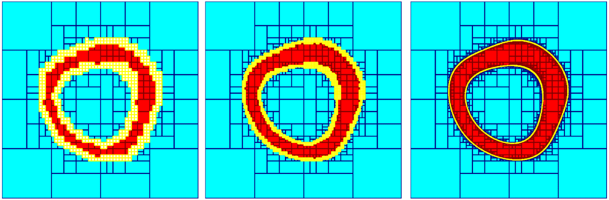
Interval Robotics
Chapter 3: Contractors

L. Jaulin

Contractors

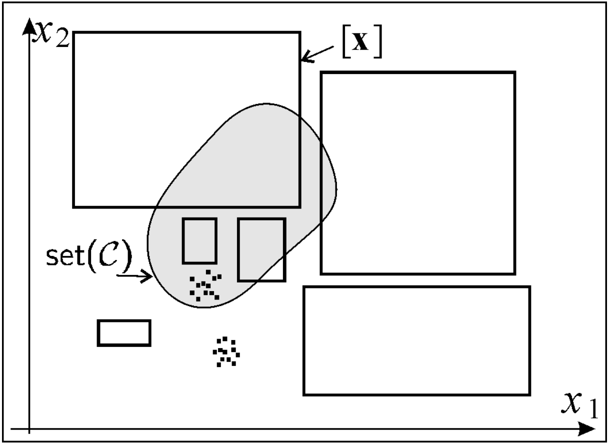


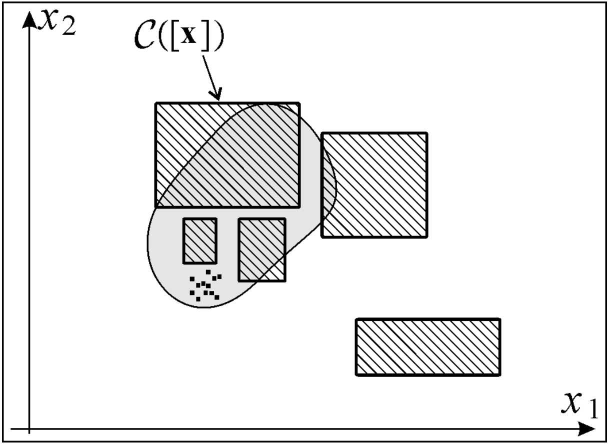
To characterize $\mathbb{X} \subset \mathbb{R}^n$, bisection algorithms bisect all boxes in all directions and become inefficient. Interval methods can still be useful if

- the solution set $\mathbb{X}$ is small (optimization problem, solving equations),
- contraction procedures are used as much as possible,
- bisections are used only as a last resort.

The operator $\mathscr{C}:\mathbb{IR}^n\rightarrow\mathbb{IR}^n$ is a *contractor* for $\mathbb{X}\subset\mathbb{R}^n$ if

$$\forall[\mathbf{x}]\in\mathbb{IR}^n,\left\{\begin{array}{ll}\mathscr{C}([\mathbf{x}])\subset[\mathbf{x}] & \text{(contractance)},\\ \mathscr{C}([\mathbf{x}])\cap\mathbb{X}=[\mathbf{x}]\cap\mathbb{X} & \text{(completeness)}.\end{array}\right.$$


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) \end{cases}$$

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) \end{cases}$$

$\mathscr{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}])$
$\mathscr{C}$ is *minimal* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathscr{C}([\mathbf{x}]) = [[\mathbf{x}] \cap \mathbb{X}]$
$\mathscr{C}$ is *thin* if	$\forall \mathbf{x} \in \mathbb{R}^n,\ \mathscr{C}(\{\mathbf{x}\}) = \{\mathbf{x}\} \cap \mathbb{X}$
$\mathscr{C}$ is *idempotent* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathscr{C}(\mathscr{C}([\mathbf{x}])) = \mathscr{C}([\mathbf{x}])$
$\mathscr{C}$ is *convergent* if	$[\mathbf{x}](k) \to \mathbf{x} \quad \Rightarrow \mathscr{C}([\mathbf{x}](k)) \to \{\mathbf{x}\} \cap \mathbb{X}$

Exercise. Replace the symbol $\bowtie$ either by $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$.

$$
\begin{array}{lcl}
\mathscr{C} \text{ minimal} & \bowtie & \mathscr{C} \text{ idempotent} \\
\mathscr{C} \text{ thin} & \bowtie & \mathscr{C} \text{ minimal} \\
\mathscr{C} \text{ minimal} & \bowtie & \mathscr{C} \text{ monotonic} \\
\mathscr{C} \text{ thin} & \bowtie & \mathscr{C} \text{ convergent}
\end{array}
$$

Solution. We have

$$\begin{array}{lcl}
\mathscr{C} \text{ minimal} & \Rightarrow & \mathscr{C} \text{ idempotent} \\
\mathscr{C} \text{ thin} & \Leftarrow & \mathscr{C} \text{ minimal} \\
\mathscr{C} \text{ minimal} & \Rightarrow & \mathscr{C} \text{ monotonic} \\
\mathscr{C} \text{ thin} & \Leftarrow & \mathscr{C} \text{ convergent}
\end{array}$$

Exercise. If $\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3,\mathscr{C}$ are contractors, do we always have

$$\begin{array}{ll}
\mathscr{C}^\infty \text{ is idempotent} & \text{(yes/no)} \\
(\mathscr{C}_1 \cap \mathscr{C}_2)^\infty = (\mathscr{C}_1 \circ \mathscr{C}_2)^\infty & \text{(yes/no)} \\
\mathscr{C}_1 \text{ minimal and } \mathscr{C}_2 \text{ minimal} \Rightarrow \mathscr{C}_1 \cup \mathscr{C}_2 \text{ minimal} & \text{(yes/no)} \\
\mathscr{C}_1 \text{ minimal and } \mathscr{C}_2 \text{ minimal} \Rightarrow \mathscr{C}_1 \cap \mathscr{C}_2 \text{ idempotent} & \text{(yes/no)} \\
\mathscr{C}_1 \sqcap (\mathscr{C}_2 \cup (\mathscr{C}_1 \cap \mathscr{C}_3)) \text{ is idempotent} & \text{(yes/no)}
\end{array}$$

Solution.

$\mathscr{C}^\infty$ is idempotent	Yes (if Scott contin
$(\mathscr{C}_1 \cap \mathscr{C}_2)^\infty = (\mathscr{C}_1 \circ \mathscr{C}_2)^\infty$	Yes
$\mathscr{C}_1$ minimal and $\mathscr{C}_2$ minimal $\Rightarrow \mathscr{C}_1 \cup \mathscr{C}_2$ minimal	Yes
$\mathscr{C}_1$ minimal and $\mathscr{C}_2$ minimal $\Rightarrow \mathscr{C}_1 \cap \mathscr{C}_2$ idempotent	No
$\mathscr{C}_1 \sqcap (\mathscr{C}_2 \cup (\mathscr{C}_1 \cap \mathscr{C}_3))$ is idempotent	Yes

Constraint projection

Exercice. Let x, y, z be 3 variables such that

$$\begin{array}{rcl} x & \in & [-\infty, 5] \\ y & \in & [-\infty, 4] \\ z & \in & [6, \infty] \\ z & = & x + y. \end{array}$$

Contract the intervals for x, y, z.

