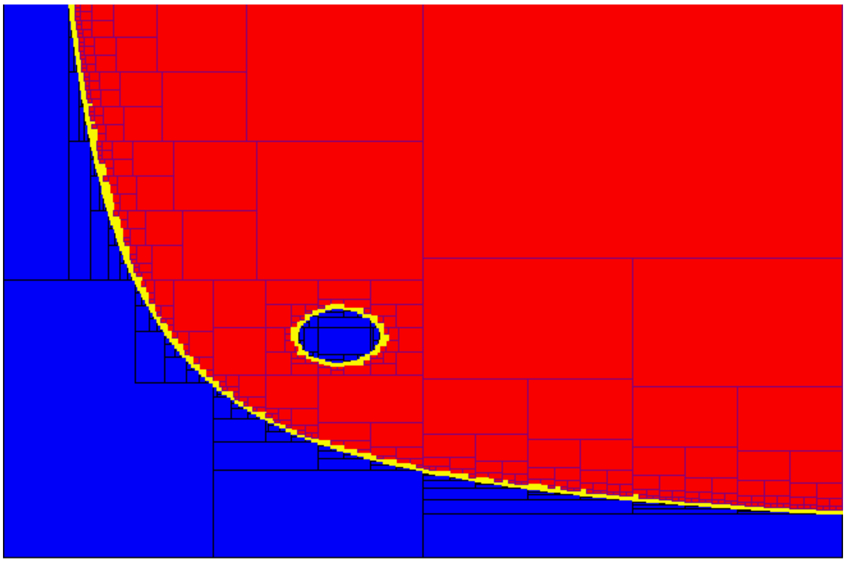
$$\mathbb{S}_{\mathsf{p}} \triangleq \{\mathbf{p} \in \mathbb{R}^{n_{\mathsf{p}}} \mid \mathbf{r}(\mathbf{p}) > \mathbf{0}\} = \mathbf{r}^{-1}\left(]0, +\infty[^{\times n}\right).$$

Example. Consider

$$
\begin{aligned}
P(s,\mathbf{p}) &= s^3 + (p_1+p_2+2)s^2 + (p_1+p_2+2)s \\
&\quad + 2p_1p_2 + 6p_1 + 6p_2 + 2.25.
\end{aligned}
$$

The Routh function is

$$
\mathbf{r}(\mathbf{p}) = \begin{pmatrix} p_1+p_2+2 \\ (p_1-1)^2+(p_2-1)^2-0.25 \\ 2(p_1+3)(p_2+3)-16+0.25 \end{pmatrix}.
$$



5 Interval polynomials

Kharitonov Theorem. The interval polynomial

$$[\mathbf{a}] = [\underline{a}_n, \overline{a}_n]s^n \times \cdots \times [\underline{a}_1, \overline{a}_1]s \times [\underline{a}_0, \overline{a}_0]$$

is robustly stable iff the four polynomials

$$\begin{array}{l}
\underline{a}_n s^n + \underline{a}_{n-1} s^{n-1} + \overline{a}_{n-2} s^{n-2} + \overline{a}_{n-3} s^{n-3} + \underline{a}_{n-4} s^{n-4} + \cdots \\
\overline{a}_n s^n + \underline{a}_{n-1} s^{n-1} + \underline{a}_{n-2} s^{n-2} + \overline{a}_{n-3} s^{n-3} + \overline{a}_{n-4} s^{n-4} + \cdots \\
\overline{a}_n s^n + \overline{a}_{n-1} s^{n-1} + \underline{a}_{n-2} s^{n-2} + \underline{a}_{n-3} s^{n-3} + \overline{a}_{n-4} s^{n-4} + \cdots \\
\underline{a}_n s^n + \overline{a}_{n-1} s^{n-1} + \overline{a}_{n-2} s^{n-2} + \underline{a}_{n-3} s^{n-3} + \underline{a}_{n-4} s^{n-4} + \cdots
\end{array}$$

are stable.

