Interval Robotics
Chapter 5: Robust estimation

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Exercise. A robot measures its own distance to three marks. The distances and the coordinates of the marks are

mark	x_i	y_i	d_i
1	0	0	$[22,23]$
2	10	10	$[10,11]$
3	30	-30	$[53,54]$

1) Define the set $\mathbb{X}$ of all feasible positions.
2) Build the contractor associated with $\mathbb{X}$.
3) Build the contractor associated with $\overline{\mathbb{X}}$.

Solution.

$$\mathbb{X} = \bigcap_{i\in\{1,2,3\}} \underbrace{\left\{(x,y) \mid (x-x_i)^2+(y-y_i)^2 \in \left[d_i^-, d_i^+\right]\right\}}_{\mathbb{X}_i}$$

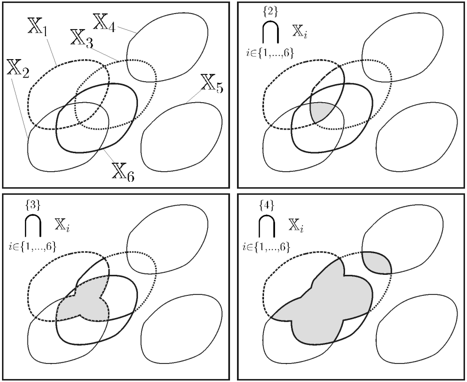
Relaxed intersection

Dealing with outliers

$$\mathscr{C} = (\mathscr{C}_1 \cap \mathscr{C}_2) \cup (\mathscr{C}_2 \cap \mathscr{C}_3) \cup (\mathscr{C}_1 \cap \mathscr{C}_3)$$

Consider m sets $\mathbb{X}_1,\dots,\mathbb{X}_m$ of $\mathbb{R}^n$. The q-relaxed intersection $\bigcap^{\{q\}} \mathbb{X}_i$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most. We have

$$\mathbf{x} \in \bigcap^{\{q\}} \mathbb{X}_i \Leftrightarrow \#\{i | \mathbf{x} \in \mathbb{X}_i\} \geq m - q$$


$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\bigcap_{i\in\{1,\dots,6\}}^{\{2\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\dots,6\}}^{\{3\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\dots,6\}}^{\{4\}} \mathbb{X}_i$

Exercise. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{1\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{6\}} \mathbb{X}_i = ?$$

Solution. we have

$$\begin{array}{rcl}
\overset{\{0\}}{\bigcap} \mathbb{X}_i & = & \emptyset \\
\overset{\{1\}}{\bigcap} \mathbb{X}_i & = & \emptyset \\
\overset{\{5\}}{\bigcap} \mathbb{X}_i & = & \bigcup \mathbb{X}_i \\
\overset{\{6\}}{\bigcap} \mathbb{X}_i & = & \mathbb{R}^2
\end{array}$$

Exercise. Consider 8 intervals: $\mathbb{X}_1 = [1,4]$,
$\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$.
Compute

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= ?, \quad \bigcap^{\{1\}} \mathbb{X}_i = ?, \quad \bigcap^{\{2\}} \mathbb{X}_i = ?, \\
\bigcap^{\{3\}} \mathbb{X}_i &= ?, \quad \bigcap^{\{4\}} \mathbb{X}_i = ?, \\
\bigcap^{\{5\}} \mathbb{X}_i &= ?, \quad \bigcap^{\{6\}} \mathbb{X}_i \mathbb{X}_i = ?.
\end{aligned}$$

Solution. For $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$, we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset, \ \bigcap^{\{1\}} \mathbb{X}_i = [3,4], \ \bigcap^{\{2\}} \mathbb{X}_i = [3,4], \\
\bigcap^{\{3\}} \mathbb{X}_i &= [2,4] \cup [6,7], \ \bigcap^{\{4\}} \mathbb{X}_i = [2,7], \\
\bigcap^{\{5\}} \mathbb{X}_i &= [1,9], \ \bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}.
\end{aligned}$$

If $\mathbb{X}_i$'s are intervals, the relaxed intersection can be computed with a complexity of $m \log m$.