Solution. We have

$$\begin{aligned} x &\in [x] = [2,5] \\ y &\in [y] = [1,4] \\ z &\in [z] = [6,9], \end{aligned}$$

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$\begin{aligned}
z = x + y \Rightarrow \quad z \in \quad & [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) \\
& = [6, \infty] \cap [-\infty, 9] = [6, 9]. \\
x = z - y \Rightarrow \quad x \in \quad & [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) \\
& = [-\infty, 5] \cap [2, \infty] = [2, 5]. \\
y = z - x \Rightarrow \quad y \in \quad & [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) \\
& = [-\infty, 4] \cap [1, \infty] = [1, 4].
\end{aligned}$$

The contractor associated with $z = x + y$ is:

Algorithm pplus(inout: $[z], [x], [y]$)

$[z] := [z] \cap ([x] + [y])$ // $z = x + y$
$[x] := [x] \cap ([z] - [y])$ // $x = z - y$
$[y] := [y] \cap ([z] - [x])$ // $y = z - x$

The contractor associated with $z = x \cdot y$ is:

Algorithm pmult (inout: $[z], [x], [y]$)

$[z] := [z] \cap ([x] \cdot [y])$ // $z = x \cdot y$
$[x] := [x] \cap ([z] \cdot 1/[y])$ // $x = z/y$
$[y] := [y] \cap ([z] \cdot 1/[x])$ // $y = z/x$

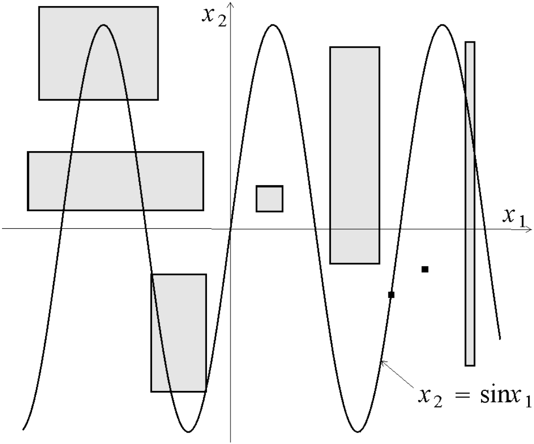
The contractor associated with $y = \exp x$ is:

	Algorithm pexp (inout: $[y], [x]$)
1	$[y] := [y] \cap \exp([x])$
2	$[x] := [x] \cap \log([y])$

Any constraint for which such a projection procedure is available will be called a *primitive constraint*.

Example. Consider the primitive equation:

$$x_2 = \sin x_1 .$$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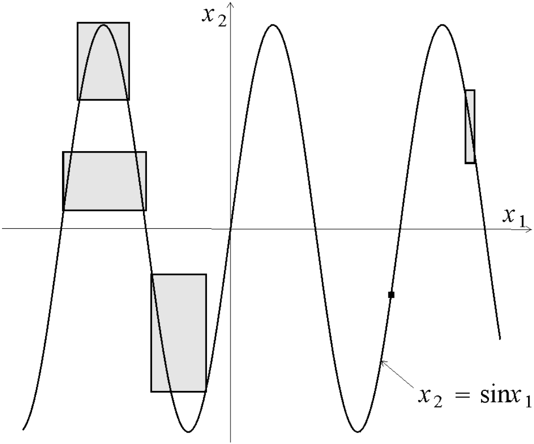
x_1
$x_2 = \sin x_1$


x_2
x_1
$x_2 = \sin x_1$

Decomposition

$$x + \sin(xy) \leq 0,$$
$$x \in [-1, 1], y \in [-1, 1]$$

Decomposition

$$x + \sin(xy) \leq 0,$$
$$x \in [-1,1], y \in [-1,1]$$

can be decomposed into

$$\left\{ \begin{array}{l} a = xy \\ b = \sin(a) \\ c = x + b \end{array} \right. , \qquad \begin{array}{ll} x \in [-1,1] & a \in [-\infty,\infty] \\ y \in [-1,1] & b \in [-\infty,\infty] \\ & c \in [-\infty,0] \end{array}$$

Forward Backward contractor

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2],$$

we decompose into

$$\begin{aligned} a &= x_1 + x_2 \\ b &= a \cdot x_3 \\ b &\in [1, 2] \end{aligned}$$

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1,2],$$

we have the following contractor:

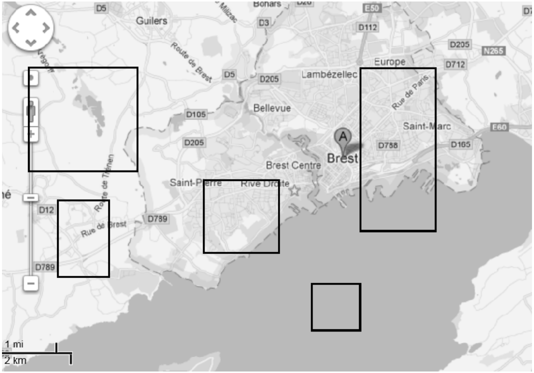
Algorithm $\mathscr{C}$ (inout$[x_1], [x_2], [x_3]$)	
$[a] = [x_1] + [x_2]$	// $a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	//$b = a \cdot x_3$
$[b] = [b] \cap [1,2]$	// $b \in [1,2]$
$[x_3] = [x_3] \cap \frac{[b]}{[a]}$	// $x_3 = \frac{b}{a}$
$[a] = [a] \cap \frac{[b]}{[x_3]}$	// $a = \frac{b}{x_3}$
$[x_1] = [x_1] \cap [a] - [x_2]$	// $x_1 = a - x_2$
$[x_2] = [x_2] \cap [a] - [x_1]$	// $x_2 = a - x_1$

Properties

$$
\begin{array}{rcl}
(\mathscr{C}_1^\infty \cap \mathscr{C}_2^\infty)^\infty & = & (\mathscr{C}_1 \cap \mathscr{C}_2)^\infty \\
(\mathscr{C}_1 \cap (\mathscr{C}_2 \cup \mathscr{C}_3)) & \supset & (\mathscr{C}_1 \cap \mathscr{C}_2) \cup (\mathscr{C}_1 \cap \mathscr{C}_3) \\
\left\{ \begin{array}{l} \mathscr{C}_1 \text{ minimal} \\ \mathscr{C}_2 \text{ minimal} \end{array} \right. & \Rightarrow & \mathscr{C}_1 \cup \mathscr{C}_2 \text{ minimal}
\end{array}
$$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.