The family of polynomials

$$\mathbb{A} = \{p_5 s^4 + (p_4 + \cos^2(p_3))s^3 + 2p_1 s^2 + p_2\sqrt{p_4}s + p_1 \mid p_1 \in 6 \pm 1,\ p_2 \in [3,4],\ p_3 \in \pm\tfrac{\pi}{4},\ p_4 \in [1,2],\ p_5 \in [1,2$$

is a subset of the interval polynomial

$$[\mathbf{a}] = [1,2]s^4 + [3/2,3]s^3 + [10,14]s^2 + [3,4\sqrt{2}]s + [5,7].$$

The Kharitonov polynomials associated with $[\mathbf{a}]$ are

$$\begin{aligned}
K_1(s) &= s^4 + \tfrac{3}{2}s^3 + 14s^2 + 4\sqrt{2}s + 5, \\
K_2(s) &= 2s^4 + \tfrac{3}{2}s^3 + 10s^2 + 4\sqrt{2}s + 7, \\
K_3(s) &= 2s^4 + 3s^3 + 10s^2 + 3s + 7, \\
K_4(s) &= s^4 + 3s^3 + 14s^2 + 3s + 5.
\end{aligned} \tag{2}$$

Since all $K_i(s)$ are stable, $[\mathbf{a}]$ is robustly stable, and so is $\mathbb{A}$.

6 Control of a sailboat

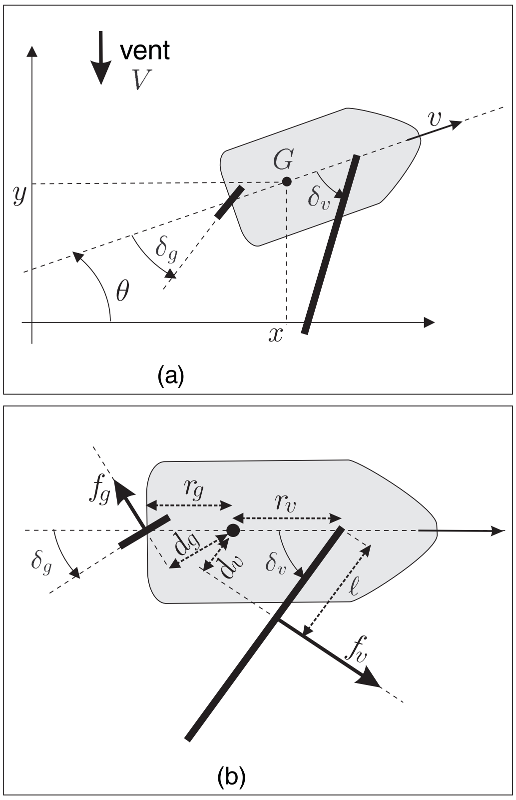
(Collaboration with P. Herrero)

A robot

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \mathbf{y} & = & \mathbf{g}\left(\mathbf{x}\right) \end{array} \right.$$

Set of feasible outputs

$$\mathbb{Y} = \{\mathbf{y} \mid \exists \left(\mathbf{x},\mathbf{u}\right), \ \mathbf{f}(\mathbf{x},\mathbf{u}) = \mathbf{0}, \mathbf{y} = \mathbf{g}\left(\mathbf{x}\right)\}.$$



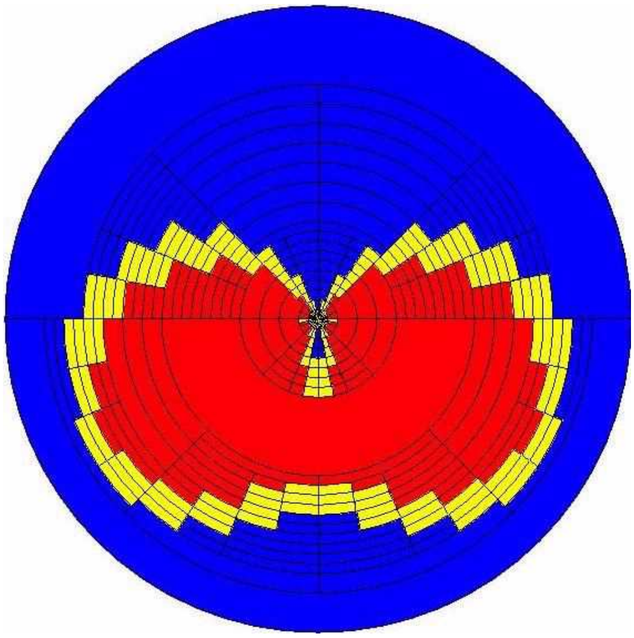
(a)

