Interval robotics

Chapter 4: Robust parameter estimation

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Exercise. A robot measures its own distance to three marks. The distances and the coordinates of the marks are

mark	x_i	y_i	d_i
1	0	0	$[22, 23]$
2	10	10	$[10, 11]$
3	30	-30	$[53, 54]$

1) Define the set $\mathbb{X}$ al all feasible positions.
2) Build the contractor associated with $\mathbb{X}$.
2) Build the contractor associated with $\overline{\mathbb{X}}$.

Solution.

$$\mathbb{X} = \bigcap_{i \in \{1,2,3\}} \underbrace{\left\{ (x,y) \ \mid \ (x-x_i)^2 + (y-y_i)^2 \in \left[d_i^-, d_i^+ \right] \right\}}_{\mathbb{X}_i}$$

$$\begin{aligned}
\overline{\mathbb{X}} &= \overline{\bigcap_{i\in\{1,2,3\}} \mathbb{X}_i} = \bigcup_{i\in\{1,2,3\}} \overline{\mathbb{X}_i} \\
&= \bigcup_{i\in\{1,2,3\}} \Big\{(x,y) \;\mid\; (x-x_i)^2 + (y-y_i)^2 \in \Big[-\infty, d_i^{-}\Big] \\
&\qquad \cup \Big\{(x,y) \;\mid\; (x-x_i)^2 + (y-y_i)^2 \in \Big[d_i^{+}, \infty\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{C} &= \bigcap_{i \in \{1,2,3\}} \mathcal{D}_{\left[d_i^-, d_i^+\right]} \\
\overline{\mathcal{C}} &= \bigcup_{i \in \{1,2,3\}} \left(\mathcal{D}_{\left[-\infty, d_i^-\right]}\right) \cup \left(\mathcal{D}_{\left[d_i^+, \infty\right]}\right)
\end{aligned}
$$

1 Relaxed intersection

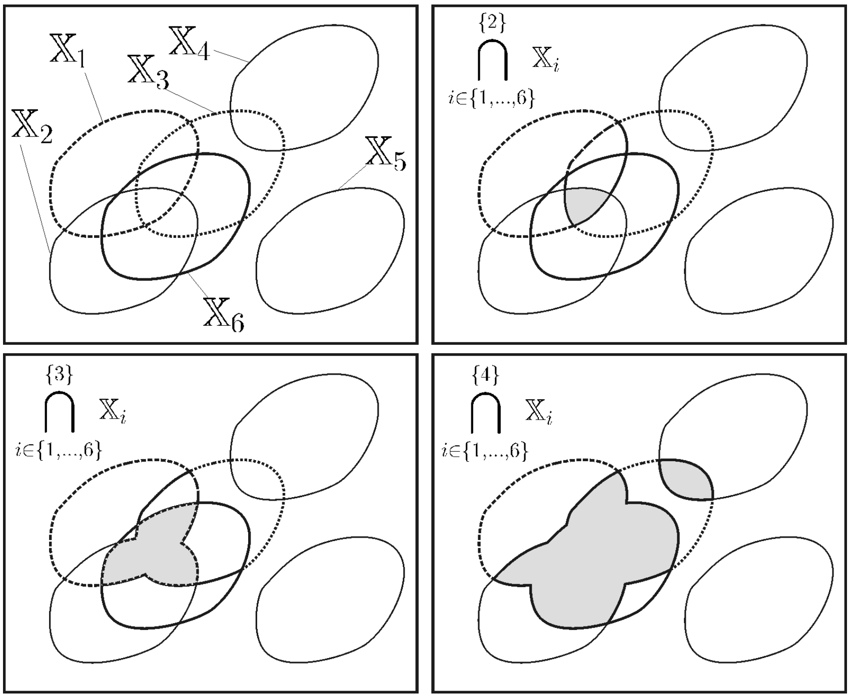
Dealing with outliers

$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Consider m sets $\mathbb{X}_{\mathbf{1}}, \dots, \mathbb{X}_m$ of $\mathbb{R}^n$. The q-relaxed intersection $\bigcap^{\{q\}} \mathbb{X}_i$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most.

We have

$$\mathbf{x} \in \bigcap^{\{q\}} \mathbb{X}_i \Leftrightarrow \# \{i | \mathbf{x} \in \mathbb{X}_i\} \geq m - q$$


$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\bigcap^{\{2\}}_{i\in\{1,\ldots,6\}} \mathbb{X}_i$
$\bigcap^{\{3\}}_{i\in\{1,\ldots,6\}} \mathbb{X}_i$
$\bigcap^{\{4\}}_{i\in\{1,\ldots,6\}} \mathbb{X}_i$

Exercise. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i \;=\; ?$$

$$\bigcap^{\{1\}} \mathbb{X}_i \;=\; ?$$

$$\bigcap^{\{5\}} \mathbb{X}_i \;=\; ?$$

$$\bigcap^{\{6\}} \mathbb{X}_i \;=\; ?$$

Solution. we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset \\
\bigcap^{\{1\}} \mathbb{X}_i &= \emptyset \\
\bigcap^{\{5\}} \mathbb{X}_i &= \bigcup \mathbb{X}_i \\
\bigcap^{\{6\}} \mathbb{X}_i &= \mathbb{R}^2
\end{aligned}$$

Exercise. Consider 8 intervals: $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?, \quad \bigcap^{\{1\}} \mathbb{X}_i = ?, \quad \bigcap^{\{2\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{3\}} \mathbb{X}_i = ?, \quad \bigcap^{\{4\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?, \quad \bigcap^{\{6\}} \mathbb{X}_i = ?.$$

Solution. For $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$, we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset, \ \bigcap^{\{1\}} \mathbb{X}_i = [3,4], \ \bigcap^{\{2\}} \mathbb{X}_i = [3,4], \\
\bigcap^{\{3\}} \mathbb{X}_i &= [2,4] \cup [6,7], \ \bigcap^{\{4\}} \mathbb{X}_i = [2,7], \\
\bigcap^{\{5\}} \mathbb{X}_i &= [1,9], \ \bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}.
\end{aligned}$$

If $\mathbb{X}_i$'s are intervals, the relaxed intersection can be computed with a complexity of $m \log m$.

Take all bounds of all intervals with their brackets.

Bounds	1	4	2	4	2	7	6	9	3	4	3	7
Brackets	[	]	[	]	[	]	[	]	[	]	[	]

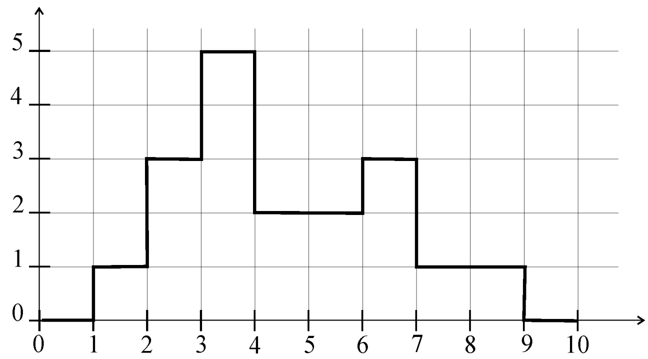
Sort the columns with respect the bounds:

Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]

Scan from left to right, counting $+1$ for '[' and -1 for ']':

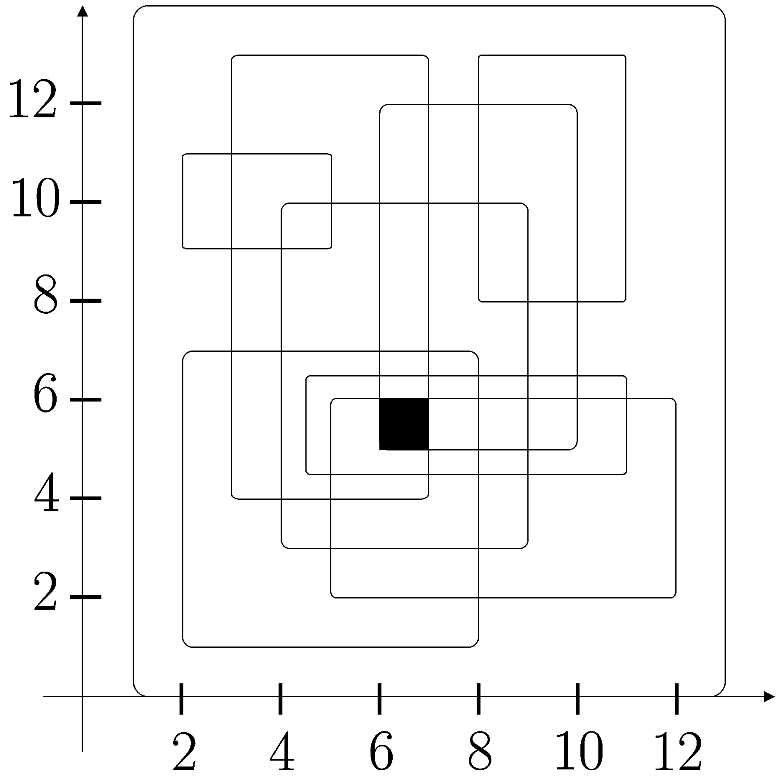
Bounds	1	2	2	3	3	4	4	4	6	7	7	9
Brackets	[	[	[	[	[	]	]	]	[	]	]	]
Sum	1	2	3	4	5	4	3	2	3	2	1	0

Read the q-intersections



Set-membership function associated with the 6 intervals

Computing the q relaxed intersection of m boxes is tractable.