Take all bounds of all intervals with their brackets.

Bounds	1	4	2	4	2	7	6	9	3	4	3	7
Brackets	[	]	[	]	[	]	[	]	[	]	[	]

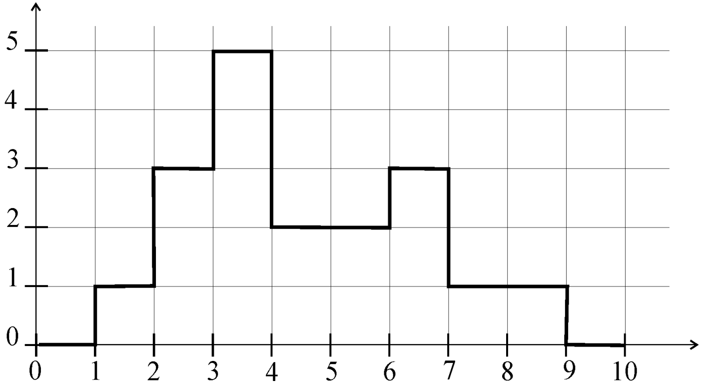
Sort the columns with respect the bounds:

Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]

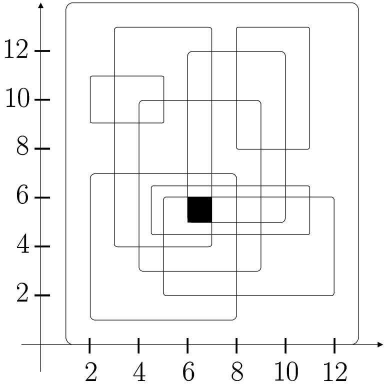
Scan from left to right, counting $+1$ for '[' and -1 for ']':

Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]
Sum	1	2	3	4	5	4	3	2	3	2	1	0

Read the q-intersections


5
4
3
2
1
0
0
1
2
3
4
5
6
7
8
9
10

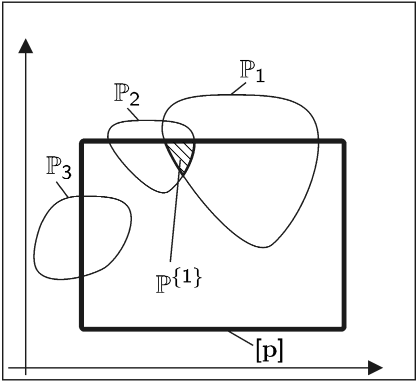
Computing the q relaxed intersection of m boxes is tractable.

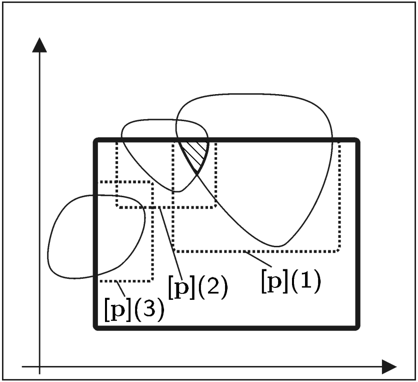

12
10
8
6
4
2
2
4
6
8
10
12

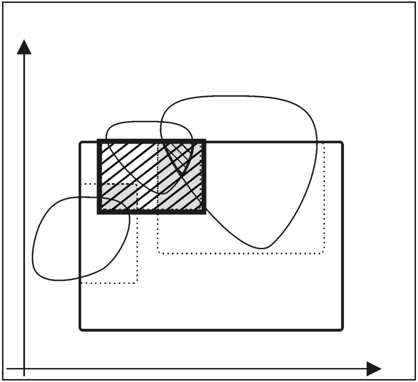
Relaxation of contractors

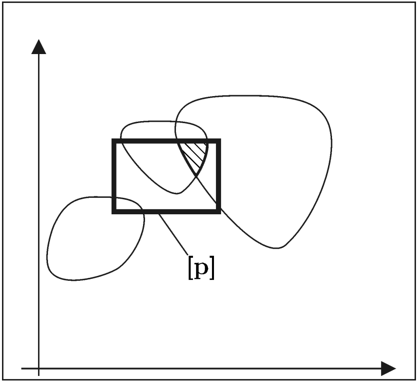
We define the q-relaxed intersection between m contractors