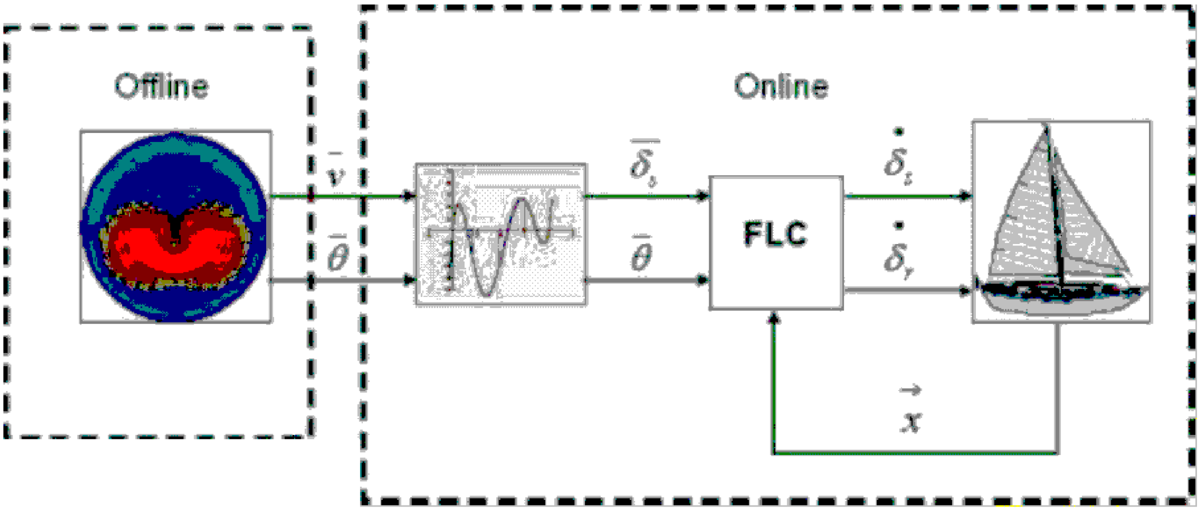
(b)

$$
\begin{cases}
\dot{x} &= v\cos\theta, \\
\dot{y} &= v\sin\theta - \beta V, \\
\dot{\theta} &= \omega, \\
\dot{\delta}_s &= u_1, \\
\dot{\delta}_r &= u_2, \\
\dot{v} &= \frac{f_s\sin\delta_s - f_r\sin\delta_r - \alpha_f v}{m}, \\
\dot{\omega} &= \frac{(\ell - r_s\cos\delta_s)f_s - r_r\cos\delta_r f_r - \alpha_\theta\omega}{J}, \\
f_s &= \alpha_s\left(V\cos\left(\theta + \delta_s\right) - v\sin\delta_s\right), \\
f_r &= \alpha_r v\sin\delta_r.
\end{cases}
$$

Polar speed diagram of a sailboat.

$$\mathbb{W} = \{ \quad (\theta, v) \mid \quad \begin{array}{l} \exists (f_s, f_r, \delta_r, \delta_s) \\ 0 = \frac{f_s \sin\delta_s - f_r \sin\delta_r - \alpha_f v}{m} \\ 0 = \frac{(\ell - r_s \cos\delta_s) f_s - r_r \cos\delta_r f_r}{J} \\ f_s = \alpha_s \left(V \cos\left(\theta + \delta_s\right) - v \sin\delta_s \right) \\ f_r = \alpha_r v \sin\delta_r \end{array} \quad \}.$$