The black box is the 2-intersection of 9 boxes

Formal definition

$$\bigcap^{\{q\}} \mathbb{X}_i = \bigcup_{\{\sigma_1,\ldots,\sigma_{m-q}\}} \mathbb{X}_{\sigma_1} \cap \cdots \cap \mathbb{X}_{\sigma_{m-q}}$$

$$\bigcup^{\{q\}} \mathbb{X}_i = \bigcap_{\{\sigma_1,\ldots,\sigma_{m-q}\}} \mathbb{X}_{\sigma_1} \cup \cdots \cup \mathbb{X}_{\sigma_{m-q}}$$

Remark

$$
\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \bigcap \mathbb{X}_i \\
\bigcup^{\{0\}} \mathbb{X}_i &= \bigcup \mathbb{X}_i
\end{aligned}
$$

De Morgan's law

$$
\begin{aligned}
\overline{\bigcap^{\{q\}} \mathbb{X}_i} &= \bigcup^{\{q\}} \overline{\mathbb{X}_i} \\
\overline{\bigcup^{\{q\}} \mathbb{X}_i} &= \bigcap^{\{q\}} \overline{\mathbb{X}_i}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
\overline{\bigcap^{\{q\}} \mathbb{X}_i} &= \overline{\bigcup_{\{\sigma_1,\ldots,\sigma_{m-q}\}} \mathbb{X}_{\sigma_1} \cap \cdots \cap \mathbb{X}_{\sigma_{m-q}}} \\
&= \bigcap_{\{\sigma_1,\ldots,\sigma_{m-q}\}} \overline{\mathbb{X}_{\sigma_1} \cap \cdots \cap \mathbb{X}_{\sigma_{m-q}}} \\
&= \bigcap_{\{\sigma_1,\ldots,\sigma_{m-q}\}} \overline{\mathbb{X}_{\sigma_1}} \cup \cdots \cup \overline{\mathbb{X}_{\sigma_{m-q}}} \\
&= \bigcup^{\{q\}} \overline{\mathbb{X}_i}.
\end{aligned}
$$

Dual rule

$$\bigcap^{\{q\}} \mathbb{X}_i = \bigcup^{\{m-q-1\}} \mathbb{X}_i$$

Proof. We have

$$
\begin{aligned}
\bigcup^{\{m-q-1\}} \mathbb{X}_i &\overset{\text{(de Morgan)}}{=} \overline{\bigcap^{\{m-q-1\}} \overline{\mathbb{X}_i}} \\
&= \overline{\left\{\mathbf{x} \mid \#\left\{i|\mathbf{x} \in \overline{\mathbb{X}_i}\right\} \geq m-(m-q-1)\right\}} \\
&= \left\{\mathbf{x} \mid \#\left\{i|\mathbf{x} \in \overline{\mathbb{X}_i}\right\} < q+1\right\}
\end{aligned}
$$

Now

$$
\#\left\{i|\mathbf{x} \in \overline{\mathbb{X}_i}\right\} + \#\left\{i|\mathbf{x} \in \mathbb{X}_i\right\} = m
$$

or equivalently

$$
\#\left\{i|\mathbf{x} \in \overline{\mathbb{X}_i}\right\} = m - \#\left\{i|\mathbf{x} \in \mathbb{X}_i\right\}
$$

Thus

$$
\begin{aligned}
\bigcup^{\{m-q-1\}} \mathbb{X}_i &= \left\{\mathbf{x} \mid m - \#\left\{i|\mathbf{x} \in \mathbb{X}_i\right\} < q+1\right\} \\
&= \left\{\mathbf{x} \mid \#\left\{i|\mathbf{x} \in \mathbb{X}_i\right\} > m-q-1\right\} \\
&= \left\{\mathbf{x} \mid \#\left\{i|\mathbf{x} \in \mathbb{X}_i\right\} \geq m-q\right\} \\
&= \bigcap^{\{q\}} \mathbb{X}_i
\end{aligned}
$$

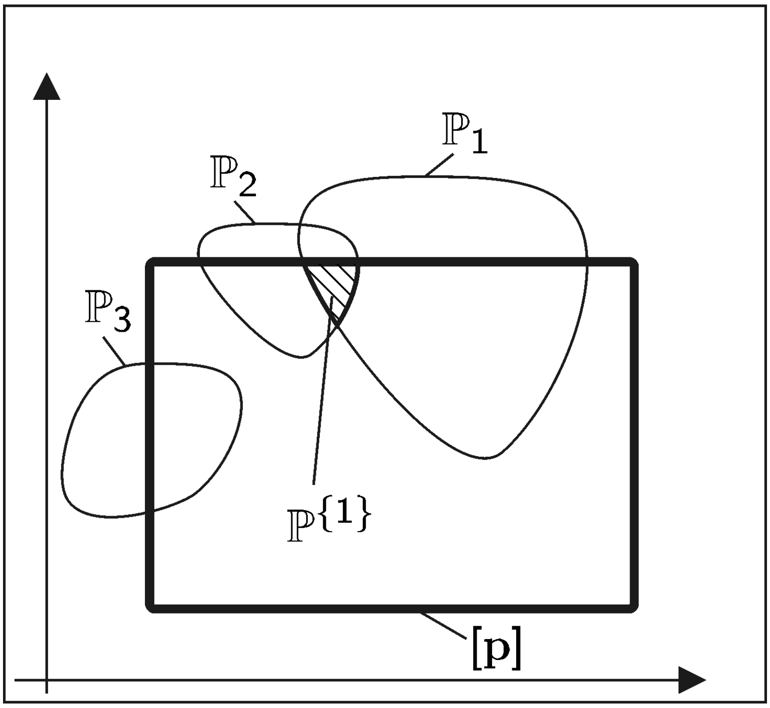
From the De Morgan's law and the dual rules, we get

$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} = \overline{\bigcup^{\{m-q-1\}} \mathbb{X}_i} = \bigcap^{\{m-q-1\}} \overline{\mathbb{X}_i}$$

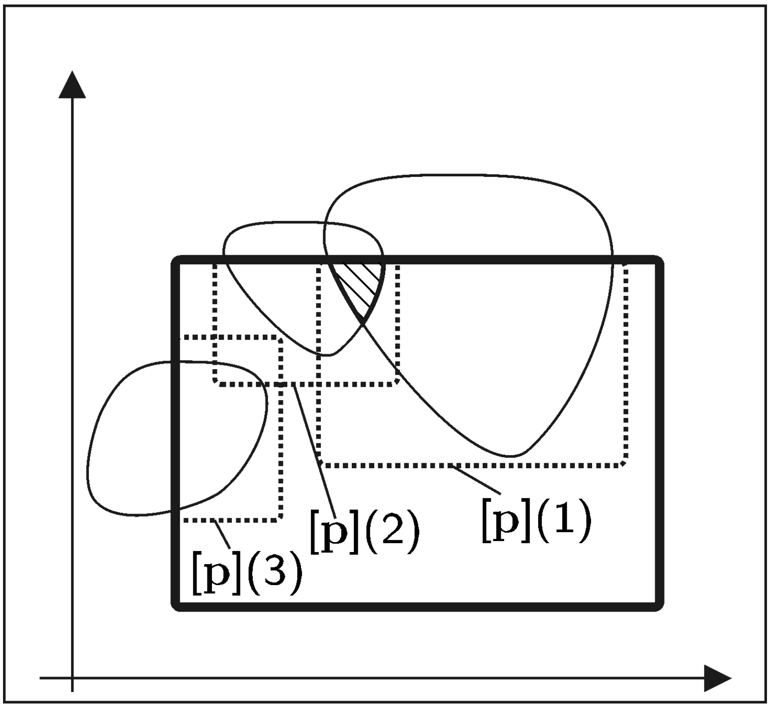
Relaxation of contractors

We define the q-relaxed intersection between m contractors

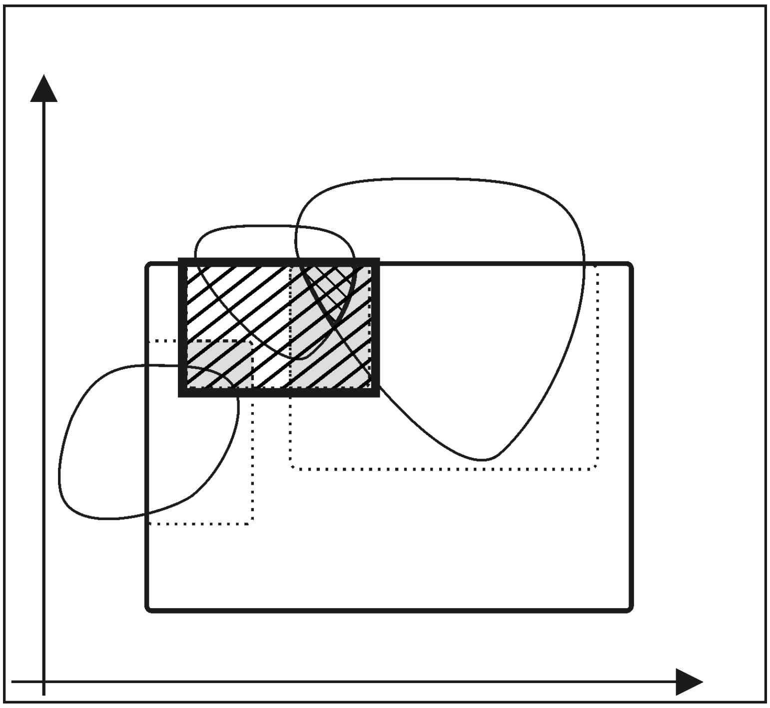
$$\mathcal{C} = \left(\bigcap_{i \in \{\mathbf{1},\dots,m\}}^{\{q\}} \mathcal{C}_i \right) \Leftrightarrow \forall\, [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\left([\mathbf{x}]\right) = \bigcap^{\{q\}} \mathcal{C}_i\left([\mathbf{x}]\right).$$

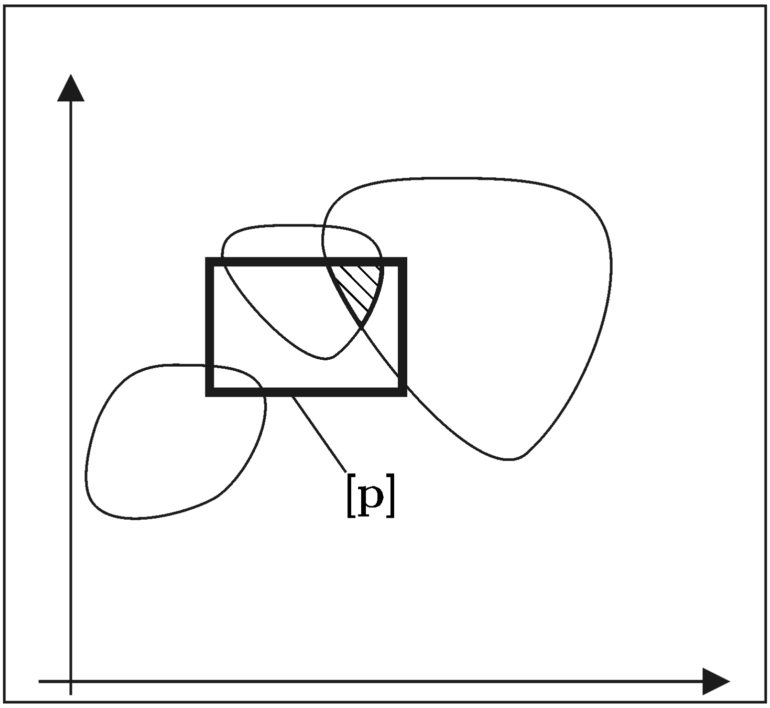


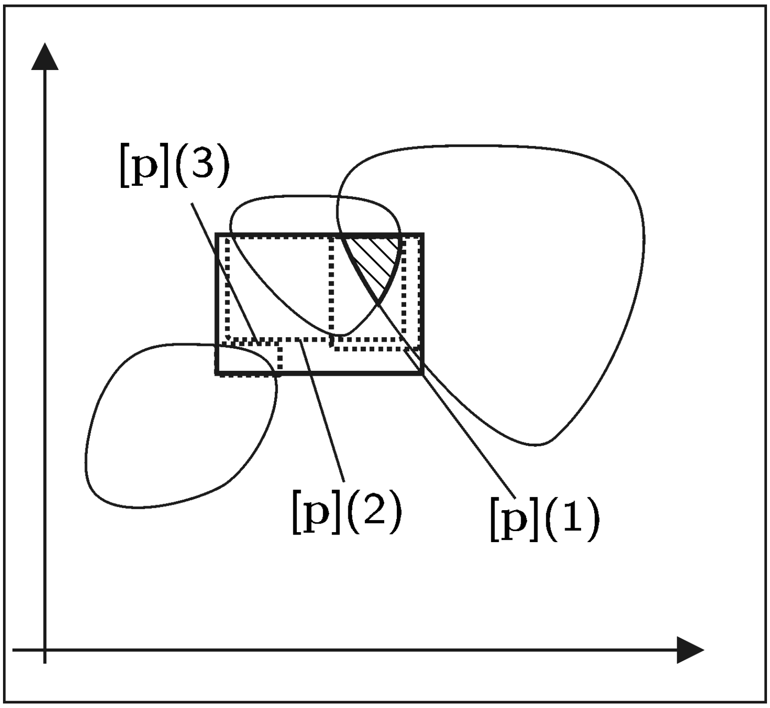
$\mathbb{P}_1$
$\mathbb{P}_2$
$\mathbb{P}_3$
$\mathbb{P}^{\{1\}}$
$[p]$

