Interval Robotics
Chapter 4: Separators

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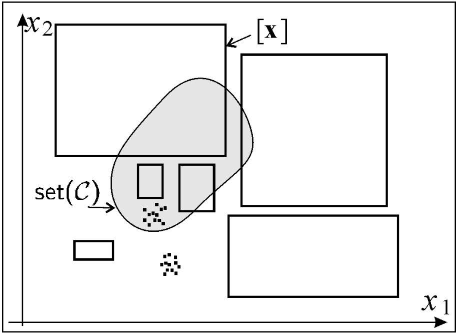
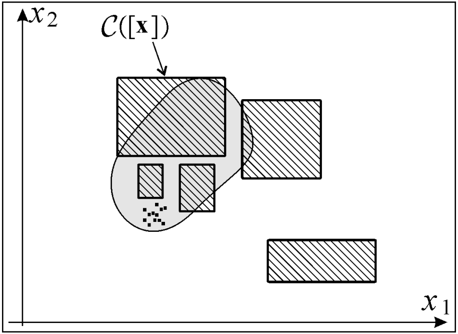
Contractors and separators

$$\begin{array}{ll} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ [\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}]) & \text{(monotonicity)} \end{array}$$


x_2
[x]
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

Inclusion

$$\mathscr{C}_1 \subset \mathscr{C}_2 \Leftrightarrow \forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathscr{C}_1([\mathbf{x}]) \subset \mathscr{C}_2([\mathbf{x}]).$$

A set $\mathbb{S}$ is *consistent* with $\mathscr{C}$ (we write $\mathbb{S} \sim \mathscr{C}$) if

$$\mathscr{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}.$$

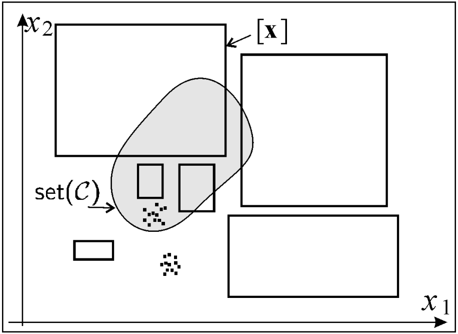
$\mathscr{C}$ is *minimal* if

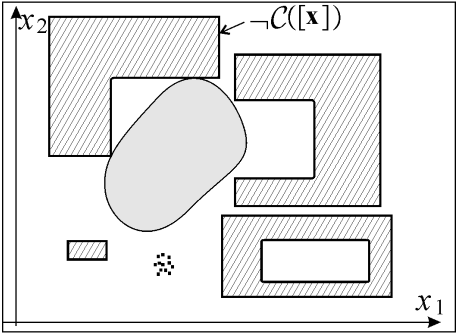
$$\left.\begin{array}{c}\mathbb{S}\sim\mathscr{C}\\ \mathbb{S}\sim\mathscr{C}_1\end{array}\right\}\Rightarrow\mathscr{C}\subset\mathscr{C}_1.$$

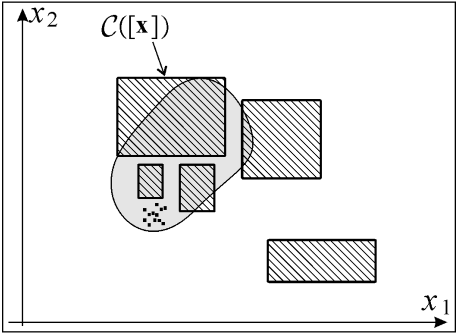
The *negation* $\neg\mathscr{C}$ of $\mathscr{C}$ is defined by

$$\neg\mathscr{C}([\mathbf{x}]) = \{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{x} \notin \mathscr{C}([\mathbf{x}])\}.$$

It is not a box in general.


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\neg\mathcal{C}([\mathbf{x}])$
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

Separators

A *separator* $\mathscr{S}$ is pair of contractors $\{\mathscr{S}^{\text{in}}, \mathscr{S}^{\text{out}}\}$ such that

$$\mathscr{S}^{\text{in}}([\mathbf{x}]) \cup \mathscr{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}] \qquad \text{(complementarity).}$$

A set $\mathbb{S}$ is *consistent* with $\mathscr{S}$ (we write $\mathbb{S} \sim \mathscr{S}$), if

$$\mathbb{S} \sim \mathscr{S}^{\text{out}} \text{ and } \overline{\mathbb{S}} \sim \mathscr{S}^{\text{in}}.$$

The *remainder* of $\mathscr{S}$ is

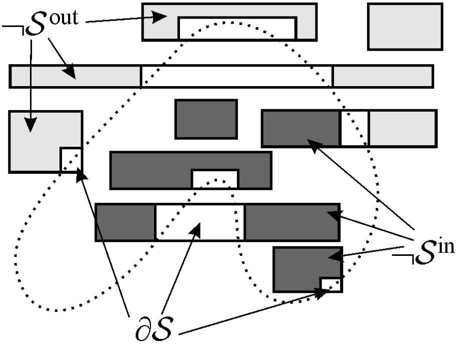
$$\partial\mathscr{S}([\mathbf{x}]) = \mathscr{S}^{\text{in}}([\mathbf{x}]) \cap \mathscr{S}^{\text{out}}([\mathbf{x}]).$$

$\partial\mathscr{S}$ is a contractor, not a separator.