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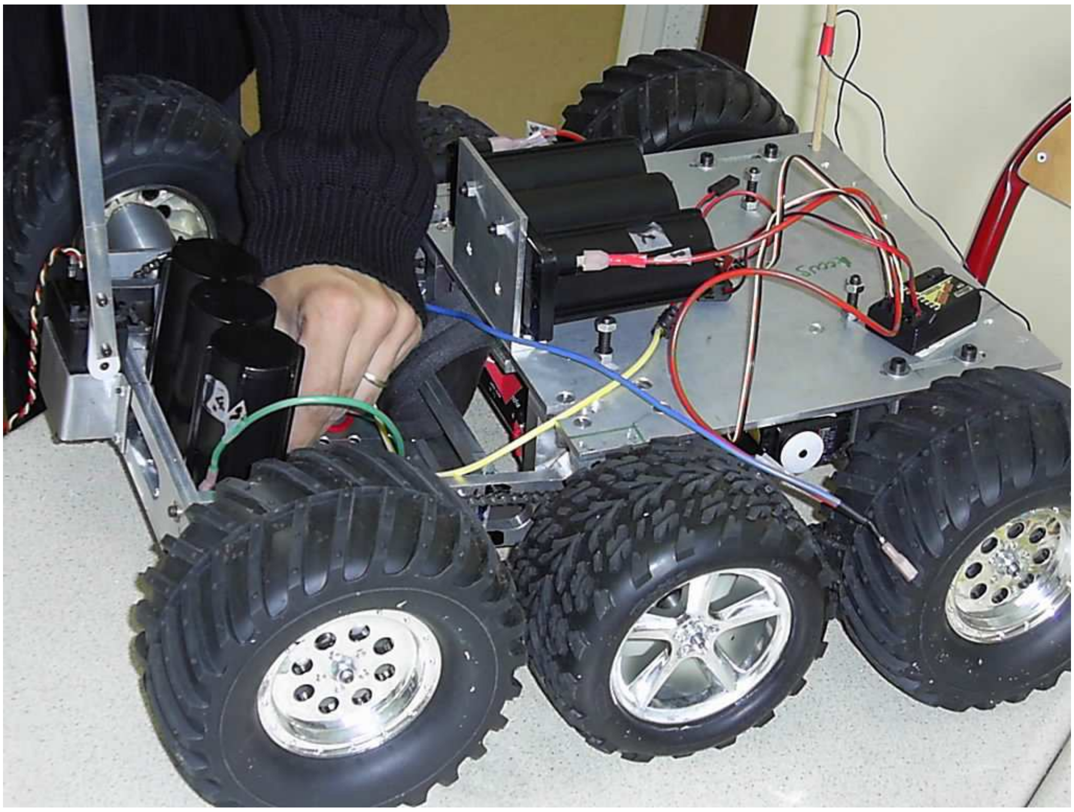
7 Control of a wheeled climbing robot

