Intervals for validating Cyber-Physical Systems

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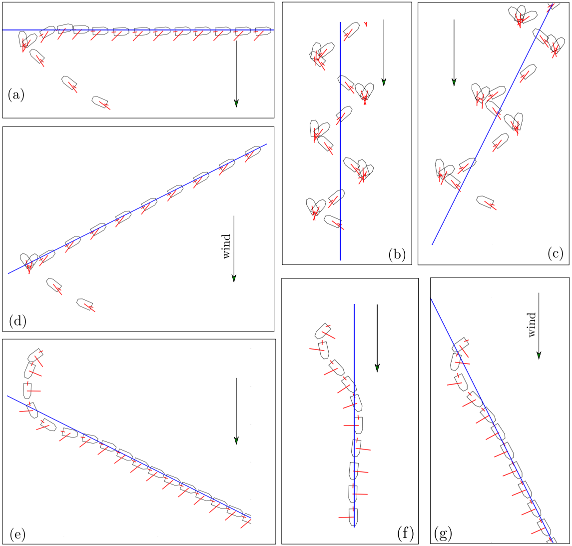
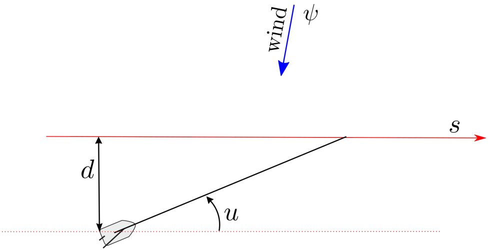
Brave

https://youtu.be/bNqiwW4p6WE

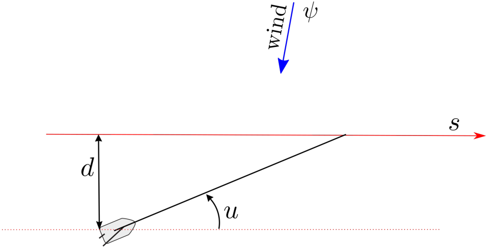

(a)
(b)
(c)
wind
(d)
(e)
(f)
wind
(g)

Easy-boat

$$\dot{d} = \sin u$$
$$(\dot{s} = \cos u)$$
$$\cos(\psi - u) + \cos\frac{\pi}{5} > 0$$

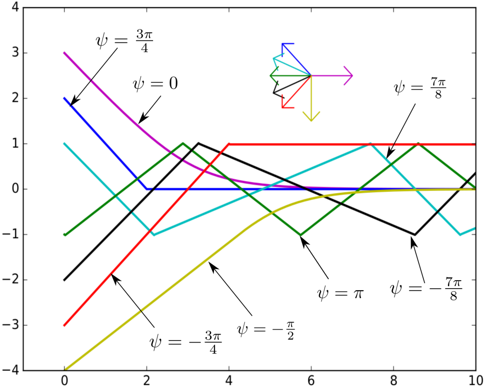


$$\dot{d} = \sin u$$
$$\cos(\psi - u) + \cos\frac{\pi}{5} > 0$$

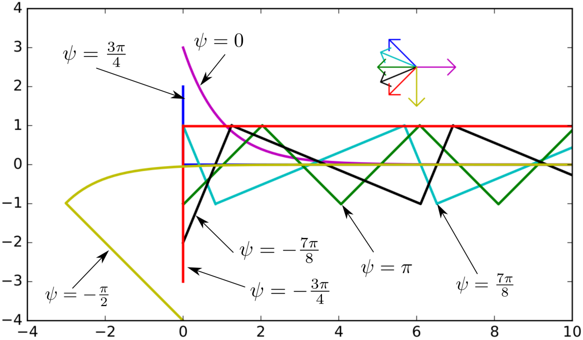


Controller in: (d,ψ,q); out: u
if $d^2 - 1 > 0$ then $q :=$ sign(d) if $\cos(\psi + \text{atan}\, d) + \cos\frac{\pi}{4} \le 0 \vee \left(d^2 \le 1 \wedge \cos\psi + \cos\frac{\pi}{4} \le 0\right)$ then $u := \pi + \psi - q\frac{\pi}{4}$ else $u := -\text{atan}\, d$

Simulations



Simulation in the (t, d)-space



Simulation in the (s, d)-space, with $\dot{s} = \cos u$

Formalism

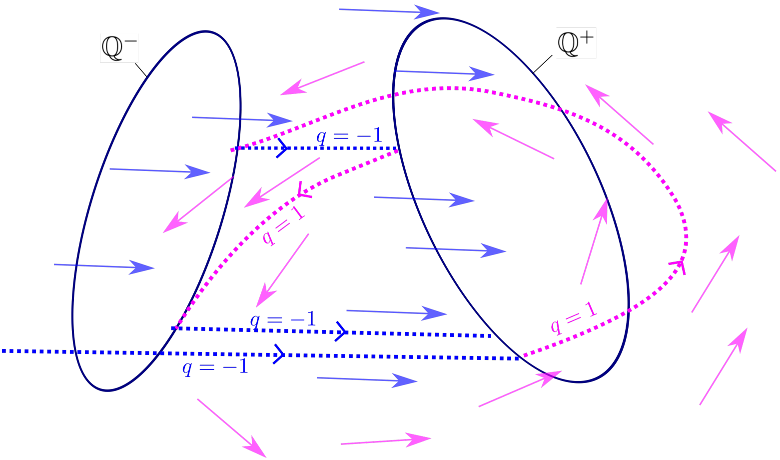
Given $\mathbb{Q}^-$, $\mathbb{Q}^+$ disjoint, two smooth functions $\mathbf{f}_a, \mathbf{f}_b : \mathbb{R}^n \to \mathbb{R}^n$.
We define [2]

