Interval Robotics
Chapter 1: Interval computation

L. Jaulin

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Notions on set theory

Exercise: If $\mathbb{X} = \{a, b, c, d\}$ and $\mathbb{Y} = \{b, c, x, y\}$, then

$$
\begin{aligned}
\mathbb{X} \cap \mathbb{Y} &= ? \\
\mathbb{X} \cup \mathbb{Y} &= ? \\
\mathbb{X} \setminus \mathbb{Y} &= ? \\
\mathbb{X} \times \mathbb{Y} &= ?
\end{aligned}
$$

Exercise: If $\mathbb{X} = \{a,b,c,d\}$ and $\mathbb{Y} = \{b,c,x,y\}$, then

$$\begin{aligned}
\mathbb{X} \cap \mathbb{Y} &= \{b,c\} \\
\mathbb{X} \cup \mathbb{Y} &= \{a,b,c,d,x,y\} \\
\mathbb{X} \setminus \mathbb{Y} &= \{a,d\} \\
\mathbb{X} \times \mathbb{Y} &= \{(a,b),(a,c),(a,x),(a,y), \\
& \quad \ldots,(d,b),(d,c),(d,x),(d,y)\}
\end{aligned}$$

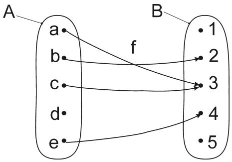
The *direct image* of $\mathbb{X}$ by f is

$$f(\mathbb{X}) \triangleq \{f(x) \mid x \in \mathbb{X}\}.$$

The *reciprocal image* of $\mathbb{Y}$ by f is

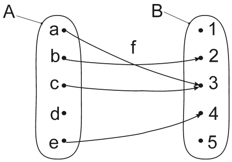
$$f^{-1}(\mathbb{Y}) \triangleq \{x \in \mathbb{X} \mid f(x) \in \mathbb{Y}\}.$$

Exercise: If f is defined as follows



$$
\begin{aligned}
f(A) &= ?. \\
f^{-1}(B) &= ?. \\
f^{-1}(f(A)) &= ? \\
f^{-1}(f(\{b,c\})) &= ?.
\end{aligned}
$$

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= \{2,3,4\} = Im(f). \\ f^{-1}(B) &= \{a,b,c,e\} = \mathsf{dom}(f). \\ f^{-1}(f(A)) &= \{a,b,c,e\} \subset A \\ f^{-1}(f(\{b,c\})) &= \{a,b,c\}. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= ? \\ f^{-1}([4,9]) &= ?. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= [4,9] \\ f^{-1}([4,9]) &= [-3,-2] \cup [2,3]. \end{aligned}$$

This is consistent with the property

$$f^{-1}(f(\mathbb{Y})) \supset \mathbb{Y}.$$

Intervals

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= ?,\\
[-1,3]\cdot[2,5] &= ?,\\
\mathsf{abs}\,([-7,1]) &= ?
\end{aligned}
$$

Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= [1,8], \\ [-1,3]\cdot[2,5] &= [-5,15], \\ \mathsf{abs}([-7,1]) &= [0,7] \end{aligned}$$

The interval extension of

$$f(x_1,x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned}[f]([x_1],[x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos[x_2] \\ &\quad + \sin[x_1] \cdot \sin[x_2] + 2.\end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic

If $\diamond \in \{+, -, \cdot, /, \max, \min\}$

$$[x] \diamond [y] \quad = \quad [\ \{x \diamond y \mid x \in [x], y \in [y]\}\ ].$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A} \subset \mathbb{R}$.

Exercise.

$$\begin{aligned} [-1,3]+[2,5] &= [?,?] \\ [-1,3]\cdot[2,5] &= [?,?] \\ [-2,6]/[2,5] &= [?,?] \end{aligned}$$

Solution.

$$\begin{aligned}
[-1,3]+[2,5] &= [1,8] \\
[-1,3].[2,5] &= [-5,15] \\
[-2,6]/[2,5] &= [-1,3]
\end{aligned}$$

Exercise. Compute

$$[-2,2]/[-1,1] \quad = [?,?]$$

Solution.