We have

$$\neg\mathscr{S}^{\mathsf{in}}([\mathbf{x}])\cup\neg\mathscr{S}^{\mathsf{out}}([\mathbf{x}])\cup\partial\mathscr{S}([\mathbf{x}])=[\mathbf{x}]\,.$$

Moreover, they do not overlap.



$\neg\mathscr{S}^{\text{in}}([\mathbf{x}])$, $\neg\mathscr{S}^{\text{out}}([\mathbf{x}])$ and $\partial\mathscr{S}([\mathbf{x}])$

Inclusion

$$\mathscr{S}_1 \subset \mathscr{S}_2 \Leftrightarrow \mathscr{S}_1^{\mathsf{in}} \subset \mathscr{S}_2^{\mathsf{in}} \text{ and } \mathscr{S}_1^{\mathsf{out}} \subset \mathscr{S}_2^{\mathsf{out}}.$$

Here $\subset$ means *more accurate.*

$\mathscr{S}$ is *minimal* if

$$\mathscr{S}_1 \subset \mathscr{S} \Rightarrow \mathscr{S}_1 = \mathscr{S}.$$

i.e., if $\mathscr{S}^{\text{in}}$ and $\mathscr{S}^{\text{out}}$ are both minimal.

Paver

We want to compute $\mathbb{X}^-, \mathbb{X}^+$ such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

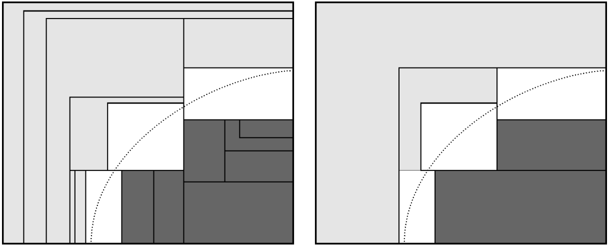
Algorithm Paver(in: $[\mathbf{x}], \mathscr{S}$; out: $\mathbb{X}^-$, $\mathbb{X}^+$)

1 $\mathscr{L} := \{[\mathbf{x}]\}$;
2 Pull $[\mathbf{x}]$ from $\mathscr{L}$;
3 $\left\{[\mathbf{x}^{\text{in}}], [\mathbf{x}^{\text{out}}]\right\} = \mathscr{S}([\mathbf{x}])$;
4 Store $[\mathbf{x}] \setminus [\mathbf{x}^{\text{in}}]$ into $\mathbb{X}^-$ and also into $\mathbb{X}^+$;
5 $[\mathbf{x}] = [\mathbf{x}^{\text{in}}] \cap [\mathbf{x}^{\text{out}}]$;
6 If $w([\mathbf{x}]) < \varepsilon$, then store $[\mathbf{x}]$ in $\mathbb{X}^+$,
7 Else bisect $[\mathbf{x}]$ and push into $\mathscr{L}$ the two childs
8 If $\mathscr{L} \neq \emptyset$, go to 2.

For the implementation, the paving is represented by a binary tree. The ith node of the tree contains two boxes: $[\mathbf{x}^{\text{in}}](i)$ and $[\mathbf{x}^{\text{out}}](i)$.

The binary tree is said to be *minimal* if for any node i_1 with brother i_2 and father j, we have

$$\begin{cases} \text{(i)} & [\mathbf{x}^{\text{in}}](i_1) \neq \emptyset,\ [\mathbf{x}^{\text{out}}](i_1) \neq \emptyset \\ \text{(ii)} & [\mathbf{x}^{\text{in}}](j) \cap [\mathbf{x}^{\text{out}}](j) = \quad \left([\mathbf{x}^{\text{in}}](i_1) \cap [\mathbf{x}^{\text{out}}](i_1)\right) \\ & \qquad\qquad\qquad\qquad \sqcup \left([\mathbf{x}^{\text{in}}](i_2) \cap [\mathbf{x}^{\text{out}}](i_2)\right) \end{cases}$$



Algebra

Contractor algebra only allows monotonic operations such as $\cup$ or $\cap$, i.e.,

$$\forall i, \mathscr{C}_i \subset \mathscr{C}_i' \Rightarrow \mathscr{C}_1 \cup (\mathscr{C}_2 \cap \mathscr{C}_3) \subset \mathscr{C}_1' \cup \left(\mathscr{C}_2' \cap \mathscr{C}_3'\right).$$

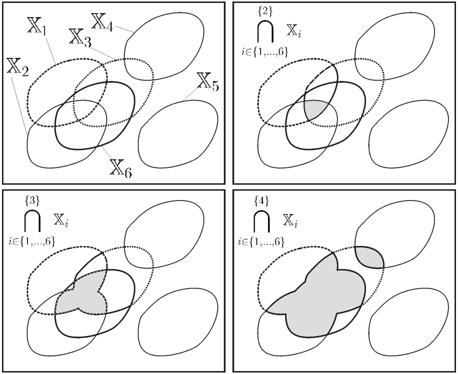
The complementary $\overline{\mathscr{C}}$ of a contractor $\mathscr{C}$, the restriction $\mathscr{C}_1 \backslash \mathscr{C}_2$, etc. cannot be defined.
Separators extend the operations allowed for contractors to non monotonic expressions.