Guilers
Bellevue
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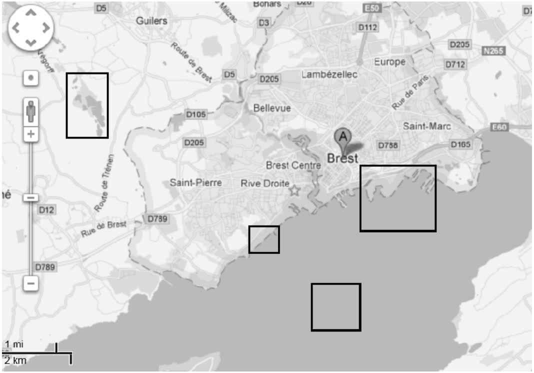
Saint-Marc
Brest
Brest Centre
Saint-Pierre
Rive Droite
1 mi
2 km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Rue de Brest
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D3
D112
E50
D205
N265
D712
D105
D788
D165
E60
D12
D789
1 mi
2 km

Propagation

A CN (Constraint Network) is composed of
1) a set of variables $\mathscr{V} = \{x_1, \ldots, x_n\}$,
2) a set of constraints $\mathscr{C} = \{c_1, \ldots, c_m\}$ and
3) a set of interval domains $\{[x_1], \ldots, [x_n]\}$.

Principle of propagation techniques: contract $[\mathbf{x}] = [x_1] \times \cdots \times [x_n]$ as follows:

$$((((((([\mathbf{x}] \sqcap c_1) \sqcap c_2) \sqcap \ldots) \sqcap c_m) \sqcap c_1) \sqcap c_2) \ldots,$$

until a steady box is reached.

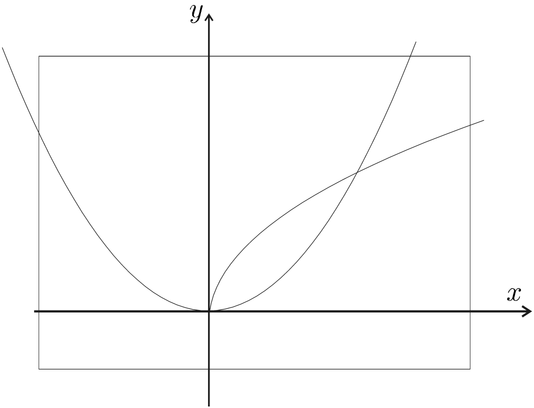
Consider the system of two equations.

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

We can build two contractors

$$\mathscr{C}_1 : \left\{ \begin{array}{l} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{array} \right. \quad \text{associated to } y = x^2$$

$$\mathscr{C}_2 : \left\{ \begin{array}{l} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{array} \right. \quad \text{associated to } y = \sqrt{x}$$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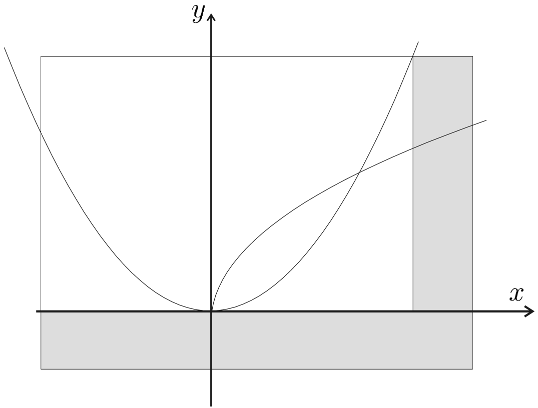
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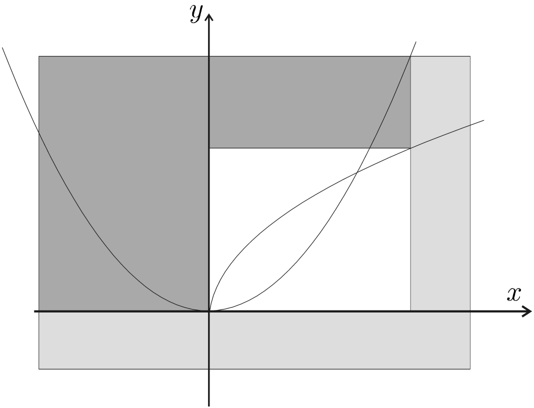
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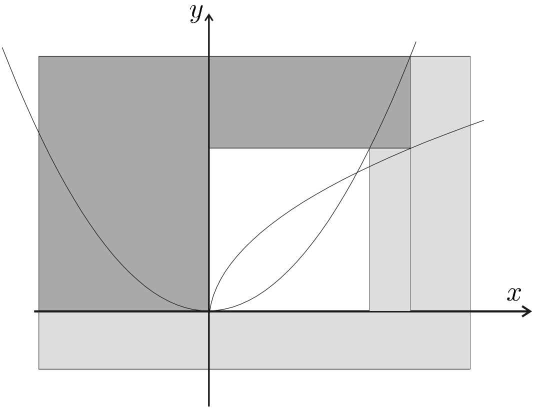
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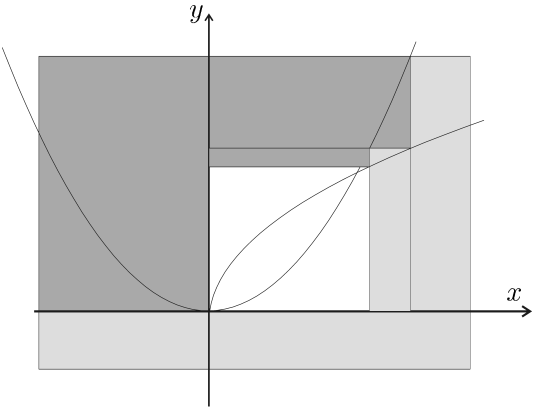
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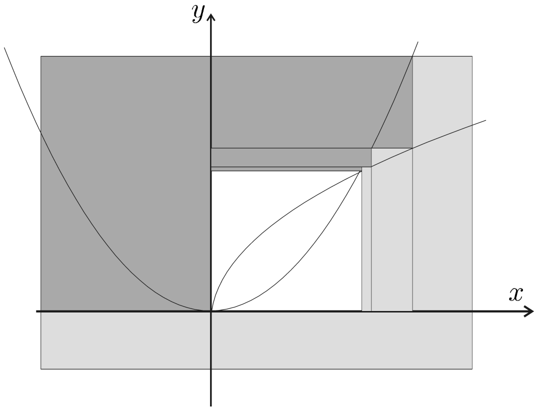
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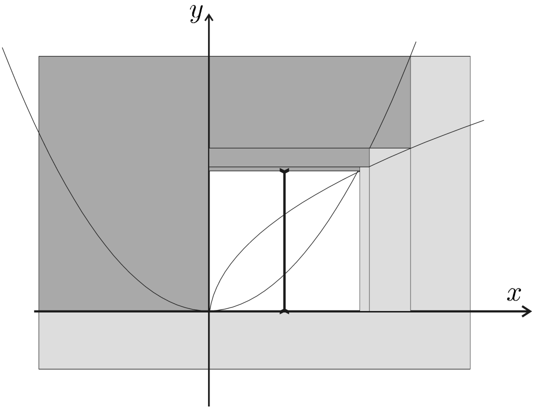
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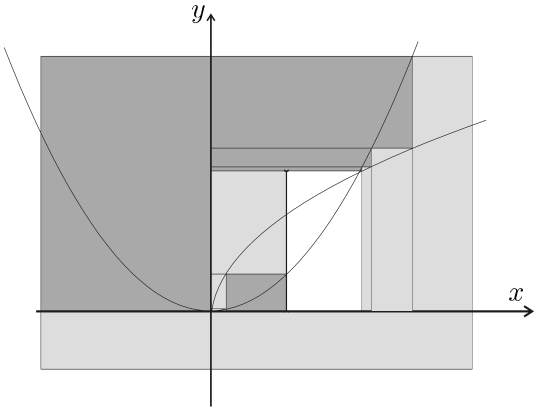
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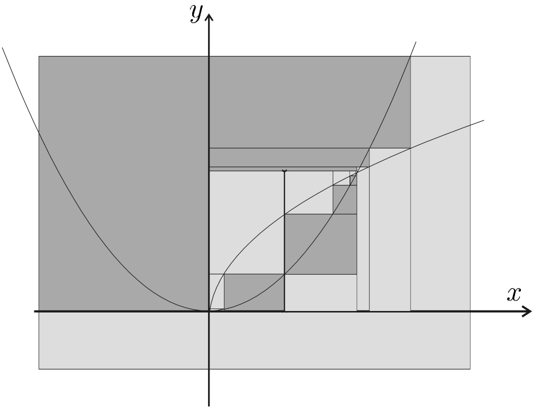
y
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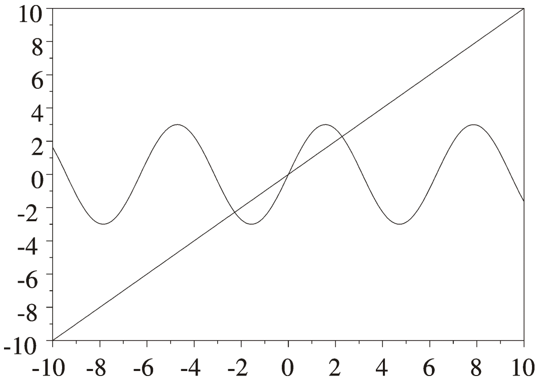
If $\mathscr{C}^*_{\mathbb{S}_1}$ and $\mathscr{C}^*_{\mathbb{S}_2}$ are two minimal contractors for $\mathbb{S}_1$ and $\mathbb{S}_2$ then