$$[-2,2]/[-1,1] \quad = [-\infty,\infty]$$

$$
\begin{aligned}
[x^-,x^+]+[y^-,y^+] = \quad & [x^-+y^-,x^++y^+],\\
[x^-,x^+]\cdot[y^-,y^+] = \quad & [x^-y^-\wedge x^+y^-\wedge x^-y^+\wedge x^+y^+,\\
& x^-y^-\vee x^+y^-\vee x^-y^+\vee x^+y^+],
\end{aligned}
$$

If $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Exercise.

$$\begin{aligned}
\sin([0,\pi]) &= ? \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = ? \\
\mathsf{abs}([-7,1]) &= ? \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = ? \\
\log([-2,-1]) &= ?.
\end{aligned}$$

Solution.

$$
\begin{aligned}
\sin([0,\pi]) &= [0,1] \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = [0,9] \\
\mathsf{abs}([-7,1]) &= [0,7] \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = [0,2] \\
\log([-2,-1]) &= \emptyset.
\end{aligned}
$$

Inclusion functions

A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

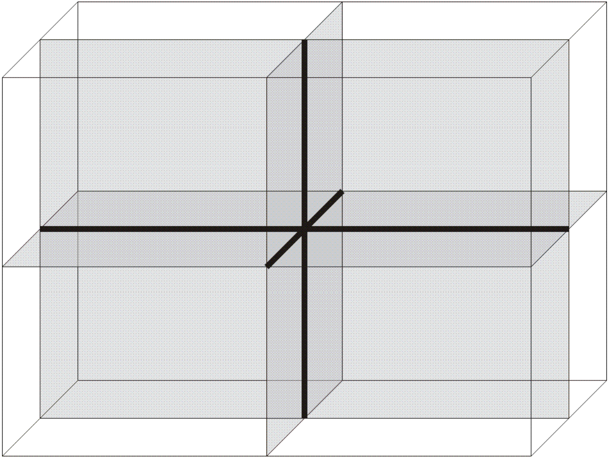
$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

The *width* $w([\mathbf{x}])$ is the length of the largest side. For instance

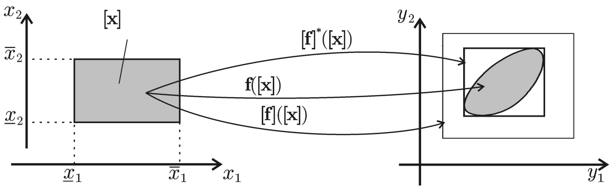
$$w([1,2] \times [-1,3]) = 4$$

The *principal plane* of $[\mathbf{x}]$ is symmetric and perpendicular to the largest side.



$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* for $\mathbf{f}$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \ \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

The inclusion function $[\mathbf{f}]$ is

monotonic	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([\mathbf{f}]([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{y}]))$
minimal	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [\mathbf{f}]([\mathbf{x}]) = [\mathbf{f}([\mathbf{x}])]$
thin	if	$w([\mathbf{x}]) = 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) = 0$
convergent	if	$w([\mathbf{x}]) \to 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) \to 0.$

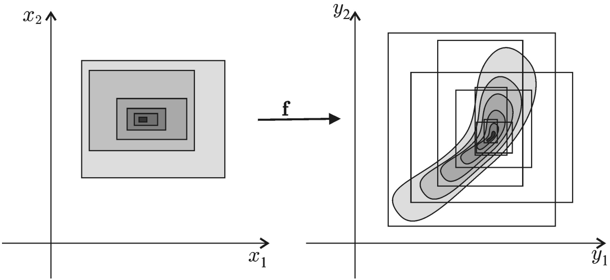
Exercise

The figure provides a nested sequence of boxes $[\mathbf{x}](k)$, their image $\mathbf{f}([\mathbf{x}])$ by a function $\mathbf{f}$ and the image by an inclusion function $[\mathbf{f}]$.

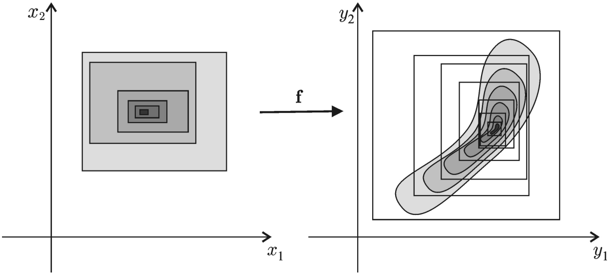
a) $[\mathbf{f}]$ is convergent.

b) $[\mathbf{f}]$ is monotonic

c) $[\mathbf{f}]$ is minimal.