The *complement* of $\mathscr{S} = \{\mathscr{S}^{\text{in}}, \mathscr{S}^{\text{out}}\}$ is

$$\overline{\mathscr{S}} = \{\mathscr{S}^{\text{out}}, \mathscr{S}^{\text{in}}\}.$$

If $\mathscr{S}_i = \left\{\mathscr{S}_i^{\text{in}}, \mathscr{S}_i^{\text{out}}\right\}, i \geq 1$, are separators, we define

$$
\begin{array}{rcll}
\mathscr{S}_1 \cap \mathscr{S}_2 & = & \left\{\mathscr{S}_1^{\text{in}} \cup \mathscr{S}_2^{\text{in}}, \mathscr{S}_1^{\text{out}} \cap \mathscr{S}_2^{\text{out}}\right\} & \text{(intersection)} \\
\mathscr{S}_1 \cup \mathscr{S}_2 & = & \left\{\mathscr{S}_1^{\text{in}} \cap \mathscr{S}_2^{\text{in}}, \mathscr{S}_1^{\text{out}} \cup \mathscr{S}_2^{\text{out}}\right\} & \text{(union)} \\
\bigcap^{\{q\}} \mathscr{S}_i & = & \left\{\bigcap^{\{m-q-1\}} \mathscr{S}_i^{\text{in}}, \bigcap^{\{q\}} \mathscr{S}_i^{\text{out}}\right\} & \text{(relaxed intersection)} \\
\mathscr{S}_1 \backslash \mathscr{S}_2 & = & \mathscr{S}_1 \cap \overline{\mathscr{S}_2}. & \text{(difference)}
\end{array}
$$


$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\bigcap_{i\in\{1,...,6\}}^{\{2\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,...,6\}}^{\{3\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,...,6\}}^{\{4\}} \mathbb{X}_i$

Theorem. If $\mathbb{S}_i$ are subsets of $\mathbb{R}^n$, we have

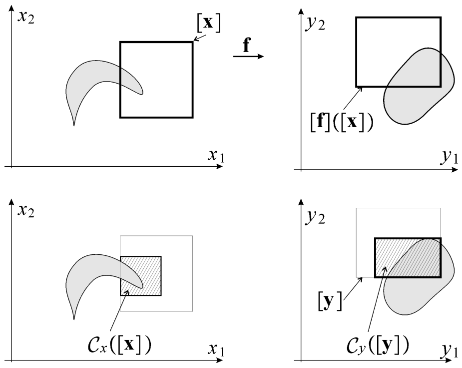
$$\begin{array}{cccc}
\text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathcal{S}_1 \cap \mathcal{S}_2 \\
\text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathcal{S}_1 \cup \mathcal{S}_2 \\
\text{(iii)} & \overline{\mathbb{S}_i} & \sim & \overline{\mathcal{S}_i} \\
\text{(iv)} & \mathbb{S}_i & \sim & \mathcal{S}_i^k,\ k \geq 0 \\
\text{(v)} & \bigcap^{\{q\}} \mathbb{S}_i & \sim & \bigcap^{\{q\}} \mathcal{S}_i \\
\text{(vi)} & \mathbb{S}_1 \backslash \mathbb{S}_2 & \sim & \mathcal{S}_1 \backslash \mathcal{S}_2.
\end{array}$$

Inversion of separators

The inverse of $\mathbb{Y} \subset \mathbb{R}^n$ by $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined as

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) = \{\mathbf{x} \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\}.$$

$\mathbf{f}$ can be a translation, rotation, homothety, projection, ...

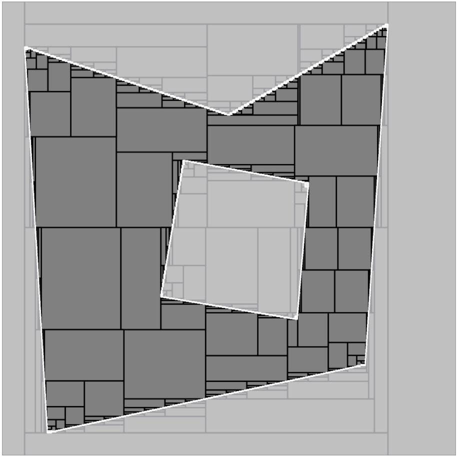

x_2
$[\mathbf{x}]$
$\mathbf{f}$
x_1
y_2
$[\mathbf{f}]([\mathbf{x}])$
y_1
x_2
$\mathcal{C}_x([\mathbf{x}])$
x_1
y_2
$[\mathbf{y}]$
$\mathcal{C}_y([\mathbf{y}])$
y_1