$$\mathscr{C}_{\mathbb{S}} = \mathscr{C}^*_{\mathbb{S}_1} \circ \mathscr{C}^*_{\mathbb{S}_2} \circ \mathscr{C}^*_{\mathbb{S}_1} \circ \mathscr{C}^*_{\mathbb{S}_2} \circ \dots$$

is a contractor for $\mathbb{S} = \mathbb{S}_1 \cap \mathbb{S}_2$, but it is not always optimal. This is the *local consistency effect.*

Exemple. Consider the system

$$\left\{ \begin{array}{rcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$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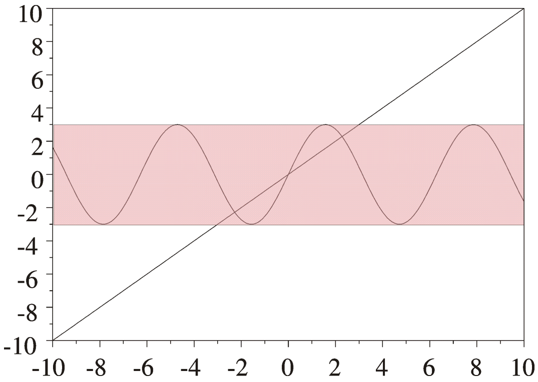
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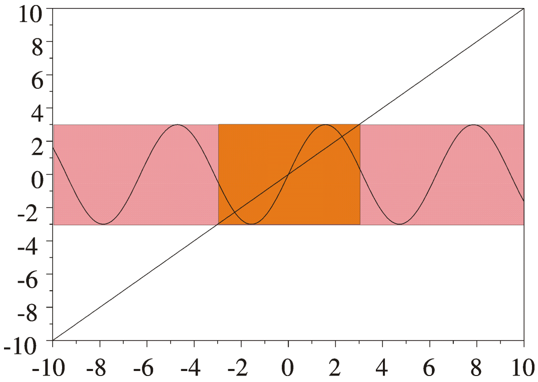
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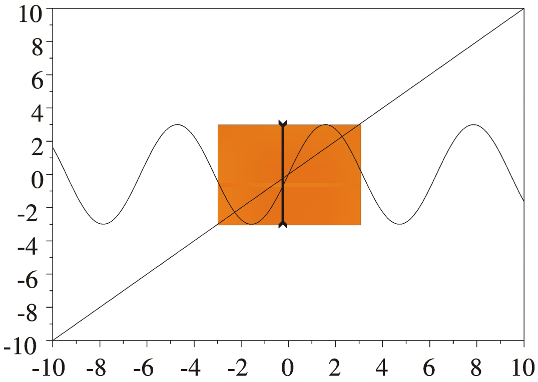
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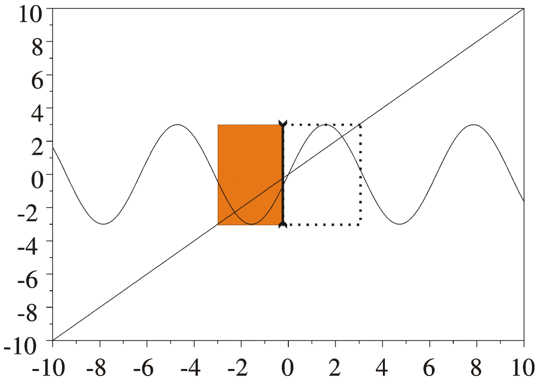
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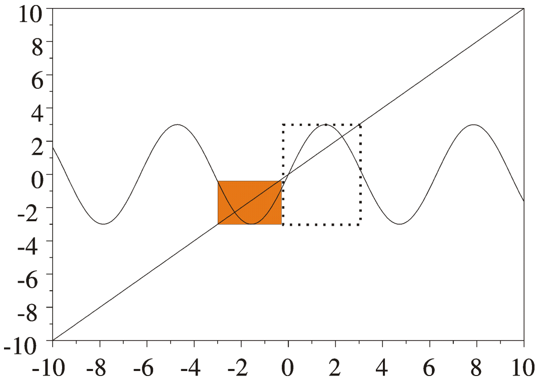
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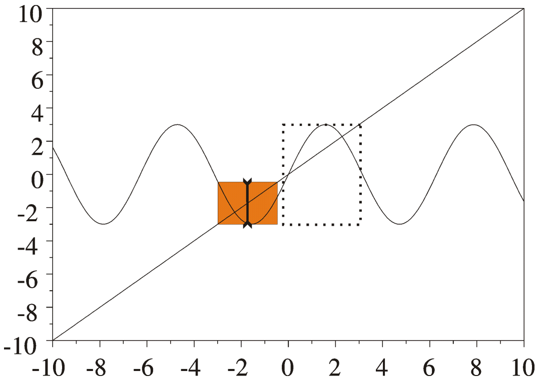
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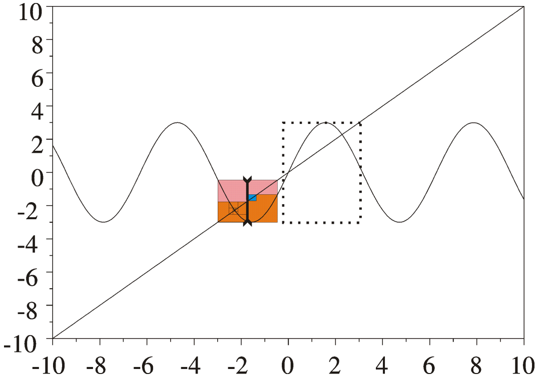
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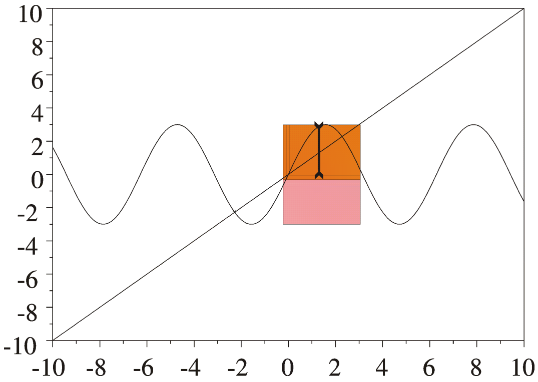
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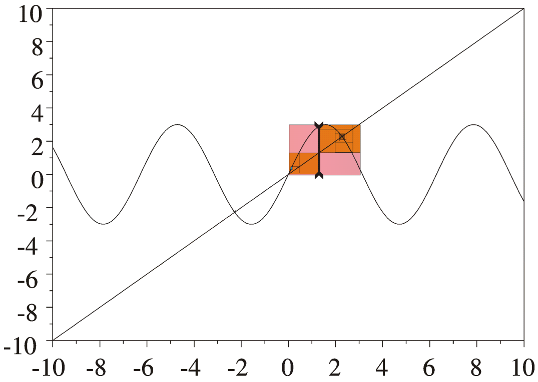
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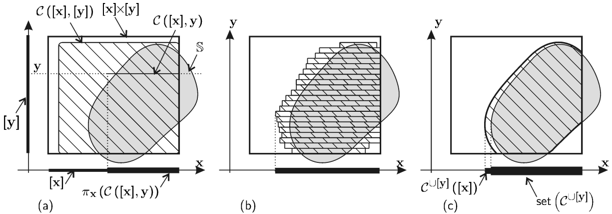
Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) = \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) = [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) = \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
repetition	$\mathcal{C}^\infty = \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$
repeat intersection	$\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$
repeat union	$\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^\infty$