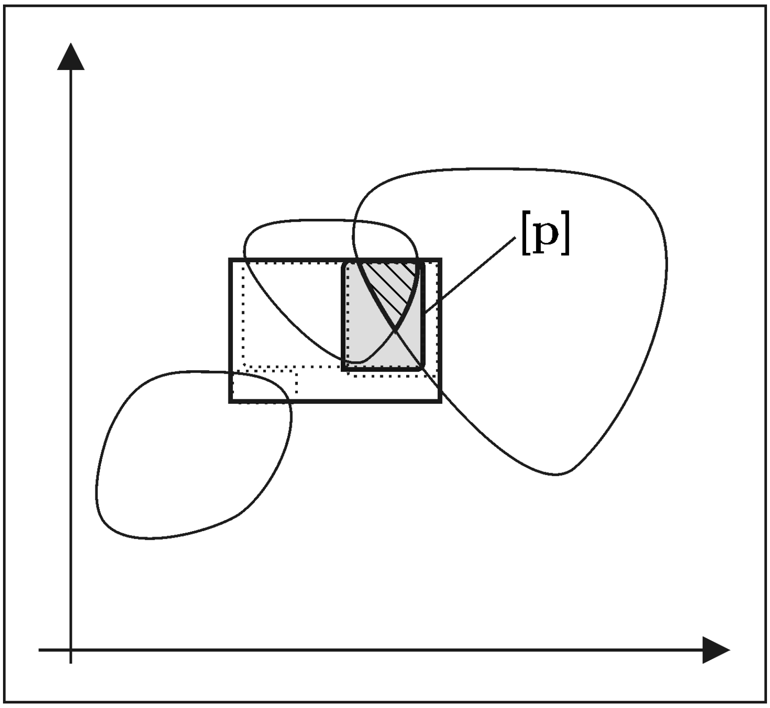


[p](1)
[p](2)
[p](3)




[p]


[p](3)
[p](2)
[p](1)

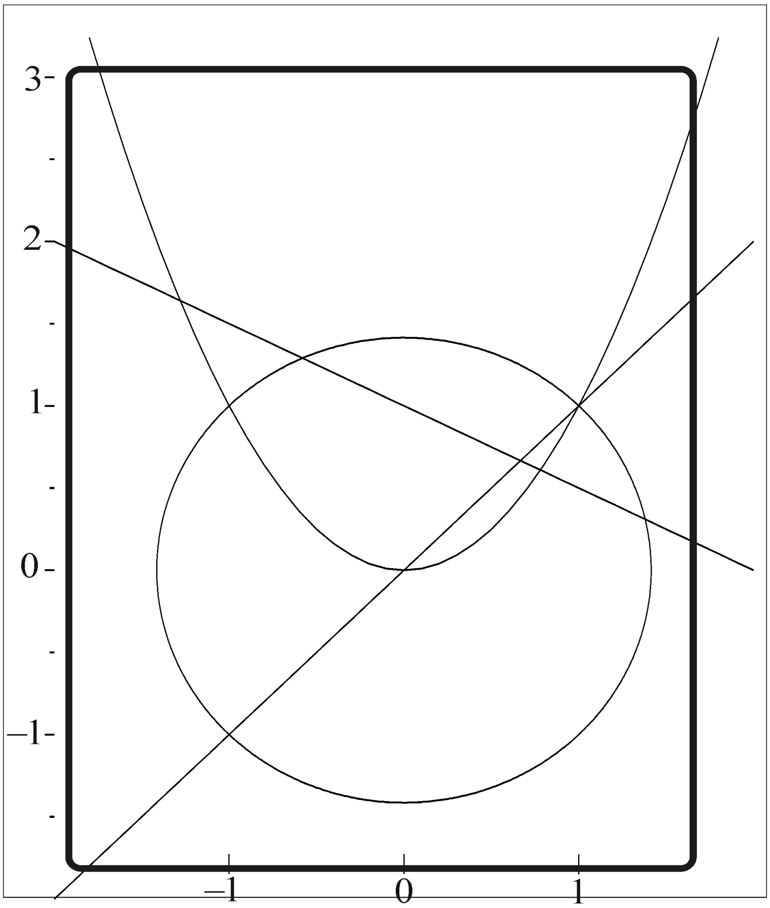

[p]

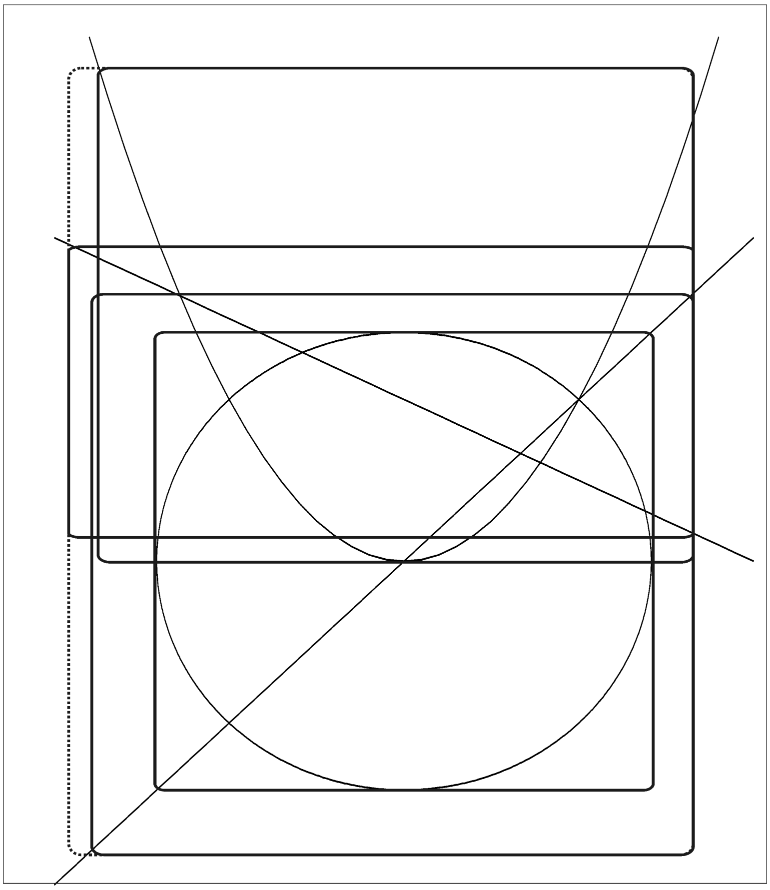
2 Solving a relaxed set of equalities

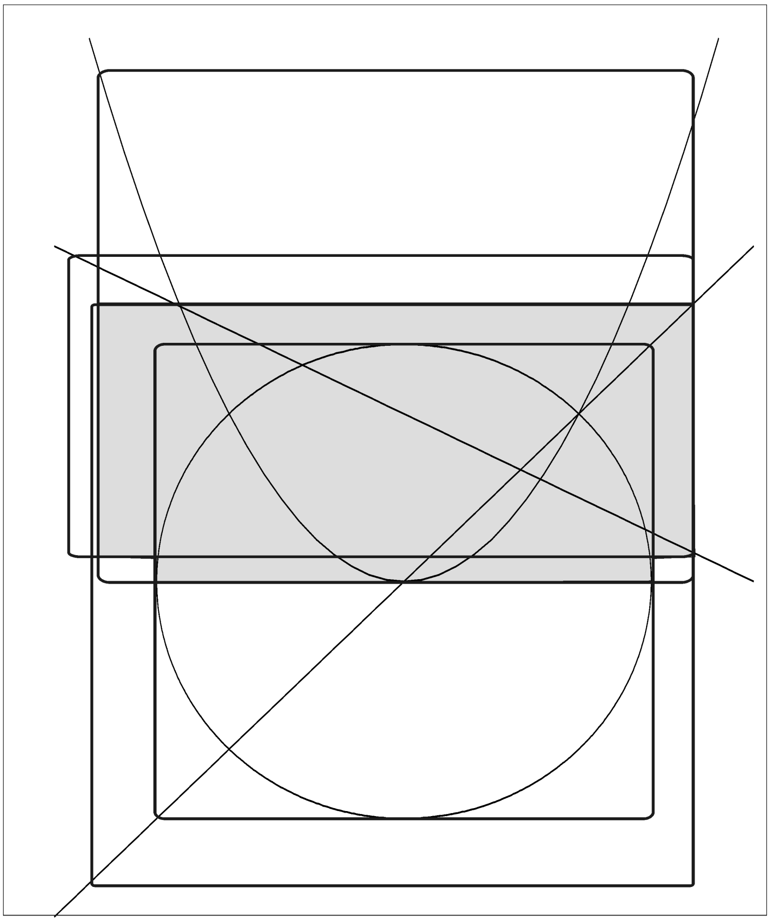
Solve

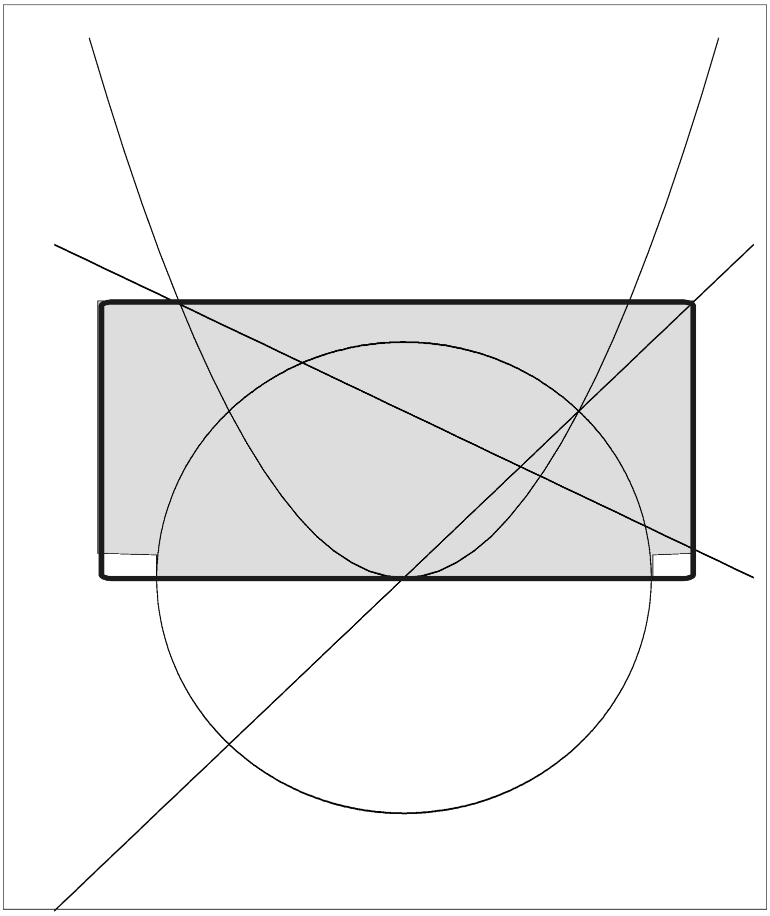
$$\begin{cases} p_2 - p_1^2 & = & 0 \\ p_2^2 + p_1^2 - 1 & = & 0 \\ p_2 - p_1 & = & 0 \\ 2p_2 + p_1 - 2 & = & 0 \end{cases}$$