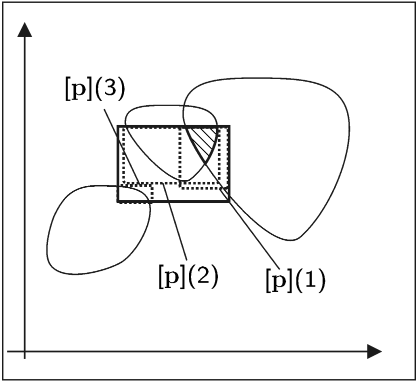
$$\mathscr{C} = \left(\bigcap_{i \in \{1,\dots,m\}}^{\{q\}} \mathscr{C}_i \right) \Leftrightarrow \forall [\mathbf{x}] \in \mathbb{IR}^n, \mathscr{C}([\mathbf{x}]) = \bigcap^{\{q\}} \mathscr{C}_i([\mathbf{x}]).$$

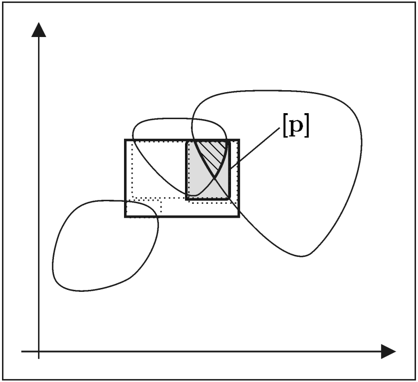

$\mathbb{P}_1$
$\mathbb{P}_2$
$\mathbb{P}_3$
$\mathbb{P}^{\{1\}}$
$[\mathbf{p}]$


[p](2)
[p](1)
[p](3)

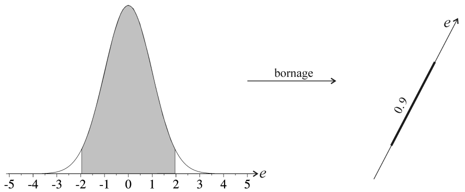


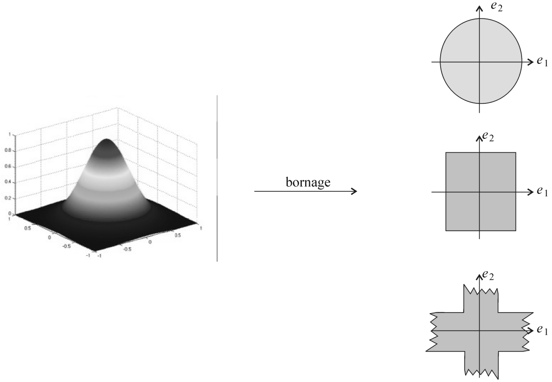

[p]


[p](3)
[p](2)
[p](1)


[p]

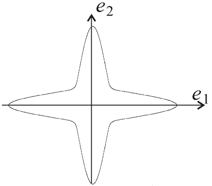
Probabilistic set estimation

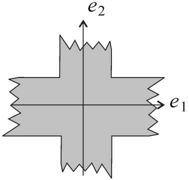

-5 -4 -3 -2 -1 0 1 2 3 4 5
e
bornage
0.9
e


bornage
e_2
e_1
e_2
e_1
e_2
e_1

$$\Pi(\mathbf{e}) \propto \left(\exp(-e_1^2) + \exp\left(-\frac{e_1^2}{10}\right)\right) * \left(\exp(-e_2^2) + \exp\left(-\frac{e_2^2}{10}\right)\right)$$