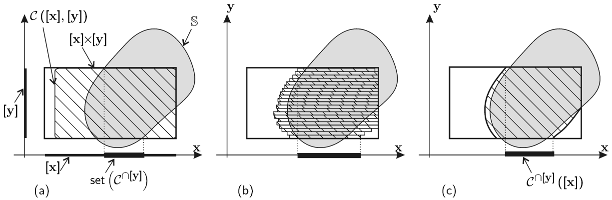
Consider the contractor $\mathscr{C}([\mathbf{x}],[\mathbf{y}])$, where $[\mathbf{x}] \in \mathbb{R}^n, [\mathbf{y}] \in \mathbb{R}^p$. We define the contractor

$$\mathscr{C}^{\cup[\mathbf{y}]}([\mathbf{x}]) \quad = \quad \left[\bigcup_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}(\mathscr{C}([\mathbf{x}],\mathbf{y}))\right] \qquad \text{(projected union)}$$


$\mathcal{C}([x],[y])$
$[x]\times[y]$
$\mathcal{C}([x],y)$
$\mathbb{S}$
y
[y]
x
[x]
$\pi_x(\mathcal{C}([x],y))$
(a)
y
x
(b)
y
x
$\mathcal{C}^{\cup[y]}([x])$
set $\left(\mathcal{C}^{\cup[y]}\right)$
(c)

and also the contractor

$$\mathscr{C}^{\cap[\mathbf{y}]}([\mathbf{x}]) = \bigcap_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}\left(\mathscr{C}([\mathbf{x}],\mathbf{y})\right), \qquad \text{(projected intersection)}$$



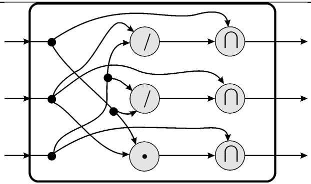
We have

$$\begin{aligned} \mathsf{set}\left(\mathscr{C}^{\cup[\mathbf{y}]}\right) &= \{\mathbf{x}, \exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \mathsf{set}(\mathscr{C})\} \\ \mathsf{set}\left(\mathscr{C}^{\cap[\mathbf{y}]}\right) &= \{\mathbf{x}, \forall \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \mathsf{set}(\mathscr{C})\}. \end{aligned}$$

A link with matrix algebra

$$\begin{array}{ccc} \text{linear application} & \rightarrow & \text{matrices} \\ \mathscr{L}:\left\{\begin{array}{lll} \alpha & = & 2a+3h \\ \gamma & = & h-5a \end{array}\right. & \rightarrow & \mathbf{A}=\left(\begin{array}{cc} 2 & 3 \\ 1 & -5 \end{array}\right) \end{array}$$

We have a matrix algebra and Matlab.
We have: var$(\mathscr{L})=\{a,h\}$, covar$(\mathscr{L})=\{\alpha,\gamma\}$.
But we cannot write: var$(\mathbf{A})=\{a,h\}$, covar$(\mathbf{A})=\{\alpha,\gamma\}$.