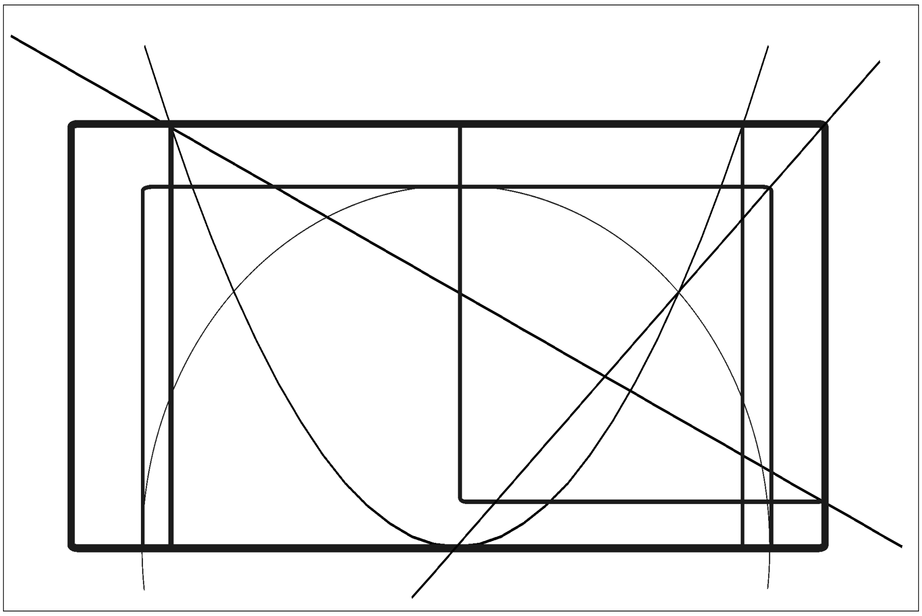
with $q = 1$.