bornage →





$$\mathbf{y} = \psi(\mathbf{p}) + \mathbf{e},$$

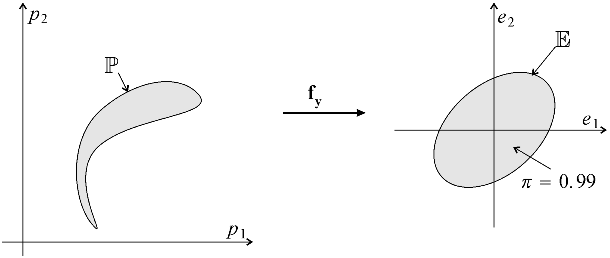
where
$\mathbf{e} \in \mathbb{E} \subset \mathbb{R}^m$ is the error vector,
$\mathbf{y} \in \mathbb{R}^m$ is the collected data vector,
$\mathbf{p} \in \mathbb{R}^n$ is the parameter vector to be estimated.

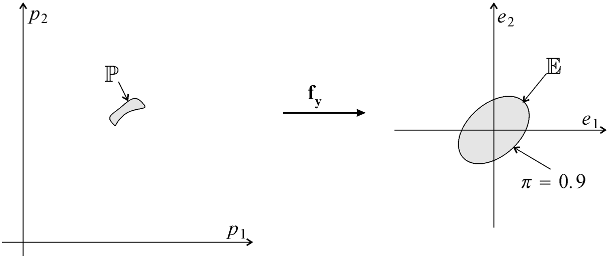
Or equivalently

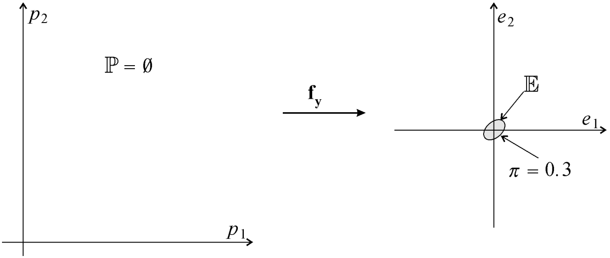
$$\mathbf{e} = \mathbf{y} - \psi(\mathbf{p}) = \mathbf{f}_{\mathbf{y}}(\mathbf{p}),$$

The *posterior feasible set* for the parameters is

$$\mathbb{P} = \mathbf{f}_{\mathbf{y}}^{-1}(\mathbb{E}).$$


p_2
$\mathbb{P}$
p_1
$\mathbf{f_y}$
e_2
$\mathbb{E}$
e_1
$\pi = 0.99$


p_2
$\mathbb{P}$
p_1
$\mathbf{f_y}$
e_2
$\mathbb{E}$
e_1
$\pi = 0.9$


p_2
$\mathbb{P} = \emptyset$
p_1
$\mathbf{f_y}$
e_2
$\mathbb{E}$
e_1
$\pi = 0.3$

Consider the error model

$$\underbrace{\begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}}_{=\mathbf{e}} = \underbrace{\begin{pmatrix} y_1 - \psi_1(\mathbf{p}) \\ \vdots \\ y_m - \psi_m(\mathbf{p}) \end{pmatrix}}_{=\mathbf{f}_{\mathbf{y}}(\mathbf{p})}$$

The data y_i is an *inlier* if $e_i \in [e_i]$ and an *outlier* otherwise. We assume that

$$\forall i,\ \Pr(e_i \in [e_i]) = \pi$$

and that all e_i's are independent.