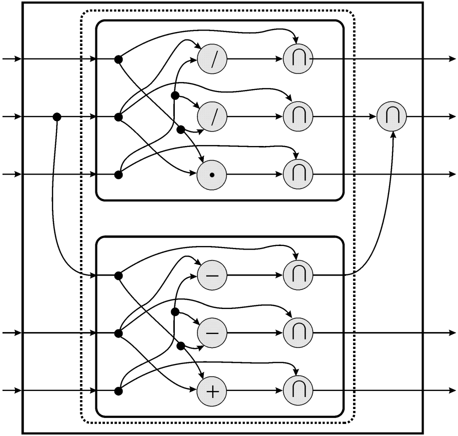
constraint	$\to$	contractor
$a \cdot b = z$	$\to$	

Contractor fusion

$$\left\{ \begin{array}{lcl} a \cdot b = z & \rightarrow & \mathscr{C}_1 \\ b + c = d & \rightarrow & \mathscr{C}_2 \end{array} \right.$$

Since b occurs in both constraints, we fuse the two contractors as:

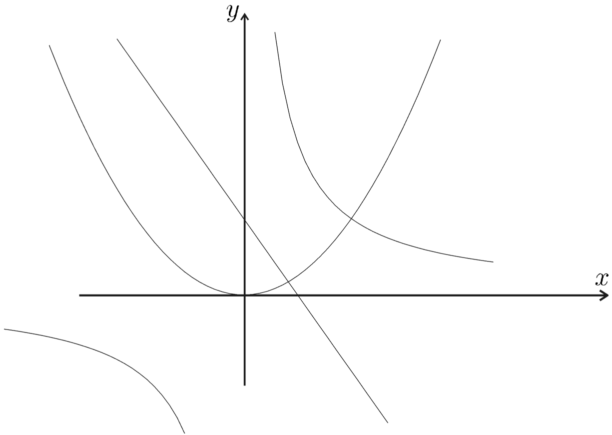
$$\begin{aligned} \mathscr{C} &= \mathscr{C}_1 \times \mathscr{C}_2 \rfloor_{(2,1)} \\ &= \mathscr{C}_1 | \mathscr{C}_2 \text{ (for short)} \end{aligned}$$

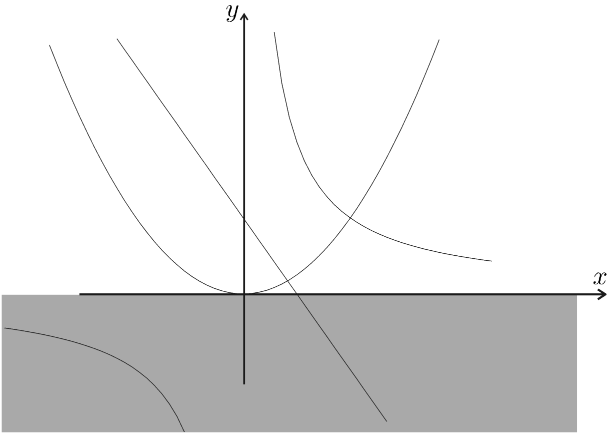


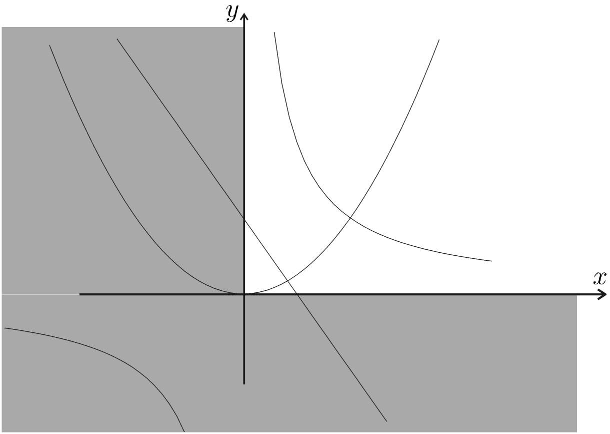
Consider the problem:

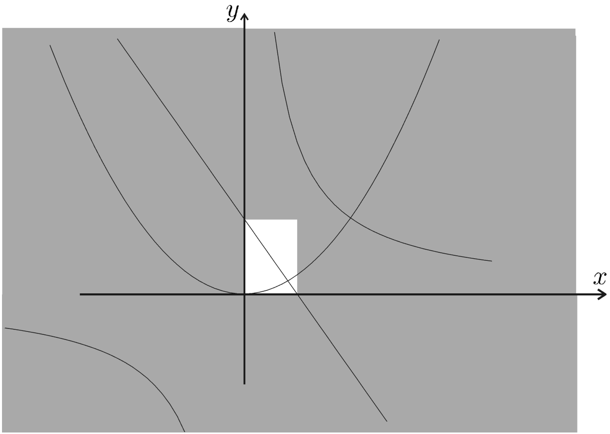
$$\begin{cases} (C_1): & y = x^2 \\ (C_2): & xy = 1 \\ (C_3): & y = -2x + 1 \end{cases}$$

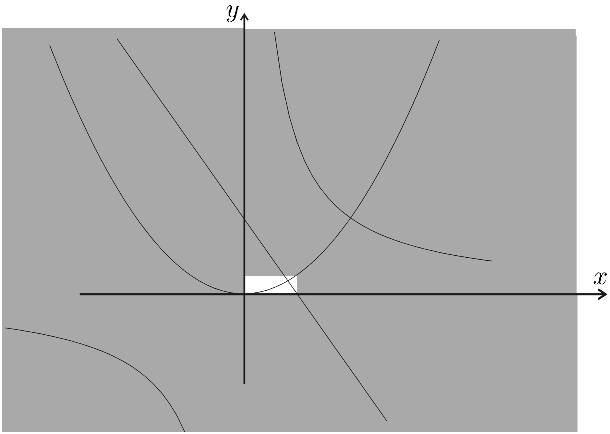
An over constrained system

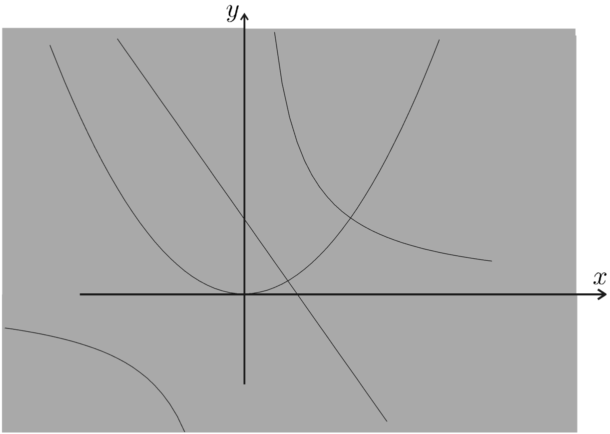

y
x


y
x


y
x


y
x


y
x


y
x

$$
\begin{array}{lll}
(C_1) \Rightarrow & y \in & [-\infty,\infty]^2 \ = \ [0,\infty] \\
(C_2) \Rightarrow & x \in & 1/[0,\infty] \ = [0,\infty] \\
(C_3) \Rightarrow & y \in & [0,\infty] \ \cap \ ((-2)\,.[0,\infty]+1) \\
& & = [0,\infty] \ \cap \ ([-\infty,1]) = [0,1] \\
& x \in & [0,\infty] \ \cap \ (-[0,1]/2+1/2) = [0,\frac{1}{2}] \\
(C_1) \Rightarrow & y \in & [0,1] \cap [0,1/2]^2 = [0,1/4] \\
(C_2) \Rightarrow & x \in & [0,1/2] \ \cap \ 1/[0,1/4] = \emptyset \\
& y \in & [0,1/4] \cap 1/\emptyset = \emptyset
\end{array}
$$