7.1 Context

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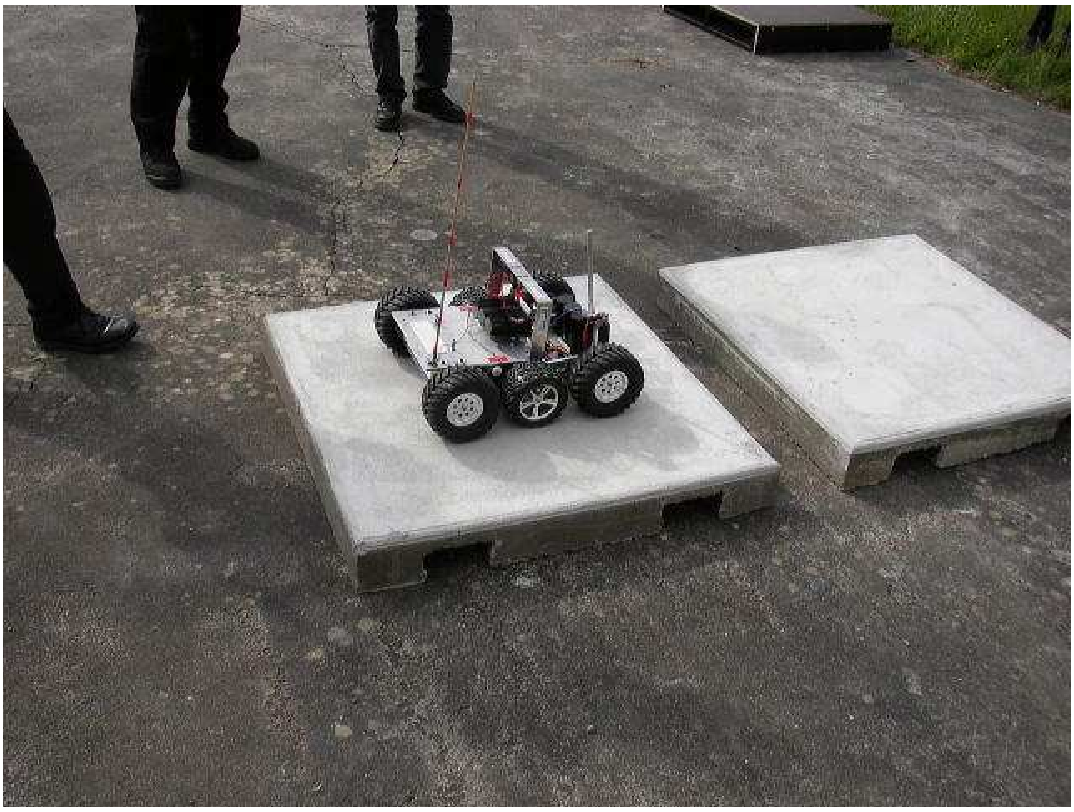
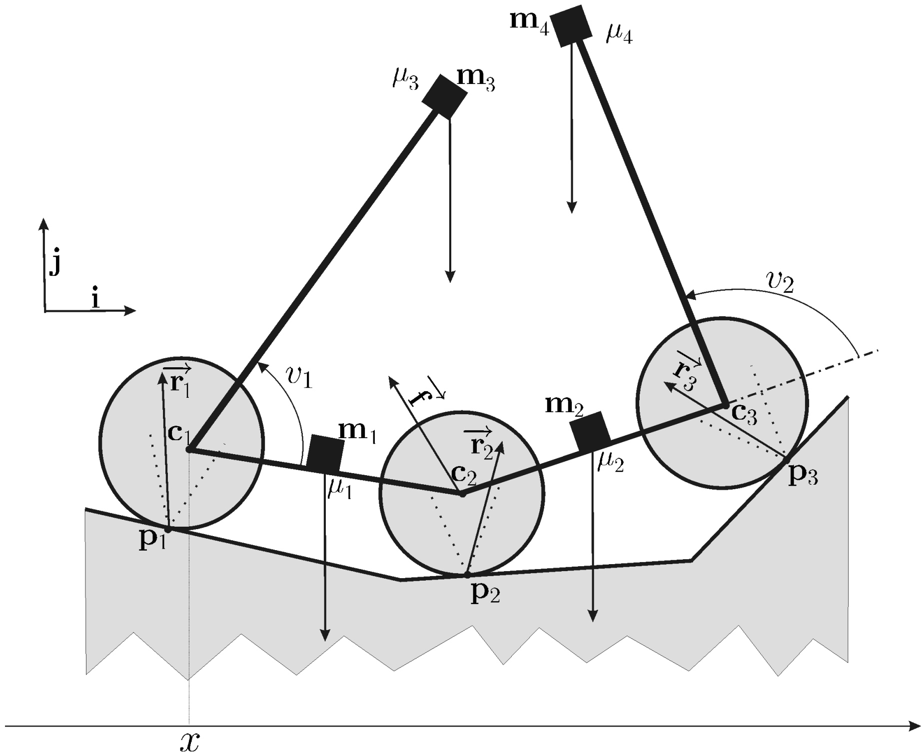
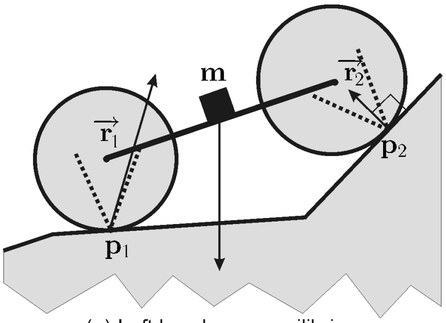


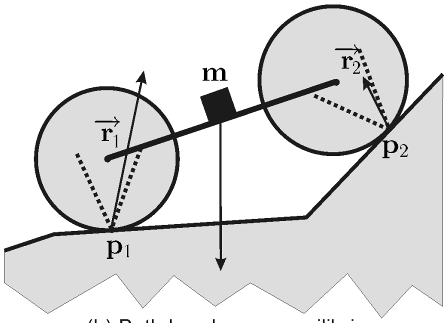
7.2 Idea



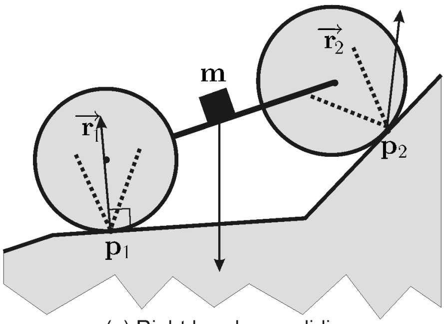
Mass transfer system to avoid any sliding