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x}, q) & = & \begin{cases} \mathbf{f}_a(\mathbf{x}, q) & \text{if } \mathbf{x} \in \mathbb{A} \\ \mathbf{f}_b(\mathbf{x}, q) & \text{if } \mathbf{x} \in \mathbb{B} = \overline{\mathbb{A}} \end{cases} \\ q & = & -1 & & \text{as soon as } \mathbf{x} \in \mathbb{Q}^- \\ & = & +1 & & \text{as soon as } \mathbf{x} \in \mathbb{Q}^+ \end{cases}$$

The pair $(\mathbf{x}, q)$ always satisfies the constraint

$$\begin{aligned} \mathbf{x} \in \mathbb{Q}^{+} &\quad\Rightarrow\quad q = 1 \\ \mathbf{x} \in \mathbb{Q}^{-} &\quad\Rightarrow\quad q = -1 \end{aligned}$$

or equivalently, $\mathbf{x} \in \overline{\mathbb{Q}^{-q}}$.


$\mathbb{Q}^-$
$\mathbb{Q}^+$
$q = -1$
$q = 1$
$q = -1$
$q = -1$
$q = 1$

With easy-boat

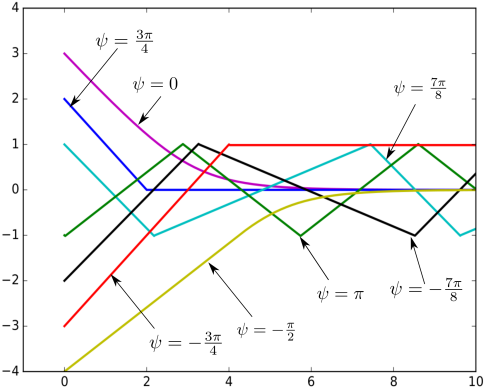
We take $\mathbf{x} = (d, \psi)$,

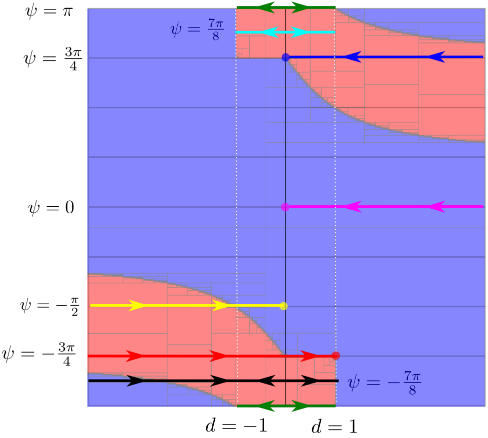
Function $\mathbf{f}(\mathbf{x}, q)$
If $\cos(x_2 + \operatorname{atan} x_1) + \cos\frac{\pi}{4} \leq 0 \vee \left(x_1^2 - 1 \leq 0 \wedge \cos x_2 + \cos\frac{\pi}{4} \leq 0\right)$
then $u := \pi + x_2 - q\frac{\pi}{4}$
else $u := -\operatorname{atan} x_1$
Return $(\sin u, 0)$

Function $\mathbf{f}(\mathbf{x}, q)$
If $\mathbf{x} \in \mathbb{A}_1 \vee (\mathbf{x} \in \mathbb{A}_2 \wedge \mathbf{x} \in \mathbb{A}_3)$
then return $\mathbf{f}_a(\mathbf{x}, q)$
else return $\mathbf{f}_b(\mathbf{x}, q)$

$$\left\{\begin{array}{lclcl} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},q) & = & \left\{\begin{array}{lc} \mathbf{f}_a(\mathbf{x},q) & \text{if } \mathbf{x}\in\mathbb{A} \\ \mathbf{f}_b(\mathbf{x},q) & \text{if } \mathbf{x}\in\mathbb{B}=\overline{\mathbb{A}} \end{array}\right. \\ q & = & -1 & & \text{as soon as } \mathbf{x}\in\mathbb{Q}^- = \{\mathbf{x}\,|\,x_1+1\le 0\} \\ & = & +1 & & \text{as soon as } \mathbf{x}\in\mathbb{Q}^+ = \{\mathbf{x}\,|\,1-x_1\le 0\} \end{array}\right.$$