We define

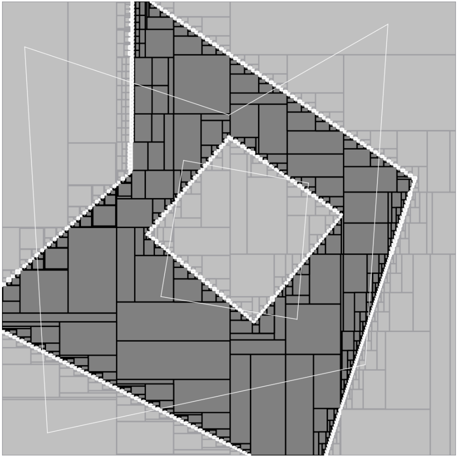
$$\mathbf{f}^{-1}(\mathscr{S}_{\mathbb{Y}}) = \left\{\mathbf{f}^{-1}(\mathscr{S}_{\mathbb{Y}}^{\mathsf{in}}), \mathbf{f}^{-1}(\mathscr{S}_{\mathbb{Y}}^{\mathsf{out}})\right\}.$$

Theorem. The separator $\mathbf{f}^{-1}(\mathscr{S}_{\mathbb{Y}})$ is a separator associated with the set $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$, i.e.,

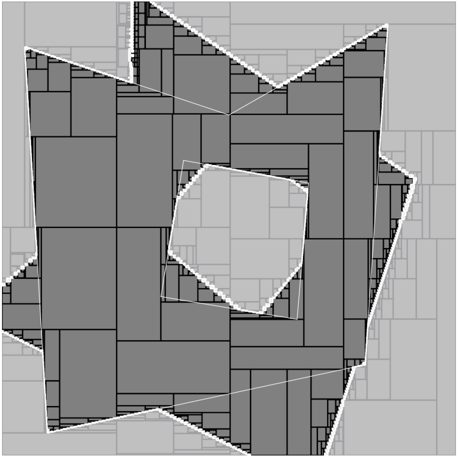
$$\mathbf{f}^{-1}(\mathbb{Y}) \sim \mathbf{f}^{-1}(\mathscr{S}_{\mathbb{Y}}).$$



$\mathbb{M}$



$\mathrm{Rot}(\mathbb{M})$



$\mathrm{Rot}(\mathbb{M}) \cup \mathbb{M}$

Atomic separators

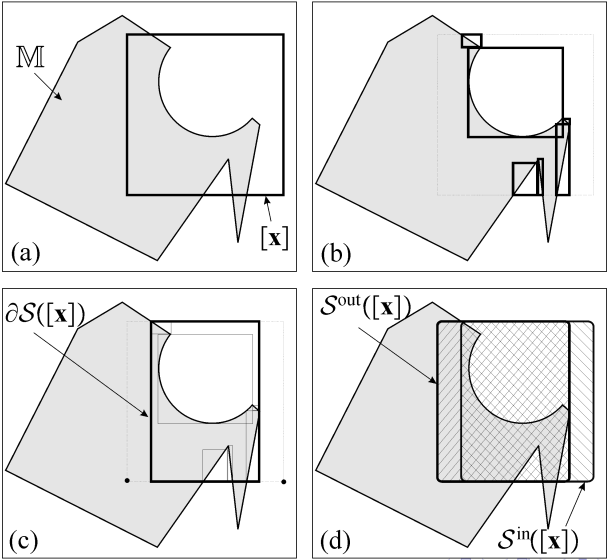
Equation-based separators

If

$$\mathbb{X} = \{f(\mathbf{x}) \leq 0\},$$

the pair $\{\mathscr{S}^{\text{in}}, \mathscr{S}^{\text{out}}\}$, where $\mathscr{S}^{\text{out}}$: $f(\mathbf{x}) \leq 0$ and $\mathscr{S}^{\text{in}}$: $f(\mathbf{x}) \geq 0$, is a separator for $\mathbb{X}$.

Database-based separators


M
[x]
(a)
(b)
∂S([x])
(c)
S^out([x])
S^in([x])
(d)

Dealing with quantifiers

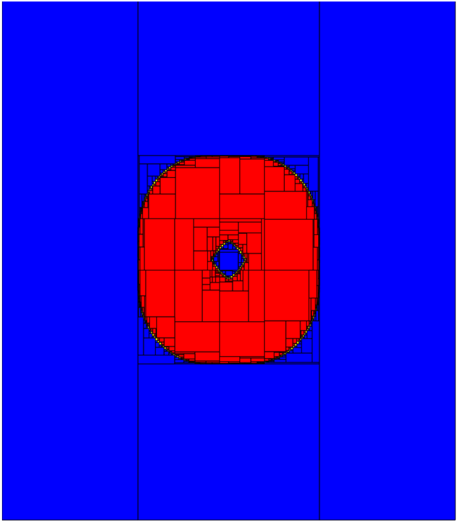
We can build a separator associated to the projection of a set defined by a separator. Consider the set

$$\mathbb{X}=\left\{\mathbf{x}\in\mathbb{R}^2\,|\,\exists\mathbf{a}\in[-1,1]^2,\,(x_1-a_1)^2+(x_2-a_2)^2\in[4,9]\right\}$$