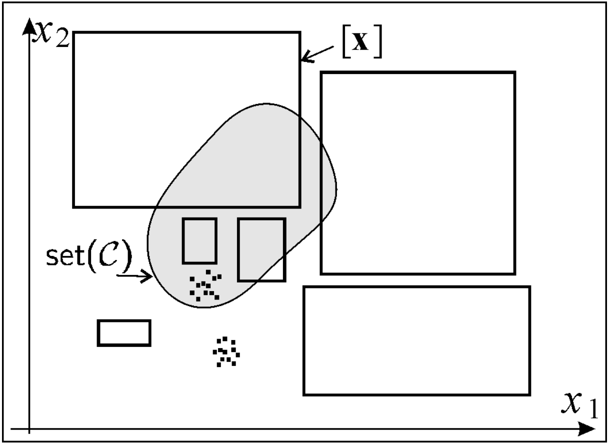
Sets and contractors

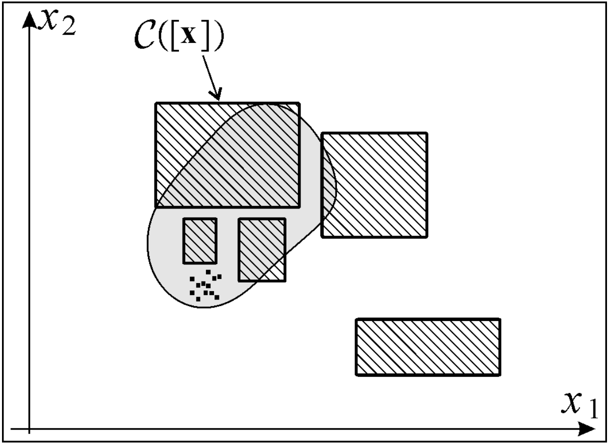
A contractor represents a set of $\mathbb{R}^n$. The set associated with a contractor $\mathscr{C}$ is

$$\mathsf{set}\,(\mathscr{C}) = \{\mathbf{x} \in \mathbb{R}^n, \mathscr{C}(\{\mathbf{x}\}) = \{\mathbf{x}\}\}\,.$$

Its domain is

$$\mathsf{dom}\,(\mathscr{C}) = \{\mathbf{x} \in \mathbb{R}^n, \mathscr{C}(\{\mathbf{x}\}) = \emptyset\}\,.$$


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

For instance, the set associated with the contractor

$$\mathscr{C}_1\begin{pmatrix}[x_1]\\ [x_2]\\ [x_3]\end{pmatrix}=\begin{pmatrix}[x_1]\cap([x_3]-[x_2])\\ [x_2]\cap([x_3]-[x_1])\\ [x_3]\cap([x_1]+[x_2])\end{pmatrix}$$

is

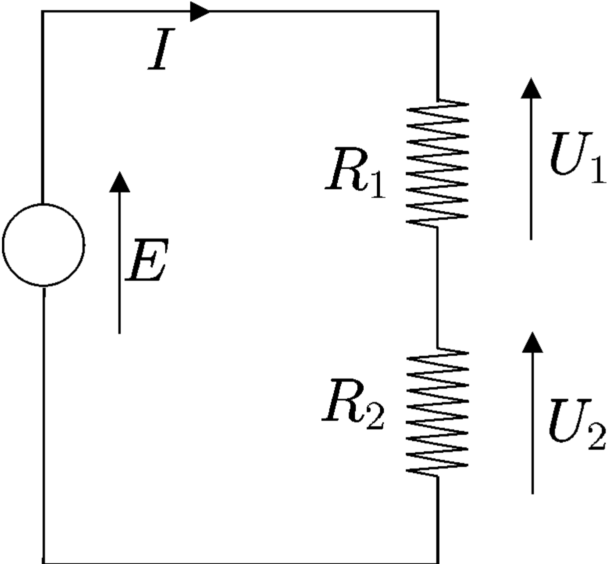
$$\mathrm{set}\,(\mathscr{C}_1)=\{(x_1,x_2,x_3)\,,x_3=x_1+x_2\}\,.$$

A contractor is also one way to represent one equation $x_3=x_1+x_2$.

Applications

Circuits

Example 1



Domains

$$\begin{aligned} E &\in [23V, 26V]; I \in [4A, 8A]; \\ U_1 &\in [10V, 11V]; U_2 \in [14V, 17V]; \\ P &\in [124W, 130W]; R_1 \in [0, \infty[\text{ and } R_2 \in [0, \infty[. \end{aligned}$$

Constraints

(i) $P = EI$, (ii) $E = (R_1 + R_2) I$, (iii) $U_1 = R_1 I$,
(iv) $U_2 = R_2 I$, (v) $E = U_1 + U_2$.

Solution set

$$\mathbb{S}=\left\{\begin{pmatrix} E \\ R_1 \\ R_2 \\ I \\ U_1 \\ U_2 \\ P \end{pmatrix} \in \begin{pmatrix} [23,26] \\ [0,\infty[\\ [0,\infty[\\ [4,8] \\ [10,11] \\ [14,17] \\ [124,130]; \end{pmatrix}, \left\{\begin{array}{l} P=EI \\ E=(R_1+R_2)I \\ U_1=R_1 I \\ U_2=R_2 I \\ E=U_1+U_2 \end{array}\right. \right\}$$

```
variables
E in [23 ,26];
I in [4,8];
U1 in [10,11];
U2 in [14 ,17];
P in [124,130];
R1 in [0 ,1e08 ];
R2 in [0 ,1e08 ];
contractor_list L
P=E*I;
E=(R1+R2)*I;
U1=R1*I;
U2=R2*I;
E=U1+U2;
end
```

```
contractor C
compose(L);
end
contractor epsilon
precision(1);
end
```

Quimper returns

$$[24;26]\times[1.846;2.307]\times[2.584;3.355]$$
$$\times[4.769;5.417]\times[10;11]\times[14;16]\times[124;130]\,,$$

i.e.,

$$\begin{array}{ll} E\in[24;26]\,, & R_1\in[1.846;2.307]\,,\\ R_2\in[2.584;3.355], & I\in[4.769;5.417]\,,\\ U_1\in[10;11]\,, & U_2\in[14;16]\,,\\ P\in[124;130]\,. & \end{array}$$

Exercise.