Solution. $[\mathbf{f}]$ is convergent, non-monotonic, non-minimal.



Convergent ? monotonic ?

Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3, 4]$, we have

$$\begin{aligned} [f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\ &= [0,16] + [-6,8] + 4 \\ &= [-2,28]. \end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{ccc} \mathbb{IR}^2 & \to & \mathbb{IR}^3 \\ ([x_1],[x_2]) & \to & \left([x_1]\cdot[x_2],[x_1]^2,[x_1]-[x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by

Algorithm $\mathbf{f}(\text{in} : \mathbf{x} = (x_1, x_2, x_3),\ \text{out} : \mathbf{y} = (y_1, y_2))$
$z := x_1$
for$k := 0$ to 100
$\quad\quad z := x_2(z + k \cdot x_3)$
next
$y_1 := z$
$y_2 := \sin(z x_1)$

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in : $[\mathbf{x}] = ([x_1],[x_2],[x_3])$, out : $[\mathbf{y}] = ([y_1],[y_2])$)

$[z] := [x_1]$
for$k := 0$ to 100
    $[z] := [x_2] \cdot ([z] + k \cdot [x_3])$
next
$[y_1] := [z]$
$[y_2] := \sin([z] \cdot [x_1])$

Is $[\mathbf{f}]$ convergent? thin? monotonic?

Centered form

If $f:\mathbb{R}^n \to \mathbb{R}$ is differentiable over $[\mathbf{x}]$, and if $\mathbf{m} = \text{mid}([\mathbf{x}])$. The mean-value theorem implies

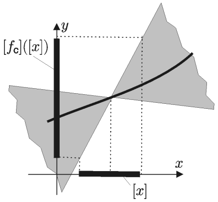
$$\forall \mathbf{x} \in [\mathbf{x}], \exists \mathbf{z} \in [\mathbf{x}] \mid f(\mathbf{x}) = f(\mathbf{m}) + \frac{df}{d\mathbf{x}}(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{m}).$$

Thus,

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \in f(\mathbf{m}) + \frac{df}{d\mathbf{x}}([\mathbf{x}]) \cdot (\mathbf{x} - \mathbf{m}),$$

Therefore, an inclusion function for $\mathbf{f}$ is

$$[f_c]([\mathbf{x}]) = f(\mathbf{m}) + \left[\frac{df}{d\mathbf{x}}\right]([\mathbf{x}]) \cdot ([\mathbf{x}] - \mathbf{m}).$$


y
[f_c]([x])
x
[x]
y
[f_c]([x])
x
[x]
y
[f_c]([x])
x
[x]

Boolean intervals

A *Boolean number* is an element of

$$\mathbb{B} = \{false, true\} = \{0, 1\}$$

If we define the relation $\leq$ as

$$0 \leq 0, \qquad 0 \leq 1, \qquad 1 \leq 1,$$

then, the set $(\mathbb{B}, \leq)$ is a lattice for which intervals can be defined.

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{?, ?, ?, ?\}.$$

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{\emptyset, 0, 1, [0, 1]\},$$

Boolean interval arithmetic

$$
\begin{aligned}
[a] \vee [b] &= \{a \vee b \mid a \in [a], b \in [b]\}, \\
[a] \wedge [b] &= \{a \wedge b \mid a \in [a], b \in [b]\}, \\
\neg [a] &= \{\neg a \mid a \in [a]\}.
\end{aligned}
$$

Exercise: Compute

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = \ ?$$

Solution: We have

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = 1 \wedge [0,1] = [0,1].$$

Inclusion test

A *test* is a function t from $\mathbb{R}^n$ to $\mathbb{B}$. An *inclusion test* $[t]$ is an inclusion function for t. Thus

$$\begin{array}{lll}([t]([\mathbf{x}])=1) & \Rightarrow & (\forall \mathbf{x}\in[\mathbf{x}],\ t(\mathbf{x})=1), \\ ([t]([\mathbf{x}])=0) & \Rightarrow & (\forall \mathbf{x}\in[\mathbf{x}],\ t(\mathbf{x})=0).\end{array}$$