We build the separator $\mathscr{S}$ for $\mathbb{X}$ as follows

$$\begin{array}{lcl}\mathscr{S}_1([\mathbf{x}],[\mathbf{a}]) & \sim & (x_1-a_1)^2+(x_2-a_2)^2\in[4,9]\\ \mathscr{S}([\mathbf{x}]) & = & \underset{[\mathbf{a}]}{\mathrm{proj}}\,\mathscr{S}_1([\mathbf{x}],[\mathbf{a}])\end{array}$$

```
from pyibex import *
from vibes import vibes
f = Function("x1","x2","a1","a2","(x1-a1)^2+(x2-a2)^2");
S1=SepFwdBwd(f,Interval(4,9))
A=IntervalVector([[-1,1],[-1,1]])
S2=SepProj(S1,A,0.001)
X0 =IntervalVector([[-10,10],[-10,10]]);
vibes.beginDrawing()
vibes.newFigure('Proj')
pySIVIA(X0,S2,0.1)
```



Example 1. Build a separator for

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], \mathbf{f}(\mathbf{x}, \mathbf{y}) \leq \mathbf{0}\}.$$

We build the separator $\mathscr{S}$ for $\mathbb{X}$ as follows

$$\begin{array}{lcl} \mathscr{S}_1([\mathbf{x}],[\mathbf{y}]) & \sim & (\mathbf{f}(\mathbf{x},\mathbf{y}) \leq \mathbf{0}) \\ \mathscr{S}([\mathbf{x}]) & = & \underset{[\mathbf{y}]}{\text{proj}} \mathscr{S}_1([\mathbf{x}],[\mathbf{y}]) \end{array}$$

Example 2. Build a separator for

$$\mathbb{X}=\{\mathbf{x}\in[\mathbf{x}]\mid\exists\mathbf{y}\in[\mathbf{y}],\forall\mathbf{z}\in[\mathbf{z}],\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})\leq\mathbf{0}\}.$$

We have

$$\mathbb{X}=\{\mathbf{x}\in[\mathbf{x}]\mid\exists\mathbf{y}\in[\mathbf{y}],\neg(\exists\mathbf{z}\in[\mathbf{z}],\neg(\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})\leq\mathbf{0}))\}.$$

We build the separator $\mathscr{S}$ for $\mathbb{X}$ as follows

$$\begin{array}{lcl}
\mathscr{S}_1([\mathbf{x}],[\mathbf{y}],[\mathbf{z}]) & \sim & (\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z}) \leq \mathbf{0}) \\
\mathscr{S}_2([\mathbf{x}],[\mathbf{y}],[\mathbf{z}]) & = & \neg\mathscr{S}_1([\mathbf{x}],[\mathbf{y}],[\mathbf{z}]) \\
\mathscr{S}_3([\mathbf{x}],[\mathbf{y}]) & = & \underset{[\mathbf{z}]}{\mathrm{proj}}\mathscr{S}_2([\mathbf{x}],[\mathbf{y}],[\mathbf{z}]) \\
\mathscr{S}_4([\mathbf{x}],[\mathbf{y}]) & = & \neg\mathscr{S}_3([\mathbf{x}],[\mathbf{y}]) \\
\mathscr{S}([\mathbf{x}]) & = & \underset{[\mathbf{y}]}{\mathrm{proj}}\mathscr{S}_4([\mathbf{x}],[\mathbf{y}])
\end{array}$$

Example 3. The problem

$$y = \min_{\mathbf{x}\in[\mathbf{x}]} f(\mathbf{x})$$

is equivalent to

$$\left\{ \begin{array}{c} \exists \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \leq y \\ \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq y. \end{array} \right.$$

The solution set is

$$\mathbb{Y} = \{y \mid (\exists \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \leq y) \text{ and } (\forall \mathbf{x} \in [\mathbf{x}] \mid f(\mathbf{x}) \geq y)\}$$

i.e.,

$$\mathbb{Y} = \{y,\ (\exists \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \leq y) \text{ and } (\neg (\exists \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) < y))\}$$

A separator $\mathscr{S}([y])$ for the solution set $\mathbb{Y}$ can be built as follows

$$\begin{array}{lll}
\mathscr{S}_1([\mathbf{x}],[y]) & \sim & (f(\mathbf{x}) \leq y) \\
\mathscr{S}_2([y]) & = & \underset{[\mathbf{x}]}{\text{proj}} \mathscr{S}_1([\mathbf{x}],[y]) \\
\mathscr{S}_3([\mathbf{x}],[y]) & \sim & (f(\mathbf{x}) < y) \\
\mathscr{S}_4([y]) & = & \underset{[\mathbf{x}]}{\text{proj}} \mathscr{S}_3([\mathbf{x}],[y]) \\
\mathscr{S}_5([y]) & = & \neg \mathscr{S}_4([y]) \\
\mathscr{S}([y]) & = & \mathscr{S}_2([y]) \wedge \mathscr{S}_5([y]).
\end{array}$$

Example. Consider the optimization problem where $\mathbf{f}:\mathbb{R}^n \mapsto \mathbb{R}^m$. The problem is

$$\mathbb{Y} = \min_{\mathbf{x}\in[\mathbf{x}]} \mathbf{f}(\mathbf{x}).$$

The set

$$\mathbb{Y} = \{\mathbf{y} \mid \exists \mathbf{x} \in [\mathbf{x}], \mathbf{f}(\mathbf{x}) \leq \mathbf{y}\} \cap \{\mathbf{y}, \forall \mathbf{x} \in [\mathbf{x}], \neg(\mathbf{f}(\mathbf{x}) < \mathbf{y})\}$$

is called the *Pareto set*. Here, $\mathbf{a} < \mathbf{b}$ means that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$.