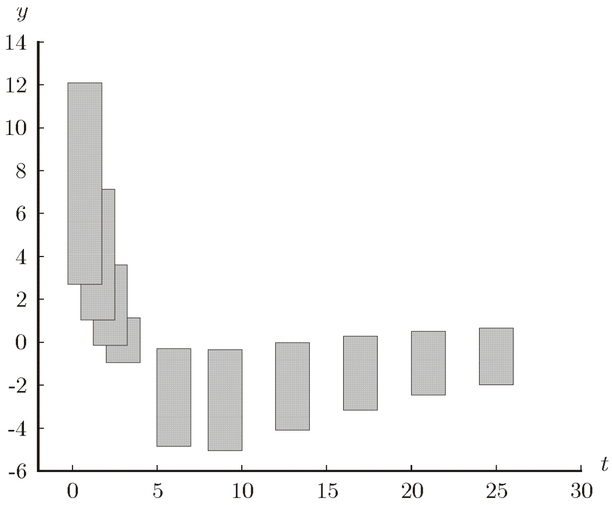
A robot measures its own distance to three marks. The distances and the coordinates of the marks are as follows

mark	x_i	y_i	d_i
1	0	0	$[22,23]$
2	10	10	$[10,11]$
3	30	-30	$[53,54]$

Build the contractor associated with the pose of the robot.

Estimation

$$y_m(\mathbf{p},t) = 20\exp(-p_1 t) - 8\exp(-p_2 t).$$


y
14
12
10
8
6
4
2
0
-2
-4
-6
t
0
5
10
15
20
25
30

i	$[t_i]$	$[y_i]$
1	$[-0.25, 1.75]$	$[2.7, 12.1]$
2	$[0.5, 2.5]$	$[1.04, 7.14]$
3	$[1.25, 3.25]$	$[-0.13, 3.61]$
4	$[2, 4]$	$[-0.95, 1.15]$
5	$[5, 7]$	$[-4.85, -0.29]$
6	$[8, 10]$	$[-5.06, -0.36]$
7	$[12, 14]$	$[-4.1, -0.04]$
8	$[16, 18]$	$[-3.16, 0.3]$
9	$[20, 22]$	$[-2.5, 0.51]$
10	$[24, 26]$	$[-2, 0.67]$

The feasible set is

$$\mathbb{S} \quad = \bigcap\nolimits_{i\in\{1,\dots,10\}} \quad \underbrace{\left\{\mathbf{p}\in\mathbb{R}^2 \mid \exists t_i\in[t_i] \mid y_m(\mathbf{p},t_i)\in[y_i]\right\}}_{\mathbb{S}_i}.$$

The complementary set is

$$\bar{\mathbb{S}} \quad = \bigcup\nolimits_{i\in\{1,\dots,10\}} \quad \underbrace{\left\{\mathbf{p}\in\mathbb{R}^2 \mid \forall t_i\in[t_i] \mid y_m(\mathbf{p},t_i)\notin[y_i]\right\}}_{\bar{\mathbb{S}}_i}$$

Define two contractors $\mathscr{C}_i(\mathbf{p},t_i)$ and $\bar{\mathscr{C}_i}(\mathbf{p},t_i)$ such that

$$\begin{cases} \mathsf{set}\left(\mathscr{C}_i(\mathbf{p},t_i)\right) & = & \{(\mathbf{p},t_i), y_m(\mathbf{p},t_i) \in [y_i]\} \\ \mathsf{set}\left(\bar{\mathscr{C}_i}(\mathbf{p},t_i)\right) & = & \{(\mathbf{p},t_i), y_m(\mathbf{p},t_i) \notin [y_i]\}. \end{cases}$$

We have

$$\begin{array}{rcl} \mathsf{set}\left(\mathscr{C}_i^{\cup[t_i]}\right) & = & \mathbb{S}_i \\ \mathsf{set}\left(\bar{\mathscr{C}_i}^{\cap[t_i]}\right) & = & \bar{\mathbb{S}}_i. \end{array}$$

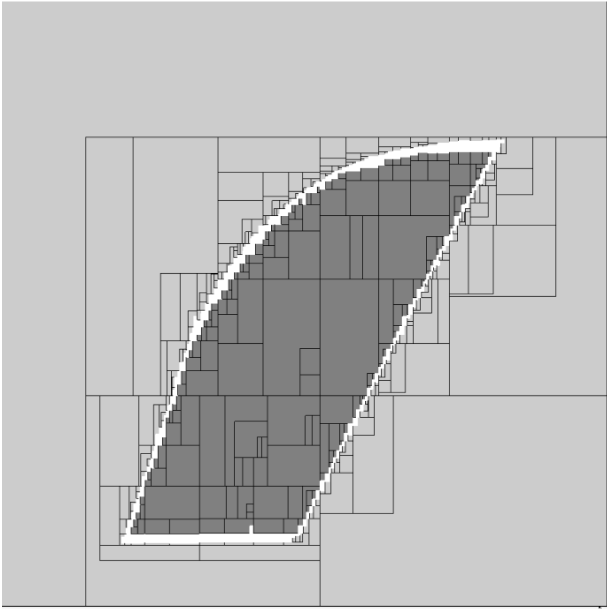
Define two contractors

$$
\begin{aligned}
\mathscr{C}([\mathbf{p}]) &= \bigcap_{i\in\{1,\dots,10\}} \mathscr{C}_i^{\cup[t_i]}([\mathbf{p}],t_i) \\
\bar{\mathscr{C}}([\mathbf{p}]) &= \bigcup_{i\in\{1,\dots,10\}} \bar{\mathscr{C}}_i^{\cap[t_i]}([\mathbf{p}],t_i).
\end{aligned}
$$

We have $\text{set}(\mathscr{C}) = \mathbb{S}$ et $\text{set}(\bar{\mathscr{C}}) = \bar{\mathbb{S}}$.

```
constant
Y[10] = [[2.7,12.1]; [1.04,7.14];
[-0.13,3.61];[-0.95,1.15];
[-4.85,-0.29];[-5.06,-0.36];
[-4.1,-0.04]; [-3.16,0.3];
[-2.5,0.51]; [-2,0.67]];
variables
p1 in [0,1.2]; p2 in [0,0.5];
parameters
t[10] in [[-0.25,1.75]; [0.5,2.5]; [1.25,3.25];
[2,4]; [5,7]; [8,10]; [12,14];
[16,18]; [20,22]; [24,26]];
function z=f(p1,p2,t)
z=20*exp(-p1*t)-8*exp(-p2*t);
end
```

```
contractor outer
inter (i=1:10,
proj_union(f(p1,p2,t[i]) in Y[i]),t[i]);
end
end
contractor inner
union (i=1:10,
proj_inter(f(p1,p2,t[i]) notin Y[i]),t[i]);
end
end
contractor epsilon
precision(0.01)
en
```



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