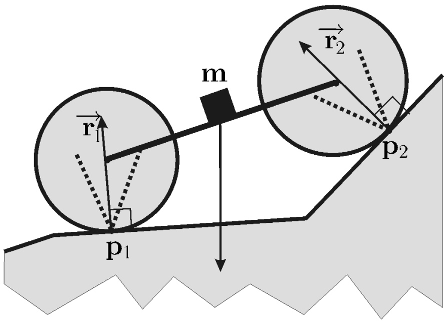
(a) Left break on : equilibrium



(b) Both breaks on : equilibrium



(c) Right break on : sliding



(d) No breaks : the robot moves

For (a), (b), (c) the fundamental principle of static is satisfied

7.3 Formalization

Consider the class of constrained dynamic robots

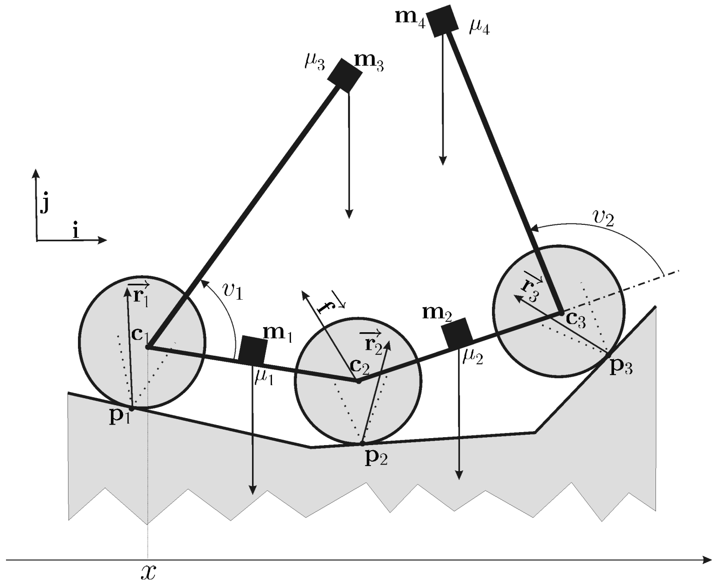
$$\begin{gathered}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{g}(\mathbf{x}(t), \mathbf{v}(t)) \leq \mathbf{0}.\end{gathered}$$

$\mathbf{u}(t)$ is the *evolution input vector*,
$\mathbf{x}(t)$ is the *state vector*,
$\mathbf{v}(t)$ is the *viable input vector*.

- If $\mathbf{g}(\mathbf{x}, \mathbf{v}) = \mathbf{A}(\mathbf{x}).\mathbf{v} + \mathbf{b}(\mathbf{x}) \leq \mathbf{0}$ then a simplex method can find a feasible $\mathbf{v}$.

- Otherwise, interval methods can be used to find a feasible $\mathbf{v}$.

7.4 Resolution



$$\dot{x} = u,$$
$$\mathbf{g}(x, v_1, v_2) \leq \mathbf{0}.$$

Fundamental principle of static. When the robot does not move,

$$\left\{\begin{array}{rcl} -\overrightarrow{\mathbf{p}_1\mathbf{m}_1}\wedge\mu_1\mathbf{j}+\overrightarrow{\mathbf{p}_1\mathbf{c}_2}\wedge\overrightarrow{\mathbf{f}}-\overrightarrow{\mathbf{p}_1\mathbf{m}_3}\wedge\mu_3\mathbf{j} & = & 0\\ \overrightarrow{\mathbf{p}_2\mathbf{m}_2}\wedge\mu_2\mathbf{j}+\overrightarrow{\mathbf{p}_2\mathbf{c}_2}\wedge\overrightarrow{\mathbf{f}}-\overrightarrow{\mathbf{p}_2\mathbf{p}_3}\wedge\overrightarrow{\mathbf{r}}_3 & & \\ \overrightarrow{\mathbf{p}_2\mathbf{m}_4}\wedge\mu_4\mathbf{j} & = & 0\\ \overrightarrow{\mathbf{r}}_1-(\mu_1+\mu_3)\mathbf{j}+\overrightarrow{\mathbf{f}} & = & 0\\ \overrightarrow{\mathbf{r}}_2-\overrightarrow{\mathbf{f}}-(\mu_2+\mu_4)\mathbf{j}+\overrightarrow{\mathbf{r}}_3 & = & 0 \end{array}\right.$$