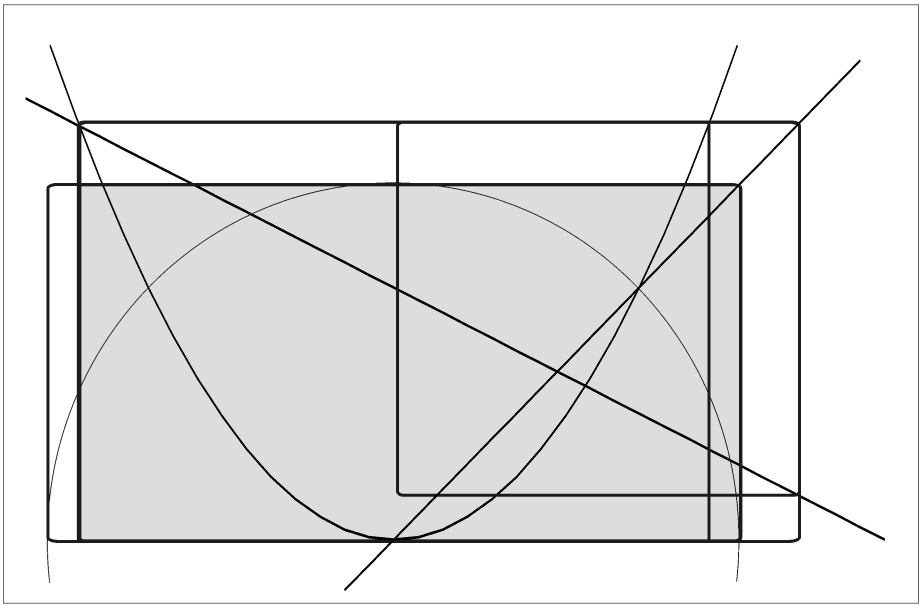

3
2
1
0
–1
–1
0
1

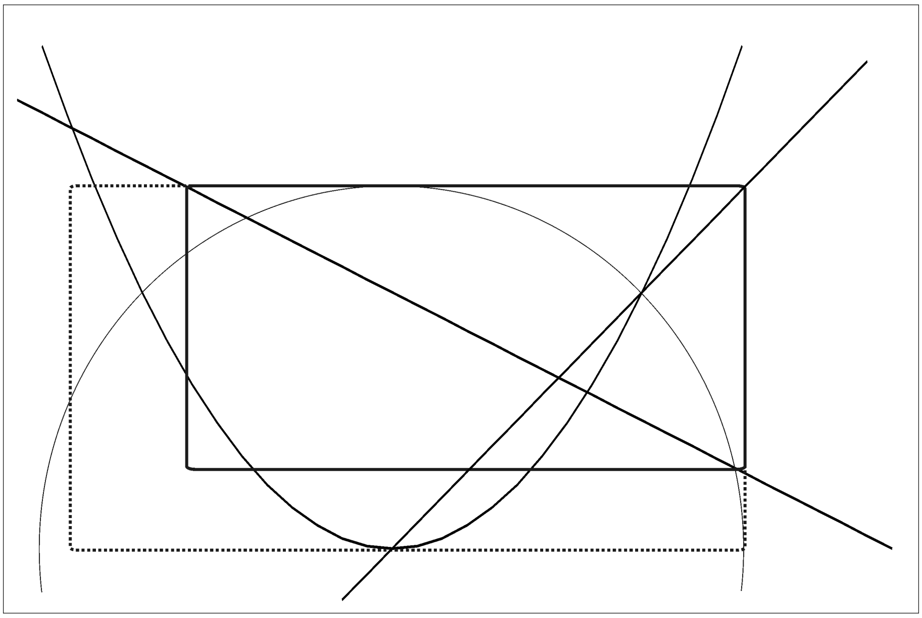


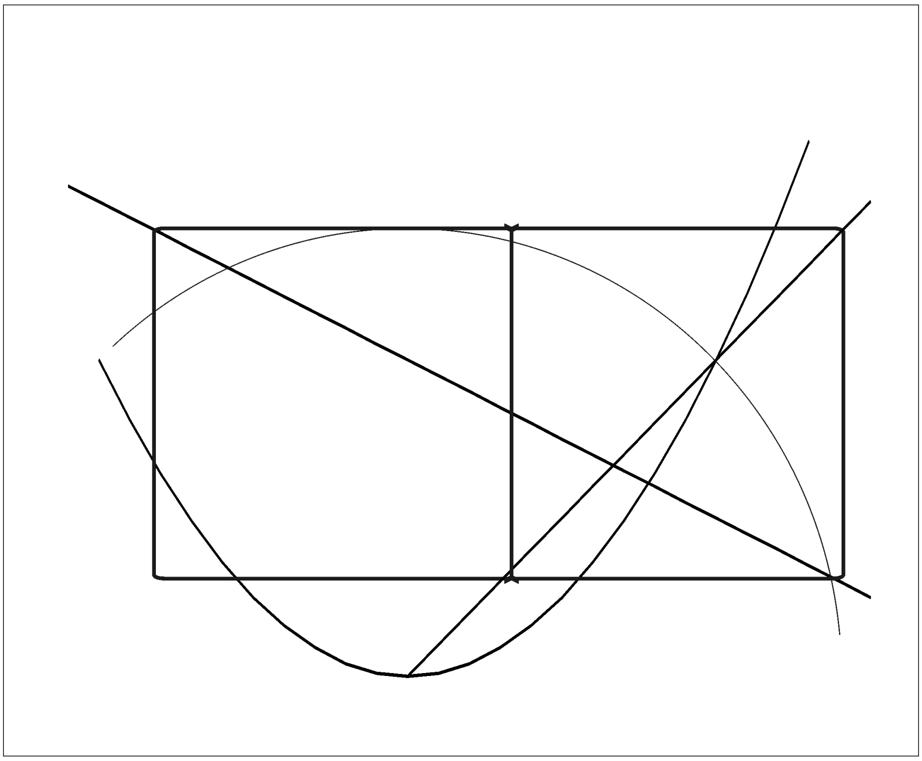


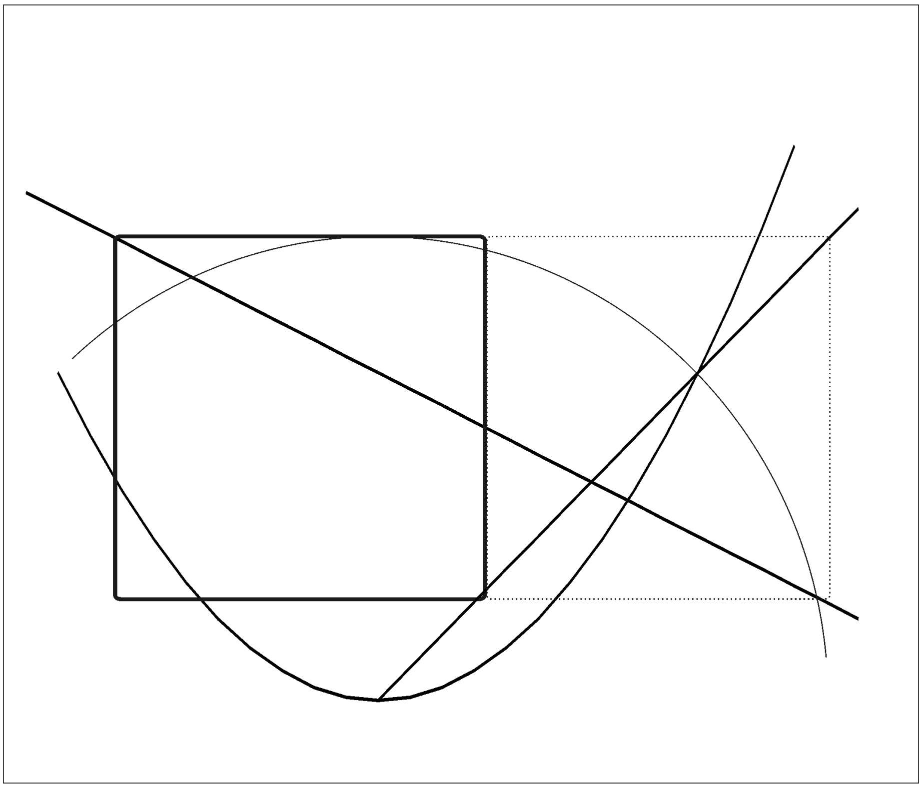


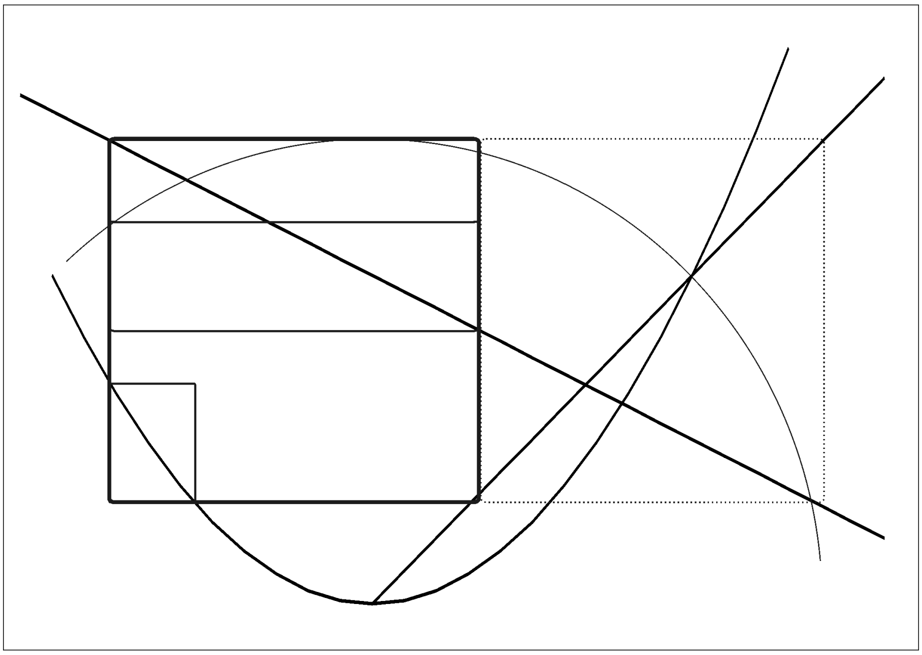


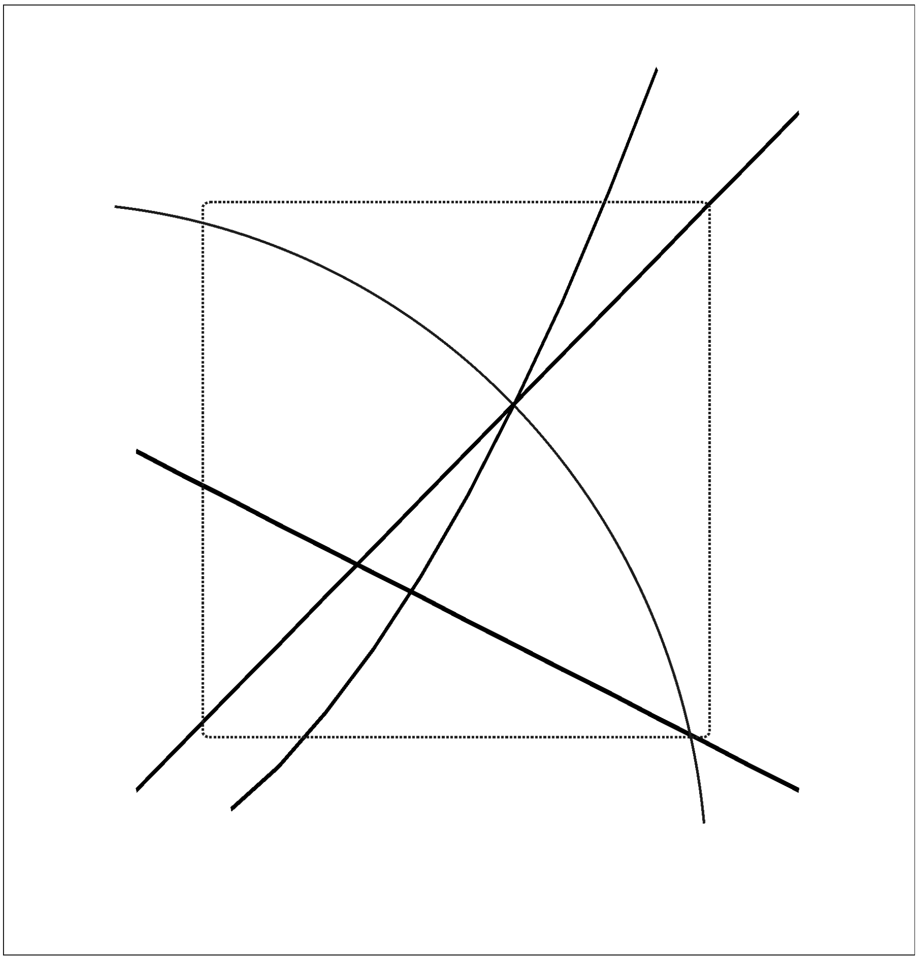


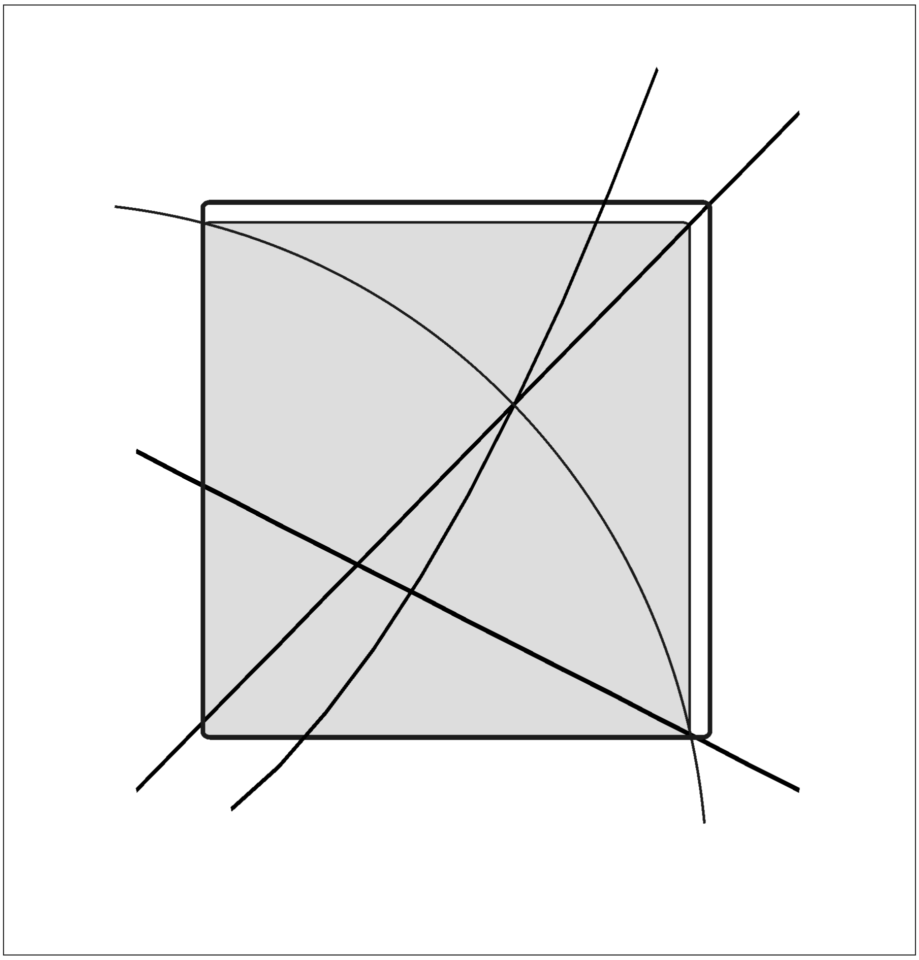


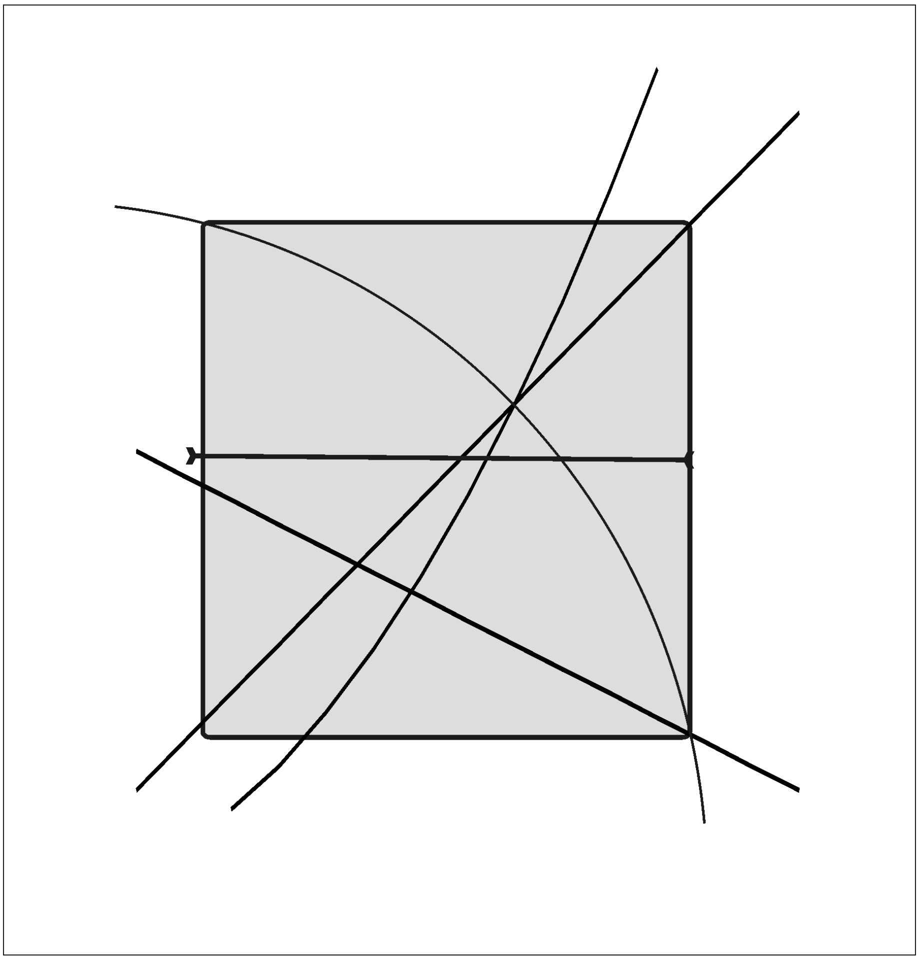


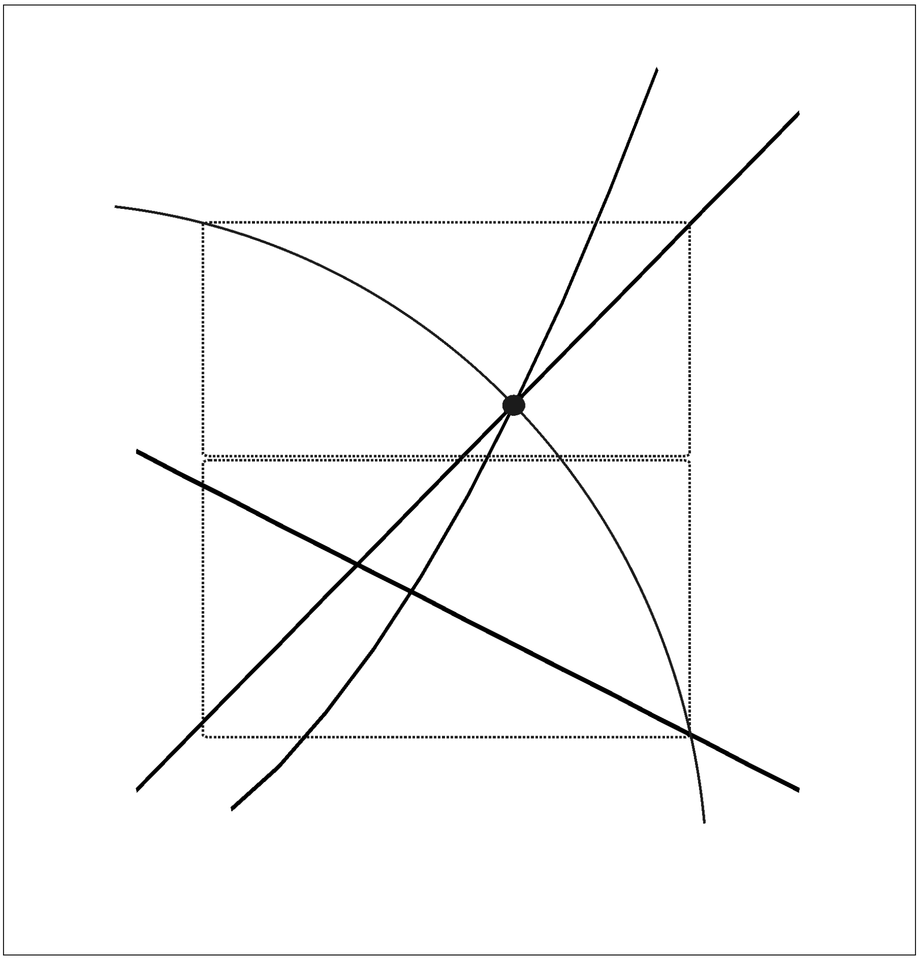




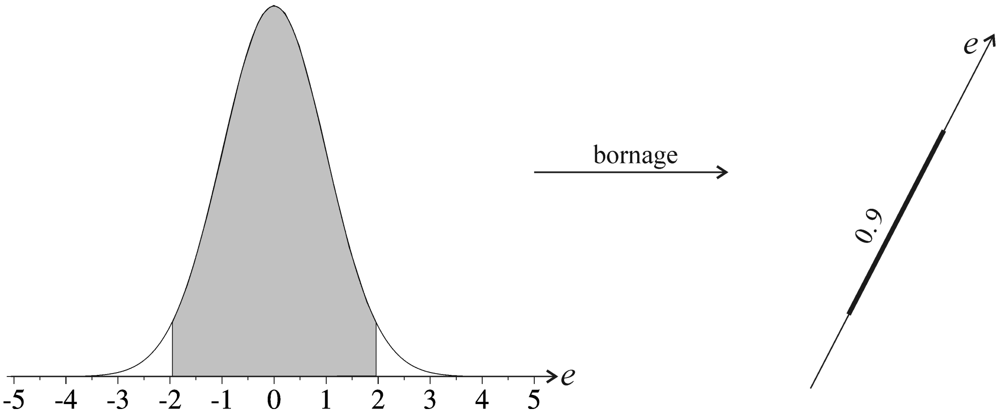


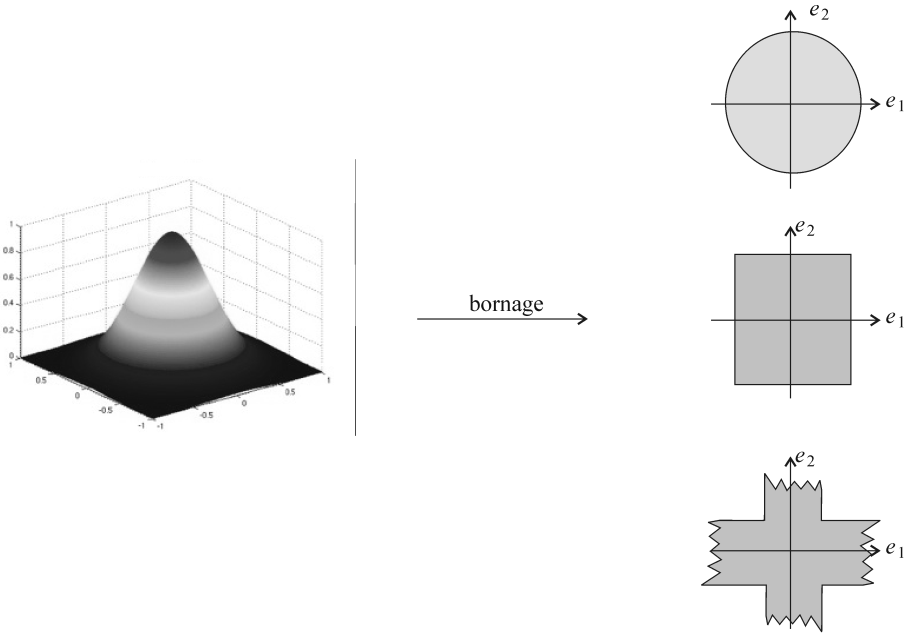






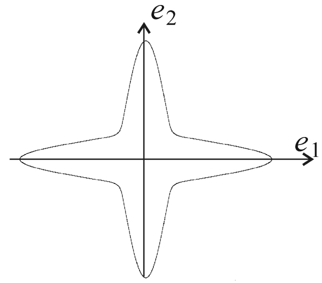
3 Probabilistic motivation

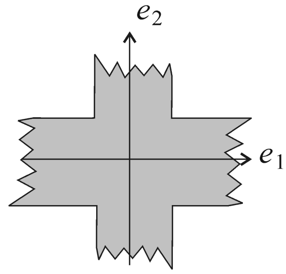

-5 -4 -3 -2 -1 0 1 2 3 4 5
e
bornage
e
0.9


bornage
e_2
e_1
e_2
e_1
e_2
e_1

$$\Pi(\mathbf{e}) \propto \left(\exp(-e_1^2) + \exp\left(-\frac{e_1^2}{10}\right)\right) * \left(\exp(-e_2^2) + \exp\left(-\frac{e_2^2}{10}\right)\right)$$

$\xrightarrow{\text{bornage}}$





Consider the error model

$$\mathbf{e} = \underbrace{\mathbf{y} - \boldsymbol{\psi}(\mathbf{p})}_{\mathbf{f}(\mathbf{y},\mathbf{p})}.$$

y_i is an *inlier* if $e_i \in [e_i]$ and an *outlier* otherwise. We assume that

$$\forall i,\ \mathsf{Pr}\left(e_i \in [e_i]\right) = \pi$$

and that all e_i's are independent.

Equivalently,

$$\begin{cases} f_1(\mathbf{y}, \mathbf{p}) \in [e_1] & \text{with a probability } \pi \\ \vdots & \vdots \\ f_m(\mathbf{y}, \mathbf{p}) \in [e_m] & \text{with a probability } \pi \end{cases}$$

Having k inliers follows a binomial distribution