Since

$$\mathbb{Y} = \{\mathbf{y} \mid \exists \mathbf{x} \in [\mathbf{x}], \mathbf{f}(\mathbf{x}) \leq \mathbf{y}\} \cap \{\mathbf{y}, \neg(\exists \mathbf{x} \in [\mathbf{x}], (\mathbf{f}(\mathbf{x}) < \mathbf{y}))\}$$

a separator $\mathscr{S}([\mathbf{y}])$ for $\mathbb{Y}$ can be built as follows

$$\begin{array}{lcl}
\mathscr{S}_1([\mathbf{x}],[\mathbf{y}]) & \sim & (\mathbf{f}(\mathbf{x}) - \mathbf{y} \leq \mathbf{0}) \\
\mathscr{S}_2([\mathbf{y}]) & = & \underset{[\mathbf{x}]}{\mathrm{proj}} \mathscr{S}_1([\mathbf{x}],[y]) \\
\mathscr{S}_3([\mathbf{x}],[\mathbf{y}]) & \sim & (\mathbf{f}(\mathbf{x}) - \mathbf{y} < \mathbf{0}) \\
\mathscr{S}_4([\mathbf{y}]) & = & \underset{[\mathbf{x}]}{\mathrm{proj}} \mathscr{S}_3([\mathbf{x}],[\mathbf{y}]) \\
\mathscr{S}_5([\mathbf{y}]) & = & \neg \mathscr{S}_4([\mathbf{y}]) \\
\mathscr{S}([\mathbf{y}]) & = & \mathscr{S}_2([\mathbf{y}]) \wedge \mathscr{S}_5([\mathbf{y}]).
\end{array}$$

Example. Consider

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) = 0\}.$$

The set $\{(\mathbf{x},\mathbf{y}) \in [\mathbf{x}] \times [\mathbf{y}], f(\mathbf{x},\mathbf{y}) = 0\}$ has an empty volume. Thus separator associated with $f(\mathbf{x},\mathbf{y}) = 0$ will never return an inner approximation.

If f is continuous

$$(\exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) = 0) \Leftrightarrow (\exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) \geq 0) \wedge (\exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) \leq 0).$$

Thus

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) \geq 0\} \cap \{\mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{y} \in [\mathbf{y}], f(\mathbf{x},\mathbf{y}) \leq 0\}.$$

A separator $\mathscr{S}$ for $\mathbb{X}$ can be built as follows

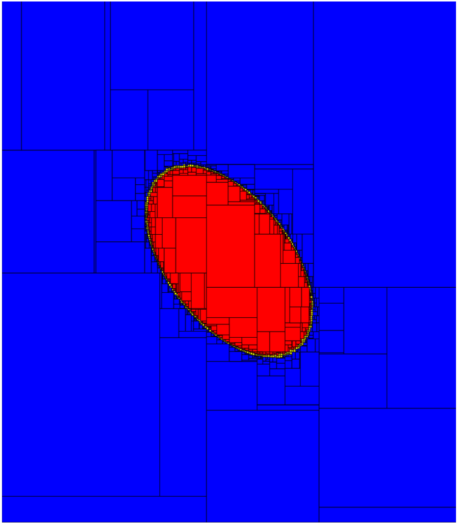
$$\begin{array}{lll} \mathscr{S}_1([\mathbf{x}],[\mathbf{y}]) & \sim & (f(\mathbf{x},\mathbf{y}) \geq 0) \\ \mathscr{S}_2([\mathbf{y}]) & = & \underset{[\mathbf{x}]}{\text{proj}} \mathscr{S}_1([\mathbf{x}],[\mathbf{y}]) \\ \mathscr{S}_3([\mathbf{x}],[\mathbf{y}]) & \sim & (f(\mathbf{x},\mathbf{y}) \leq 0) \\ \mathscr{S}_4([\mathbf{y}]) & = & \underset{[\mathbf{x}]}{\text{proj}} \mathscr{S}_3([\mathbf{x}],[\mathbf{y}]) \\ \mathscr{S}([\mathbf{y}]) & = & \mathscr{S}_2([\mathbf{y}]) \wedge \mathscr{S}_4([\mathbf{y}]). \end{array}$$

Condiser for example the set

$$\mathbb{X} = \left\{ \mathbf{x} \in [-10, 10]^{\times 2} \mid \exists y \in [y], x_1^2 + x_1 \cdot x_2 + x_2^2 + y^2 = 10 \right\}$$

```
from pyibex import *
from vibes import vibes
f = Function("x1","x2","y", "x1^2+x1*x2+x2^2+y^2-10")
S1=SepFwdBwd(f,Interval(0,1000))
Y=Interval(-10,10)
S2=SepProj(S1,Y,0.001)
S3=~S1
S4=SepProj(S3,Y,0.001)
S=S2&S4

X0 =IntervalVector([[-10,10],[-10,10]]);
vibes.beginDrawing()
vibes.newFigure('Proj')
pySIVIA(X0,S,0.1)
```