A scalar decomposition yields

$$\left\{\begin{array}{rcl} -\mu_1\left(m_{1x}-p_{1x}\right)+\left(c_{2x}-p_{1x}\right)f_y & & \\ -\left(c_{2y}-p_{1y}\right)f_x-\mu_3\left(m_{3x}-p_{1x}\right) & = & 0 \\ \mu_2\left(m_{2x}-p_{2x}\right)+\left(c_{2x}-p_{2x}\right)f_y & & \\ -\left(c_{2y}-p_{2y}\right)f_x-\left(p_{3x}-p_{2x}\right)r_{3y} & & \\ +\left(p_{3y}-p_{2y}\right)r_{3x}+\mu_4\left(m_{4x}-p_{2x}\right) & = & 0 \\ r_{1x}+f_x & = & 0 \\ r_{1y}-\mu_1-\mu_3+f_y & = & 0 \\ r_{2x}-f_x+r_{3x} & = & 0 \\ r_{2y}-f_y-\mu_2-\mu_4+r_{3y} & = & 0 \end{array}\right.$$

In a matrix form as

$$\mathbf{A}_1(x).\mathbf{y} = \mathbf{b}_1(x),$$

where

$$\mathbf{A}_1(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & p_{1y}-c_{2y} & c_{2x}-p_{1x} & -\mu_3 & 0 \\ 0 & 0 & 0 & 0 & p_{2y}-p_{3y} & p_{3x}-p_{2x} & c_{2y}-p_{2y} & p_{2x}-c_{2x} & 0 & -\mu_4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{b}_1(x) = \begin{pmatrix} \mu_1 (m_{1x} - p_{1x}) - \mu_3 p_{1x} \\ \mu_2 (m_{2x} - p_{2x}) - \mu_4 p_{2x} \\ 0 \\ \mu_1 + \mu_3 \\ 0 \\ \mu_2 + \mu_4 \end{pmatrix}$$

and

$$\mathbf{y} = \left(r_{1x}, r_{1y}, r_{2x}, r_{2y}, r_{3x}, r_{3y}, f_x, f_y, m_{3x}, m_{4x}\right)^{\mathsf{T}}.$$

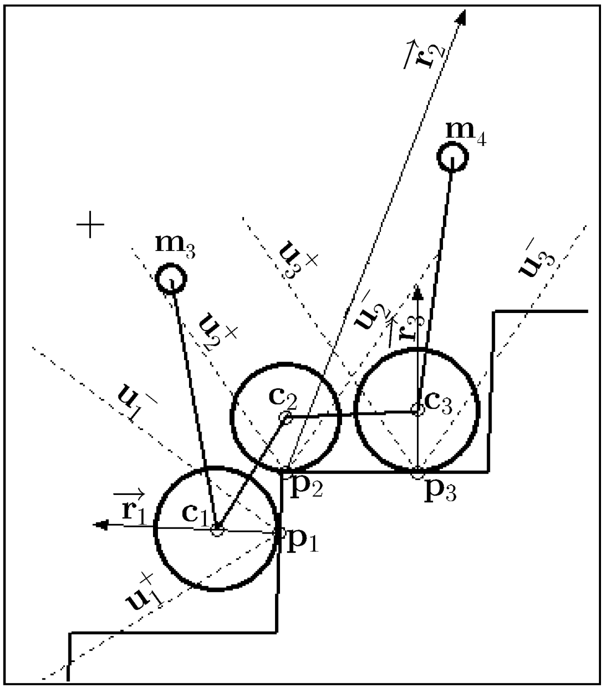
We have 10 unknowns for 6 equations: our robot has a second order hyperstatic equilibrium.

Non-sliding conditions. None of the wheels will slide if all $\overrightarrow{\mathbf{r}}_i$ belong to their Coulomb cone:

$$\mathbf{A}_2(x).\mathbf{y} \leq \mathbf{0},$$

where $\mathbf{A}_2(x)$ is given by

$$\begin{pmatrix} u_{1y}^{-} & -u_{1x}^{-} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_{1y}^{+} & u_{1x}^{+} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{2y}^{-} & -u_{2x}^{-} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -u_{2y}^{+} & u_{2x}^{+} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{3y}^{-} & -u_{3x}^{-} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u_{3y}^{+} & u_{3x}^{+} & 0 & 0 & 0 & 0 \end{pmatrix}$$



A configuration where the middle wheel is almost sliding.