An *inclusion test* $[t_{\mathbb{A}}]$ for a set $\mathbb{A}$ of $\mathbb{R}^n$ is an inclusion test for the test $(\mathbf{x} \in \mathbb{A})$. We have

$$\begin{array}{lll} [t_{\mathbb{A}}]([\mathbf{x}]) = 1 & \Rightarrow & ([\mathbf{x}] \subset \mathbb{A}), \\ [t_{\mathbb{A}}]([\mathbf{x}]) = 0 & \Rightarrow & ([\mathbf{x}] \cap \mathbb{A} = \emptyset). \end{array}$$

$[t]$ is *monotonic*	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([t]([\mathbf{x}]) \subset [t]([\mathbf{y}]))$
$[t]$ is *minimal*	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [t]([\mathbf{x}]) = t([\mathbf{x}])$
$[t]$ is *thin*	if	$\forall \mathbf{x} \in \mathbb{R}^n,\ [t](\mathbf{x}) \neq [0,1].$

If $\mathbb{A}$ and $\mathbb{B}$ are two sets, we have

$$
\begin{aligned}
t_{\mathbb{A}\cap\mathbb{B}} &= t_{\mathbb{A}} \wedge t_{\mathbb{B}} \\
t_{\mathbb{A}\cup\mathbb{B}} &= t_{\mathbb{A}} \vee t_{\mathbb{B}} \\
t_{\neg\mathbb{A}} &= \neg t_{\mathbb{A}} = 1 - t_{\mathbb{A}}.
\end{aligned}
$$

Thus, we define

$$
\begin{aligned}
[t_{\mathbb{A}\cap\mathbb{B}}]([\mathbf{x}]) &= ([t_{\mathbb{A}}]\wedge[t_{\mathbb{B}}])([\mathbf{x}]) = [t_{\mathbb{A}}]([\mathbf{x}])\wedge[t_{\mathbb{B}}]([\mathbf{x}]),\\
[t_{\mathbb{A}\cup\mathbb{B}}]([\mathbf{x}]) &= ([t_{\mathbb{A}}]\vee[t_{\mathbb{B}}])([\mathbf{x}]) = [t_{\mathbb{A}}]([\mathbf{x}])\vee[t_{\mathbb{B}}]([\mathbf{x}]),\\
[t_{\neg\mathbb{A}}]([\mathbf{x}]) &= \neg[t_{\mathbb{A}}]([\mathbf{x}]) = 1-[t_{\mathbb{A}}]([\mathbf{x}]).
\end{aligned}
$$

Exercise: Consider the test

$$t: \begin{array}{ccc} \mathbb{R}^2 & \to & \{0,1\} \\ (x_1,x_2)^\mathsf{T} & \mapsto & (x_1+x_2^2 \leqslant 5). \end{array}$$

The minimal inclusion test $[t]$ associated with t is

$$[t]([\mathbf{x}]) = \begin{cases} 1 & \text{if} \quad ? \\ 0 & \text{if} \quad ? \\ [0,1] & \text{if} \quad ? \end{cases}$$

Solution: Consider the test

$$t: \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \{0,1\} \\ (x_1, x_2)^\mathsf{T} & \mapsto & (x_1 + x_2^2 \leqslant 5). \end{array}$$

The minimal inclusion test $[t]$ associated with t is

$$[t]([\mathbf{x}]) = \begin{cases} 1 & \text{if} \quad [x_1] + [x_2]^2 \in \,]-\infty, 5], \\ 0 & \text{if} \quad [x_1] + [x_2]^2 \in \,]5, \infty[\\ [0,1] & \text{otherwise.} \end{cases}$$

Relations

Relation (or binary constraint)

A relation in $\mathbb{X}$ is a subset of $\mathbb{X} \times \mathbb{X}$.

Example. If $\mathbb{X} = \{a,b,c,d\}$, the set

$$\mathscr{R} = \{(a,a),(a,b),(b,a),(b,c),(d,d)\}$$

is a relation or a binary constraint in $\mathbb{X}$.
Since $\mathbb{X}$ is finite, it can be represented a directed graph.