Global optimization

Consider the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \text{ s.t. } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}.$$

Its epigraph is defined by

$$\mathbb{S} = \{(\mathbf{x},a) \in \mathbb{R}^n \times \mathbb{R} \mid a \geq f(\mathbf{x}) \text{ and } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}\}$$

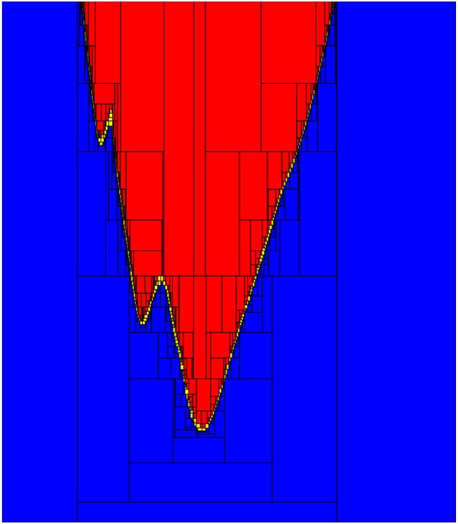
Define the *i*th *profile* of the epigraph

$$\mathbb{S}_i = \{(x_i,a) \in \mathbb{R} \times \mathbb{R} \mid \exists (x_1,\ldots,x_{i-1},x_i,\ldots,x_n) \mid a \geq f(\mathbf{x}) \text{ and } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}\}, i$$

Example.
Consider, the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} x_1^2 - x_2 + \sin x_1 x_2 \ \ \text{s.t.}\ \ x_1 + x_2 \in [1,2].$$

```python
from pyibex import *
from vibes import vibes
f = Function("x1","a","x2", "x1^2-x2+sin(x1*x2)-a")
g = Function("x1","a","x2", "x1+x2")
S1=SepFwdBwd(f,Interval(-1000,0))
S2=SepFwdBwd(g,Interval(1,2))
S3=S1&S2
S=SepProj(S3,Interval(-100,100),0.01)
vibes.beginDrawing()
pySIVIA(IntervalVector([[-5,5],[-5,5]]),S,0.1)
```

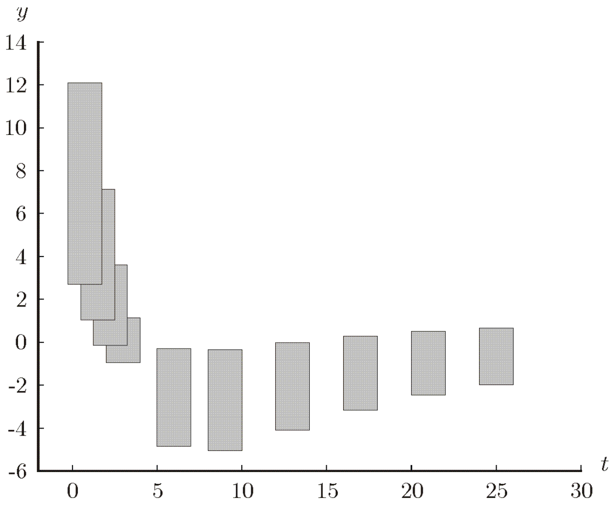


Bounded error estimation with uncertain times

Model:

$$\phi(\mathbf{p},t) = 20\exp(-p_1 t) - 8\exp(-p_2 t)$$

Data


y
14
12
10
8
6
4
2
0
-2
-4
-6
0
5
10
15
20
25
30
t

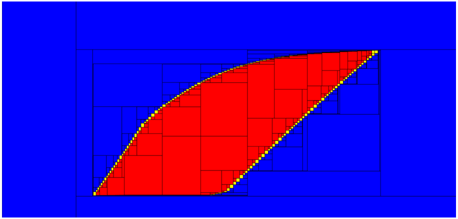
i	$\check{t}_i$	$[\check{t}_i]$	$[\check{y}_i]$
1	0.75	$[-0.25, 1.75]$	$[2.7, 12.1]$
2	1.5	$[0.5, 2.5]$	$[1.04, 7.14]$
3	2.25	$[1.25, 3.25]$	$[-0.13, 3.61]$
4	3	$[2, 4]$	$[-0.95, 1.15]$
5	6	$[5, 7]$	$[-4.85, -0.29]$
6	9	$[8, 10]$	$[-5.06, -0.36]$
7	13	$[12, 14]$	$[-4.1, -0.04]$
8	17	$[16, 18]$	$[-3.16, 0.3]$
9	21	$[20, 22]$	$[-2.5, 0.51]$
10	25	$[24, 26]$	$[-2, 0.67]$

The posterior feasible set is

$$\mathbb{S}_{\mathsf{p}} = \{\mathbf{p} \in [\mathbf{p}] \mid \exists t_1 \in [t_1], \ldots, \exists t_{10} \in [t_{10}], \phi(\mathbf{p}, t_1) \in [y_1], \ldots, \phi(\mathbf{p}, t_{10}) \in [y_{10}]\}$$