Collision avoidance. The pendulums should not intersect the ground or the robot itself

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .\mathbf{y} \in \begin{pmatrix} [m_{3x}^{\mathsf{min}}, m_{3x}^{\mathsf{max}}] \\ [m_{4x}^{\mathsf{min}}, m_{4x}^{\mathsf{max}}] \end{pmatrix}.$$

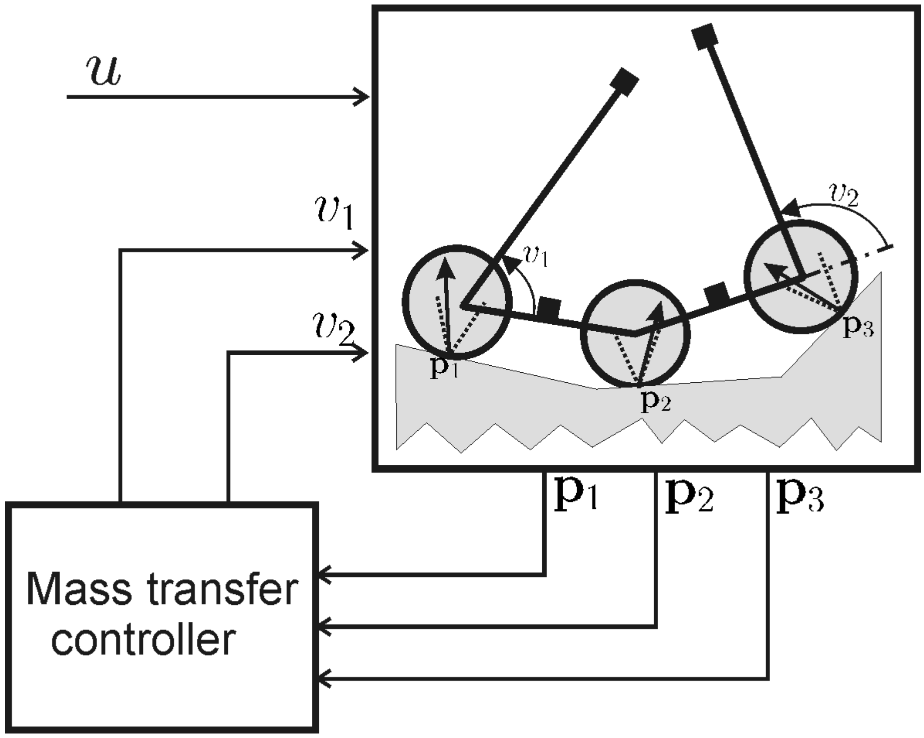
7.5 All constraints

Our robot can be described by

$$\begin{array}{lrcl} \text{(i)} & \dot{x} & = & u \\ \text{(ii)} & \mathbf{g}(x, v_1, v_2) & \leq & \mathbf{0} \end{array}$$

where (ii) is equivalent to

$$\exists \mathbf{y} = \begin{pmatrix} r_{1x}, r_{1y} \\ r_{2x}, r_{2y} \\ r_{3x}, r_{3y} \\ f_x, f_y \\ m_{3x}, m_{4x} \end{pmatrix}, \left\{ \begin{array}{lcl} \mathbf{A}_1(x).\mathbf{y} & = & \mathbf{b}_1(x) \\ \mathbf{A}_2(x).\mathbf{y} & \leq & \mathbf{0} \\ \mathbf{A}_3(x).\mathbf{y} & \leq & \mathbf{b}_3(x) \end{array} \right.$$



u
v_1
v_2
v_1
v_2
$\mathbf{p}_1$
$\mathbf{p}_2$
$\mathbf{p}_3$
$\mathbf{p}_1$
$\mathbf{p}_2$
$\mathbf{p}_3$
Mass transfer controller

Angle friction coefficient: $\phi = 0.54$

Radius of the wheels: $\rho_1 = 85$mm, $\rho_2 = 75$mm, $\rho_3 = 85$mm

Lengths of the pendulums: $\ell_1 = \ell_2 = 350$mm

Weights of the platforms: $\mu_1 = \mu_2 = 70$N

Weights and the pendulum masses: $\mu_3 = \mu_4 = 20$N.

Height and the width of the stairs: 220mm and 280mm