Equivalently,

$$\begin{cases} y_1 - \psi_1(\mathbf{p}) \in [e_1] & \text{with a probability } \pi \\ \vdots & \vdots \\ y_m - \psi_m(\mathbf{p}) \in [e_m] & \text{with a probability } \pi \end{cases}$$

The probability of having k inliers is

$$\frac{m!}{k!\,(m-k)!}\pi^k.\,(1-\pi)^{m-k}\,.$$

The probability of having strictly more than q outliers is thus

$$\gamma(q,m,\pi) = \sum_{k=0}^{m-q-1} \frac{m!}{k!\,(m-k)!}\pi^k.\,(1-\pi)^{m-k}.$$

Denote by $\mathbb{E}^{\{q\}}$ the set of all $\mathbf{e} \in \mathbb{R}^m$ consistent with at least $m-q$ error intervals $[e_i]$.
For $m=3$, we have

$$\begin{array}{rcl}
\mathbb{E}^{\{0\}} & = & [e_1]\times[e_2]\times[e_3] \\
\mathbb{E}^{\{1\}} & = & ([e_1]\cap[e_2]) \cup ([e_2]\cap[e_3])\cup([e_1]\cap[e_3]) \\
\mathbb{E}^{\{2\}} & = & [e_1]\cup[e_2]\cup[e_3] \\
\mathbb{E}^{\{3\}} & = & \mathbb{R}^3.
\end{array}$$

$$\mathbb{P}^{\{q\}} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}^{\{q\}}\right) = \bigcap_{i\in\{1,\dots,m\}}^{\{q\}} f_{y_i}^{-1}\left([e_i]\right).$$

Application to localization

A robot measures distances to three beacons.

i	x_i	y_i	$[d_i]$
1	1	3	$[1,2]$
2	3	1	$[2,3]$
3	-1	-1	$[3,4]$

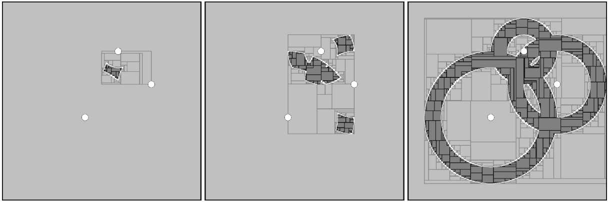
The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5] \right\},$$

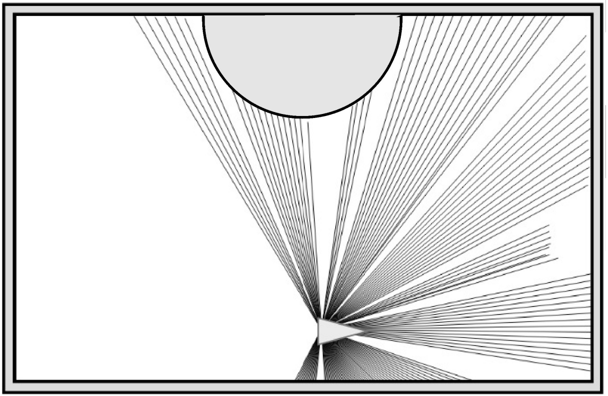
where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$.

$$\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{0\}}\right) = 0.729$$
$$\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{1\}}\right) = 0.972$$
$$\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{2\}}\right) = 0.999$$



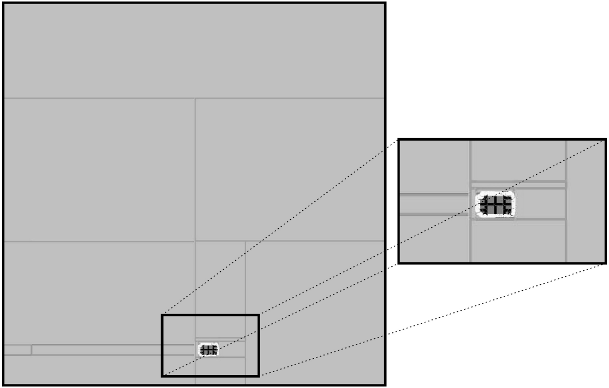
With real data


ENSTA



For $q = 16, m = 143, \pi = 0.95$, the probability of being wrong is

$$\alpha = \gamma(q, m, \pi) = 8.46 \times 10^{-4}.$$



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2. Localization with intervals : [3]
3. IAMOOC [2]

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