$$\frac{m!}{k!\,(m-k)!}\pi^k.\,(1-\pi)^{m-k}\,.$$

The probability of having more than q outliers is thus

$$\gamma\left(q, m, \pi\right) \stackrel{\text{def}}{=} \sum_{k=0}^{m-q-1} \frac{m!}{k!\left(m-k\right)!} \pi^k . \left(1-\pi\right)^{m-k} .$$

Example. If $m = 1000$, $q = 900, \pi = 0.2$, we get $\gamma\left(q, m, \pi\right) = 7.04 \times 10^{-16}$. Thus having more than 900 outliers can be seen as a rare event.

4 Robust bounded error estimation

$$\mathbb{S} = \bigcap_{i}^{\{q\}} \{\mathbf{p} \in \mathbb{R}^n \mid f_i(\mathbf{p}) \in [y_i]\}$$

We build the following contractors

$$\begin{array}{rcl}
\mathcal{C}_i & : & f_i(\mathbf{p}) \in [y_i] \\
\overline{\mathcal{C}_i} & : & f_i(\mathbf{p}) \notin [y_i] \\
\mathcal{C} & = & \bigcap\limits_i^{\{q\}} \mathcal{C}_i \\
\overline{\mathcal{C}} & = & \overline{\bigcap\limits_i^{\{q\}} \mathcal{C}_i} = \bigcup\limits_i^{\{q\}} \overline{\mathcal{C}_i} = \bigcap\limits^{\{n-q-1\}} \overline{\mathcal{C}_i}
\end{array}$$

Then we call a paver with $\overline{\mathcal{C}}$ and $\mathcal{C}$.

5 Testcase

Generation of data. $m = 500$ data

$$\begin{cases} y_i = p_1 \sin(p_2 t_i) + e_i, \text{ with a probability } 0.2. \\ y_i = r_1 \exp(r_2 t_i) + e_i, \text{ with a probability } 0.2. \\ y_i = n_i \end{cases}$$

where $t_i = 0.02.i$, $i \in \{1, 500\}$, $e_i : \mathcal{U}([-0.1, 0.1])$ and $n_i : \mathcal{N}(2, 3)$. We took $\mathbf{p}^* = (2, 2)^\mathsf{T}$ and $\mathbf{r}^* = (4, -0.4)^\mathsf{T}$.