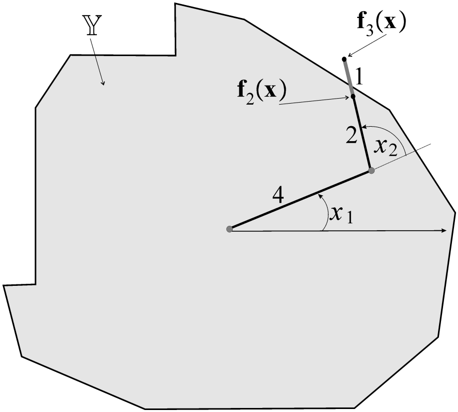
```
from pyibex import *
from vibes import vibes
f = Function("p1", "p2", "t", "20*exp(-p1*t)-8*exp(-p2*t)");
Y = [
    Interval(2.7,12.1),    Interval(1.04,7.14),    Interval(-0.13,3.61),
    Interval(-0.95,1.15), Interval(-4.85,-0.29), Interval(-5.06,-0.36),
    Interval(-4.1,-0.04), Interval(-3.16,0.3),    Interval(-2.5,0.51),
    Interval(-2,0.67)]
T = [
    Interval(-0.25,1.75), Interval(0.5,2.5),      Interval(1.25,3.25),
    Interval(2,4),         Interval(5,7),          Interval(8,10),
    Interval(12,14),       Interval(16,18),        Interval(20,22),
    Interval(24,26)]

seps = []
for Yi,Ti in zip(Y,T):
    S1=SepFwdBwd(f,Yi)
    S2=SepProj(S1,Ti,0.001)
    seps.append(S2)
S=SepInter(seps)
vibes.beginDrawing()
pySIVIA(IntervalVector([[0,1.2],[0,0.5]]), S, 0.01)
vibes.endDrawing()
```



Path planning

Wire loop game : a metal loop on a handle and a curved wire. The player holds the loop in one hand and attempts to guide it along the curved wire without touching.


𝕐
f3(x)
1
f2(x)
2
x2
4
x1

The feasible configuration space is

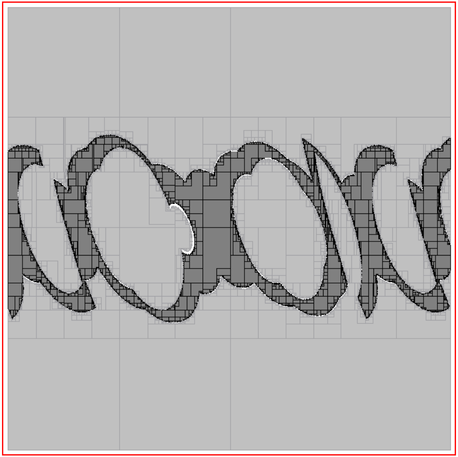
$$\mathbb{M} = \{(x_1, x_2) \in [-\pi, \pi] \mid \mathbf{f}_2(\mathbf{x}) \in \mathbb{Y} \text{ and } \mathbf{f}_3(\mathbf{x}) \notin \mathbb{Y}\}$$

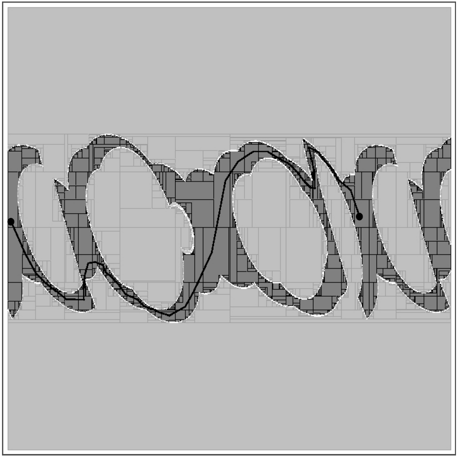
where

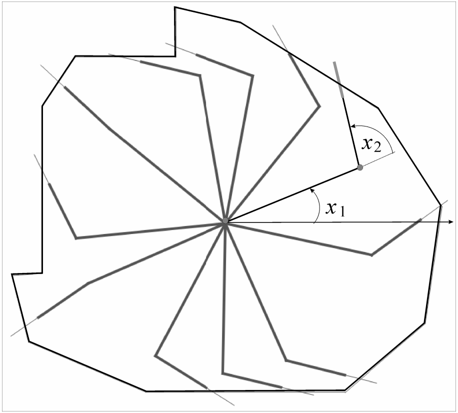
$$\mathbf{f}_\ell(\mathbf{x}) = 4\begin{pmatrix} \cos x_1 \\ \sin x_1 \end{pmatrix} + \ell \begin{pmatrix} \cos(x_1 + x_2) \\ \sin(x_1 + x_2) \end{pmatrix}.$$

A separator for $\mathbb{M}$ is

$$\mathscr{S}_{\mathbb{M}} = \mathbf{f}_2^{-1}(\mathscr{S}_{\mathbb{Y}}) \cap \mathbf{f}_3^{-1}\left(\overline{\mathscr{S}_{\mathbb{Y}}}\right).$$






x_2
x_1

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2. Separators [1]
3. IAMOOC [3]

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