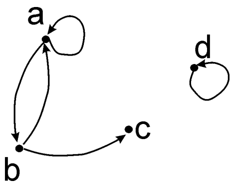
Exercise: Draw the graph associated to the relation

$$\mathscr{R} = \{(a,a),(a,b),(b,a),(b,c),(d,d)\}$$

Solution: The graph associated to the relation

$$\mathscr{R} = \{(a,a),(a,b),(b,a),(b,c),(d,d)\}$$

is given below



Exercise: Give the Boolean matrix associated to the relation

$$\mathscr{R} = \{(a,a),(a,b),(b,a),(b,c),(d,d)\}$$

Solution: The Boolean matrix associated to the relation

$$\mathscr{R} = \{(a,a),(a,b),(b,a),(b,c),(d,d)\}$$

is

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Properties of relations: Consider a binary relation $\mathscr{R}$ over a set $\mathbb{X}$.

- $\mathscr{R}$ is *reflexive*: ...
- $\mathscr{R}$ is *symmetric*: ....
- $\mathscr{R}$ is *antisymmetric*: ....
- $\mathscr{R}$ is *transitive:* ...
- $\mathscr{R}$ is *total*: ...
- $\mathscr{R}$ is an *equivalence relation* ....

Properties of relations: Consider a binary relation $\mathscr{R}$ over a set $\mathbb{X}$.

- $\mathscr{R}$ is *reflexive*: $\forall x \in \mathbb{X},\ (x,x) \in \mathscr{R}$.
- $\mathscr{R}$ is *symmetric*: $(x,y) \in \mathscr{R} \Rightarrow (y,x) \in \mathscr{R}$.
- $\mathscr{R}$ is *antisymmetric*: $(x,y) \in \mathscr{R}$ and $(y,x) \in \mathscr{R} \Rightarrow x = y$.
- $\mathscr{R}$ is *transitive*: $(x,y) \in \mathscr{R},\ (y,z) \in \mathscr{R} \Rightarrow (x,z) \in \mathscr{R}$.
- $\mathscr{R}$ is *total*: $\forall (x,y) \in \mathbb{X}^2,\ (x,y) \in \mathscr{R}$ or $(y,x) \in \mathscr{R}$.
- $\mathscr{R}$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Exercise: Give the Boolean matrix associated to the symmetric closure of the relation

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: The Boolean matrix associated to the symmetric closure of the relation $\mathbf{R}$ is

$$\mathbf{S} = \mathbf{R} + \mathbf{R}^{\mathsf{T}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise: Give the smallest equivalence relation enclosing the relation

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

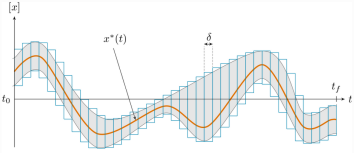
.

Solution: The smallest equivalence relation enclosing $\mathbf{R}$ is $(\mathbf{R}+\mathbf{R}^\mathsf{T})^*$ where

$$\mathbf{A}^* = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\dots$$

Here, we get

$$\left(\mathbf{R}+\mathbf{R}^{\mathsf{T}}\right)^{*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{2} + \ldots. = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



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Definition

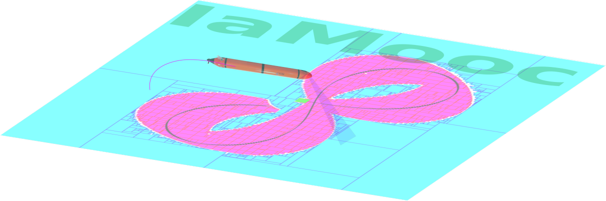
A tube $[\mathbf{x}](\cdot)$ is defined as an envelope enclosing an uncertain trajectory $\mathbf{x}(\cdot):\mathbb{R}\to\mathbb{R}^n$. It is built as an interval of two functions $[\mathbf{x}^-(\cdot),\mathbf{x}^+(\cdot)]$ such that $\forall t,\ \mathbf{x}^-(t)\leqslant\mathbf{x}^+(t)$. A trajectory $\mathbf{x}(\cdot)$ belongs to the tube $[\mathbf{x}](\cdot)$ if $\forall t,\ \mathbf{x}(t)\in[\mathbf{x}](t)$. Fig. 1 illustrates a tube implemented with a set of boxes. This sliced implementation is detailed hereinafter.

Fig. 1 A tube $[x](\cdot)$ represented by a set of slices. This representation can be used to enclose signals such as $x^*(\cdot)$.

Code example:

```
float timestep = 0.1;
Interval domain(0,10);
Tube x(domain, timestep, Function("t", "(t-5)^2 + [-0.5,0.5]"));
```

http://www.codac.io/



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