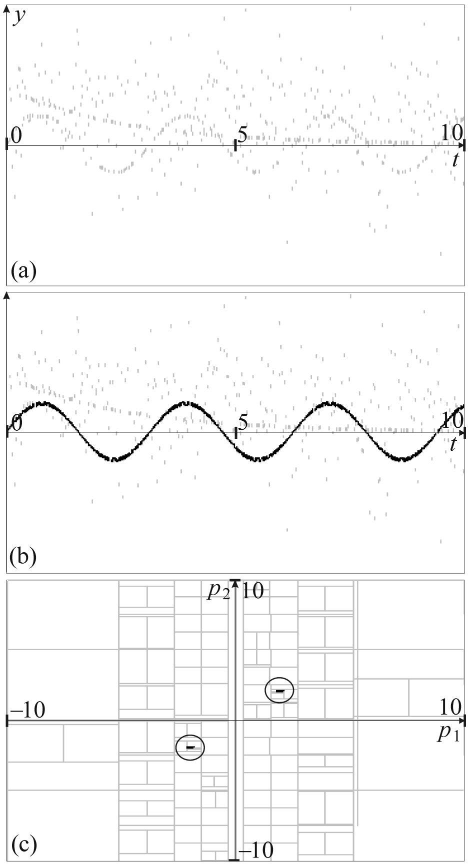
Estimation. We only know that

$$y_i = p_1 \sin(p_2 t_i) + e_i, \text{with a probability } 0.2.$$

We want

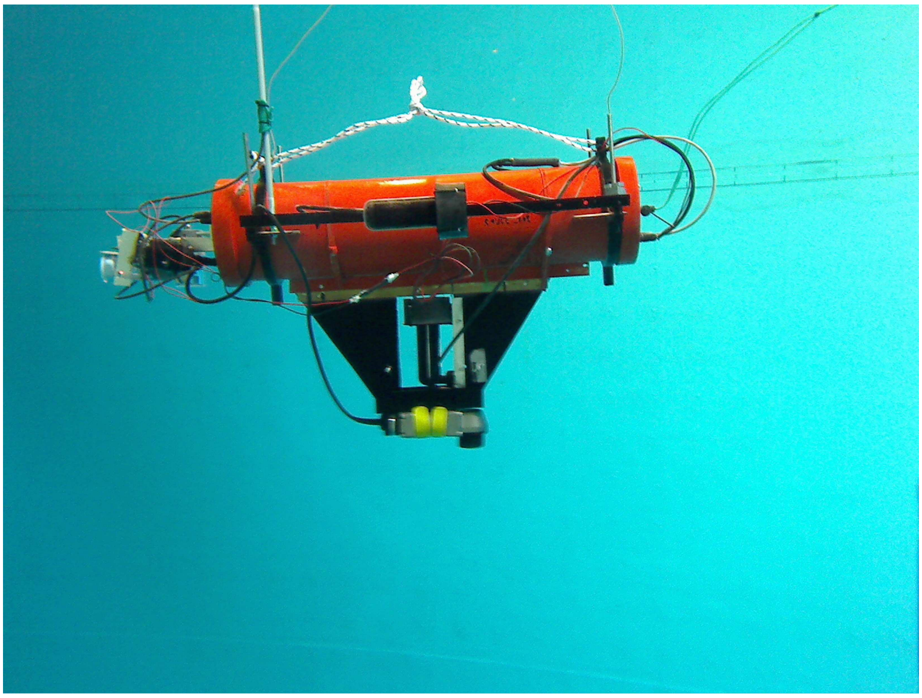
$$\Pr\left(\mathbf{p}^* \in \widehat{\mathbb{P}}\right) \geq 0.95$$

Since $\gamma(414, 500, 0.2) = 0.0468$ and $\gamma(413, 500, 0.2) = 0.12$, we should assume $q = 414$ outliers.

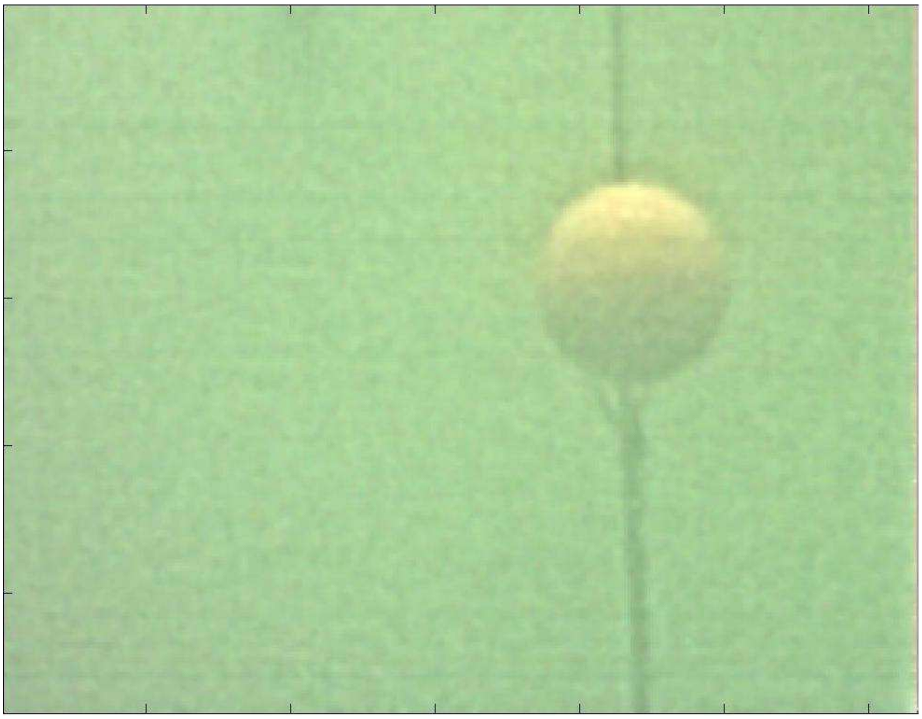


y
0
5
10
t
(a)
0
5
10
t
(b)
p2
10
−10
10
p1
−10
(c)

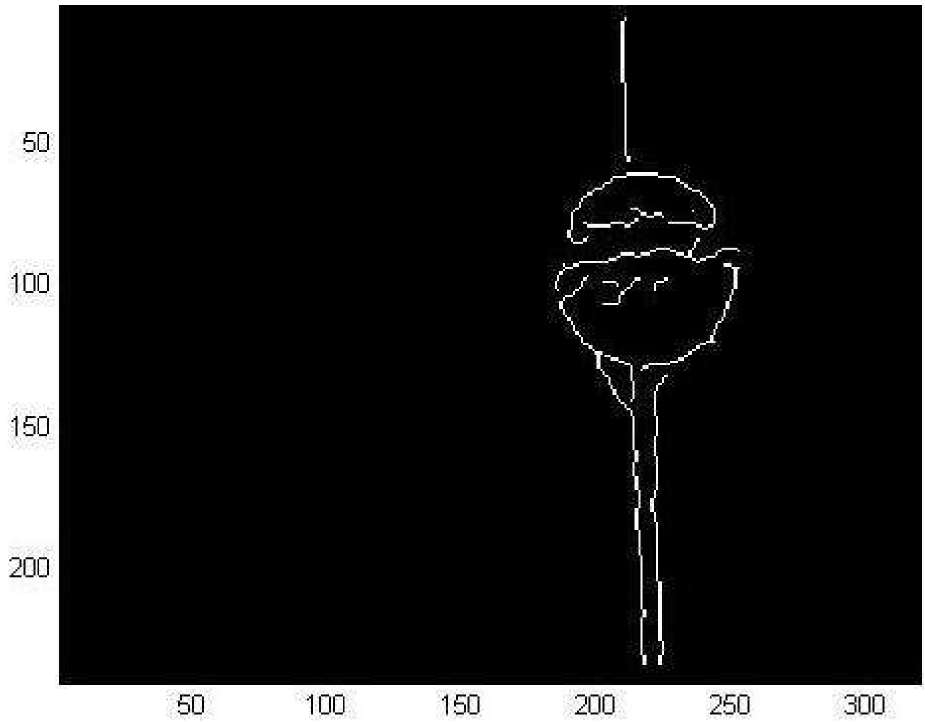
6 Shape detection



Sauc'isse robot swimming inside a pool



A spheric buoy seen by Sauc'isse


50
100
150
200
50
100
150
200
250
300

50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
600
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1

An *implicit parameter set estimation problem* amounts to characterizing

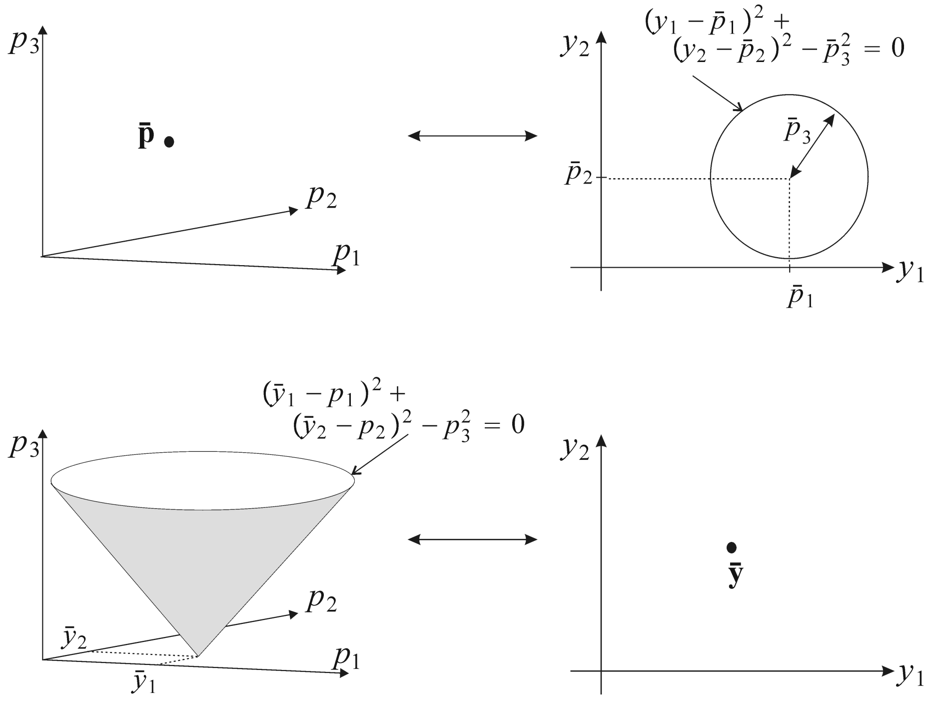
$$\mathbb{P} = \bigcap_{i \in \{1,\ldots,m\}} \underbrace{\{\mathbf{p} \in \mathbb{R}^n, \exists \mathbf{y} \in [\mathbf{y}](i), \mathbf{f}(\mathbf{p}, \mathbf{y}) = \mathbf{0}\}}_{\mathbb{P}_i}$$

where $\mathbf{p}$ is the parameter vector, $[\mathbf{y}](i)$ is the ith measurement box and $\mathbf{f}$ is the model function.

Consider the *shape function* $\mathbf{f}(\mathbf{p}, \mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^2$ corresponds to a pixel and $\mathbf{p}$ is the shape vector.

Example (circle):

$$f(\mathbf{p}, \mathbf{y}) = (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2.$$



p_3
$\bar{\mathbf{p}}$
p_2
p_1
$(y_1 - \bar{p}_1)^2 +$
$(y_2 - \bar{p}_2)^2 - \bar{p}_3^2 = 0$
y_2
$\bar{p}_3$
$\bar{p}_2$
y_1
$\bar{p}_1$
$(\bar{y}_1 - p_1)^2 +$
$(\bar{y}_2 - p_2)^2 - p_3^2 = 0$
p_3
p_2
$\bar{y}_2$
p_1
$\bar{y}_1$
y_2
$\bar{\mathbf{y}}$
y_1

The *shape* associated with $\mathbf{p}$ is

$$\mathcal{S}(\mathbf{p}) \overset{\text{def}}{=} \left\{ \mathbf{y} \in \mathbb{R}^2, \mathbf{f}(\mathbf{p}, \mathbf{y}) = \mathbf{0} \right\}.$$

Consider a set of (small) boxes in the image

$$\mathcal{Y} = \{[\mathbf{y}](1), \ldots, [\mathbf{y}](m)\}.$$

Each box is assumed to intersect the shape we want to extract.

In our buoy example,

- $\mathcal{Y}$ corresponds to edge pixel boxes.

- $f(\mathbf{p}, \mathbf{y}) = (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2$.

- $\mathbf{p} = (p_1, p_2, p_3)^\mathsf{T}$ where p_1, p_2 are the coordinates of the center of the circle and p_3 its radius.

Now, in our shape extraction problem, a lot of $[\mathbf{y}](i)$ are outlier.

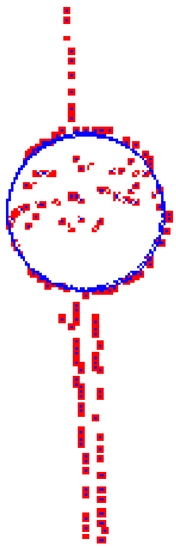
The q relaxed feasible set is

$$\mathbb{P}^{\{q\}} \stackrel{\text{def}}{=} \bigcap_{i \in \{\mathbf{1},\ldots,m\}}^{\{q\}} \left\{ \mathbf{p} \in \mathbb{R}^n, \exists \mathbf{y} \in [\mathbf{y}](i), \mathbf{f}\left(\mathbf{p},\mathbf{y}\right) = \mathbf{0} \right\}.$$

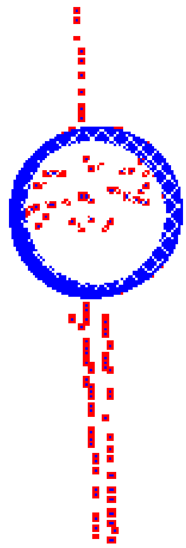
An optimal contractor for the set

$$\left\{\mathbf{p} \in [\mathbf{p}], \exists \mathbf{y} \in [\mathbf{y}], (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2 = 0\right\}.$$

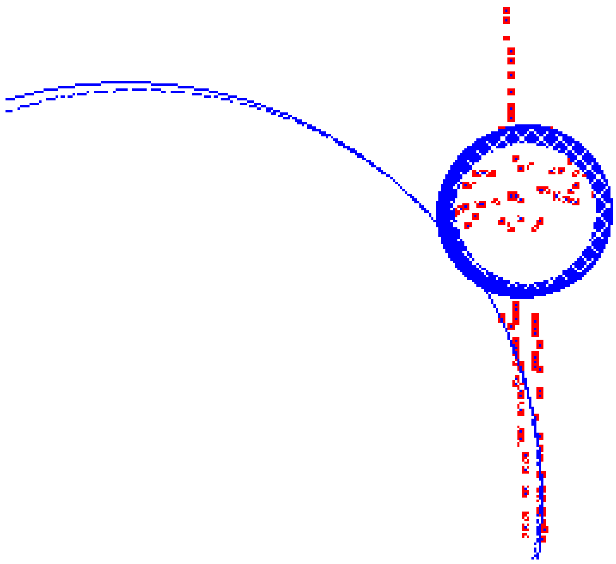
FB(in: $[\mathbf{y}], [\mathbf{p}]$, out: $[\mathbf{p}]$)
$[d_1] := [y_1] - [p_1]\,;$
$[d_2] := [y_2] - [p_2]\,;$
$[c_1] := [d_1]^2\,;$
$[c_2] := [d_2]^2\,;$
$[c_3] := [p_3]^2\,;$
$[e] := [0,0] \cap ([c_1] + [c_2] - [c_3])\,;$
$[c_1] := [c_1] \cap ([e] - [c_2] + [c_3])\,;$
$[c_2] := [c_2] \cap ([e] - [c_1] + [c_3])\,;$
$[c_3] := [c_3] \cap ([c_1] + [c_2] - [e])\,;$
$[\bar{p}_3] := [p_3] \cap \sqrt{[c_3]};$
$[d_2] := [d_2] \cap \sqrt{[c_2]};$
$[d_1] := [d_1] \cap \sqrt{[c_1]};$
$[p_2] := [p_2] \cap ([y_2] - [d_2])\,;$
$[p_1] := [p_1] \cap ([y_1] - [d_1])\,;$



$q = 0.70\ m$ (i.e. 70% of the data can be outlier)



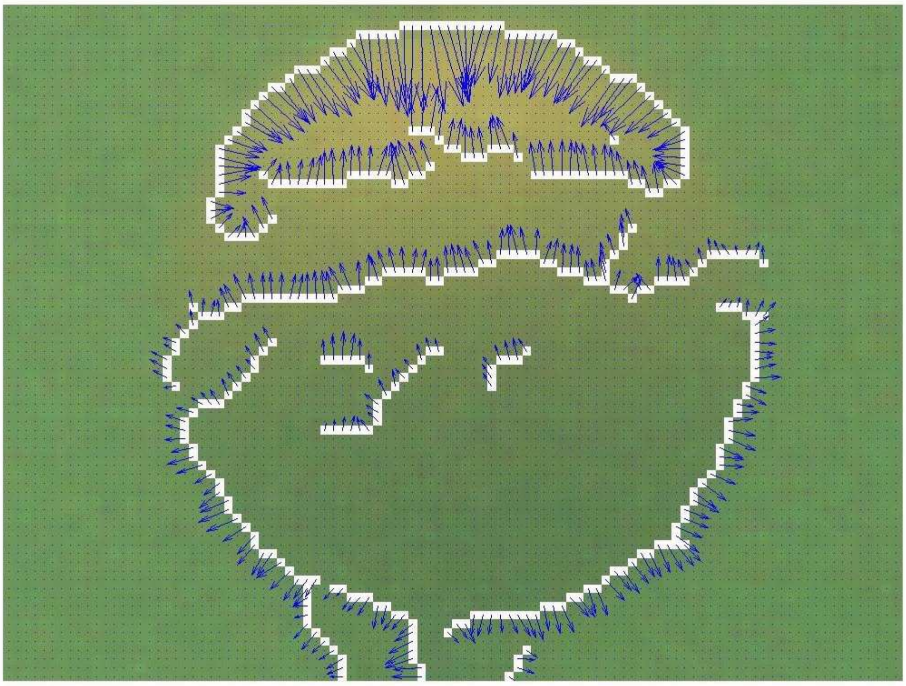
$q = 0.80\ m$ (i.e. 80% of the data can be outlier)



$q = 0.81\ m$ (i.e. 81% of the data can be outlier)

O'Gorman and Clowes (1976), in the context of the Hough transform (1972):

the local gradient of the image intensity is orthogonal to the edge.



Now, $\mathbf{y} = (y_1, y_2, y_3)^\mathsf{T}$ where y_3 is the direction of the gradient.

The gradient condition is

$$\det \begin{pmatrix} \frac{\partial f(\mathbf{p},\mathbf{y})}{\partial y_1} & \cos(y_3) \\ \frac{\partial f(\mathbf{p},\mathbf{y})}{\partial y_2} & \sin(y_3) \end{pmatrix} = 0.$$

For $f(\mathbf{p},\mathbf{y}) = (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2$, we get

$$\mathbf{f}(\mathbf{p},\mathbf{y}) = \begin{pmatrix} (y_1 - p_1)^2 + (y_2 - p_2)^2 - p_3^2 \\ (y_1 - p_1)\sin(y_3) - (y_2 - p_2)\cos(y_3) \end{pmatrix}.$$

New outliers: the edge points that are on the shape, but that do not satisfy the gradient condition.

The computing time is now 2 seconds instead of 15 seconds.

The Hough transform is defined by

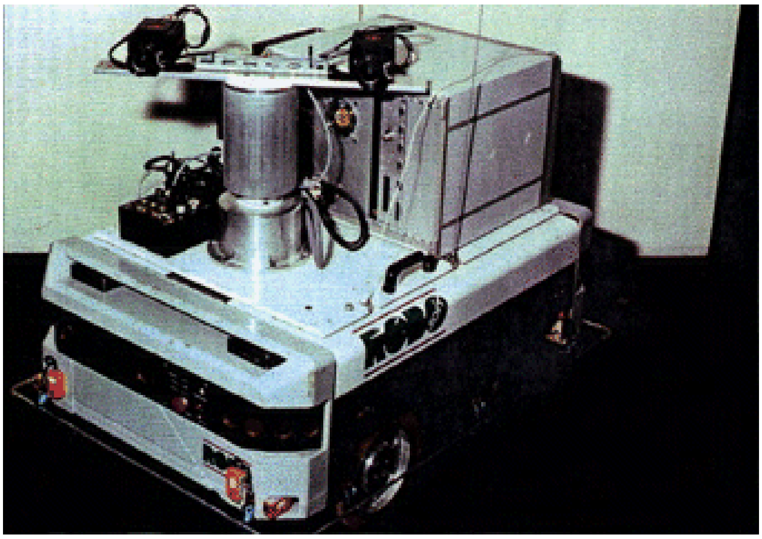
$$\eta(\mathbf{p}) = \text{card}\{i \in \{\mathbf{1}, \ldots, m\}, \exists \mathbf{y} \in [\mathbf{y}](i), \mathbf{f}(\mathbf{p}, \mathbf{y}) = \mathbf{0}\}.$$

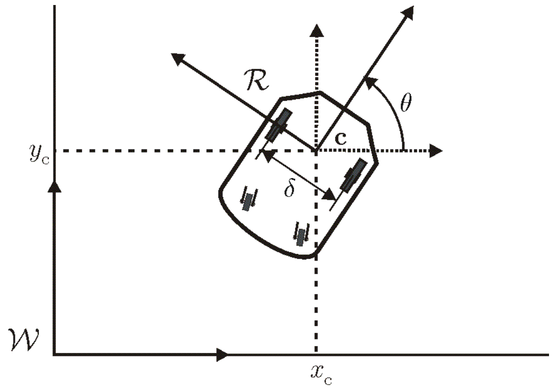
Hough method keeps all $\mathbf{p}$ such that $\eta(\mathbf{p}) \geq m - q$.

Instead, our approach solves $\eta(\mathbf{p}) \geq m - q$.

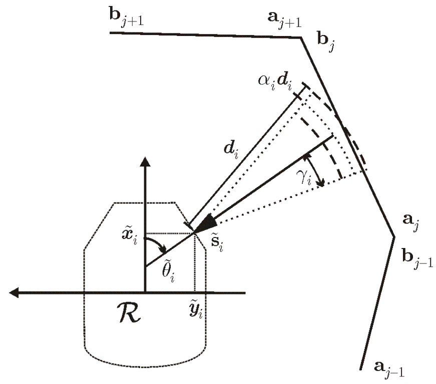
7 Static localization

Robot with 24 ultrasonic telemeters





$\mathcal{R}$
θ
$\mathbf{c}$
y_{c}
δ
$\mathcal{W}$
x_{c}



$\mathbf{b}_{j+1}$
$\mathbf{a}_{j+1}$
$\mathbf{b}_j$
$\alpha_i \boldsymbol{d}_i$
$\boldsymbol{d}_i$
γ_i
$\mathbf{a}_j$
$\tilde{\boldsymbol{x}}_i$
$\tilde{\mathbf{s}}_i$
$\mathbf{b}_{j-1}$
$\tilde{\theta}_i$
$\mathcal{R}$
$\tilde{\boldsymbol{y}}_i$
$\mathbf{a}_{j-1}$

After set inversion