$$
\begin{aligned}
\mathbf{x} &= (d, \psi) \\
\mathbf{f}_a(\mathbf{x}, q) &= \begin{pmatrix} \sin\left(\pi + x_2 - q\frac{\pi}{4}\right) \\ 0 \end{pmatrix} \\
\mathbf{f}_b(\mathbf{x}) &= \begin{pmatrix} \sin\left(-\operatorname{atan} x_1\right) \\ 0 \end{pmatrix} \\
\mathbb{A}_1 &= \left\{\mathbf{x} \,|\, \cos\left(x_2 + \operatorname{atan} x_1\right) + \cos\frac{\pi}{4} \le 0\right\} \\
\mathbb{A}_2 &= \left\{\mathbf{x} \,|\, x_1^2 - 1 \le 0\right\} \\
\mathbb{A}_3 &= \left\{\mathbf{x} \,|\, \cos x_2 + \cos\frac{\pi}{4} \le 0\right\} \\
\mathbb{A} &= \mathbb{A}_1 \cup \left(\mathbb{A}_2 \cap \mathbb{A}_3\right) \\
\mathbb{Q}^- &= \{\mathbf{x} \,|\, x_1 + 1 \le 0\} \\
\mathbb{Q}^+ &= \{\mathbf{x} \,|\, 1 - x_1 \le 0\}
\end{aligned}
$$


$\psi = \frac{3\pi}{4}$
$\psi = 0$
$\psi = \frac{7\pi}{8}$
$\psi = \pi$
$\psi = -\frac{7\pi}{8}$
$\psi = -\frac{\pi}{2}$
$\psi = -\frac{3\pi}{4}$
4
3
2
1
0
−1
−2
−3
−4
0
2
4
6
8
10


$\psi = \pi$
$\psi = \frac{7\pi}{8}$
$\psi = \frac{3\pi}{4}$
$\psi = 0$
$\psi = -\frac{\pi}{2}$
$\psi = -\frac{3\pi}{4}$
$\psi = -\frac{7\pi}{8}$
$d = -1$
$d = 1$

Two problems

Two problems:

- Capture : The boat will be captured by its corridor [1]
- Characterize of the sliding surface.

Capture

The set $\mathbb{C} = \{\mathbf{x} | V(\mathbf{x}) \leq 0\}$, with $V : \mathbb{R}^n \to \mathbb{R}$ is a *capture set* if all trajectories that enter inside $\mathbb{C}$ stays inside forever.

The *Lie derivative* of V with respect to $\mathbf{f}$ is

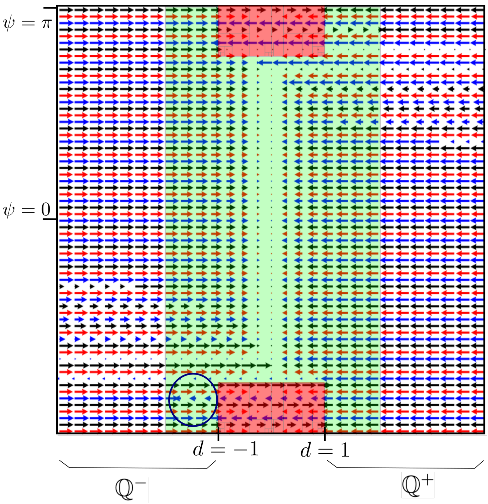
$$\mathscr{L}_{\mathbf{f}}^{V}(\mathbf{x}) = \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

The Lie set as

$$\mathbb{L}_{\mathbf{f}}^{V} = \left\{ \mathbf{x} | \mathscr{L}_{\mathbf{f}}^{V}(\mathbf{x}) \leq 0 \right\}.$$

Take $V(\mathbf{x}) = x_1^2 - 4$. We have

$$\begin{array}{rcccl} \mathscr{L}_a^V(\mathbf{x},q) & = & \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_a(\mathbf{x},q) & = & 2x_1 \cdot \sin(\frac{q\pi}{4} - x_2) \\ \mathscr{L}_b^V(\mathbf{x}) & = & \frac{dV}{d\mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}_b(\mathbf{x},q) & = & \frac{-2x_1^2}{\sqrt{x_1^2+1}} \end{array}$$



Fields $\mathbf{f}_a(\mathbf{x}, q)$, $\mathbf{f}_b(\mathbf{x})$, the sets $\mathbb{C}$ (green) and $\mathbb{V}$ (red)

Sliding surface

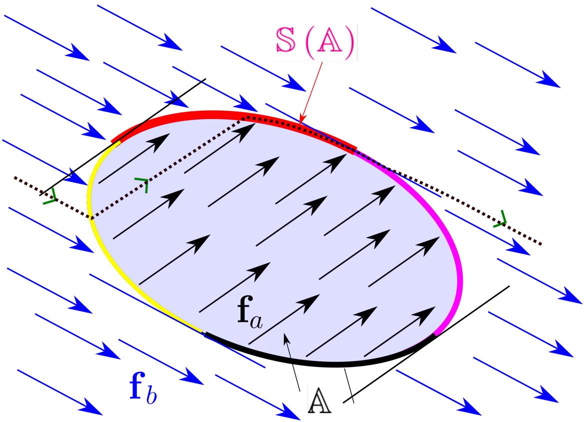
The *sliding surface* $\mathbb{S}(\mathbb{A})$ for $\mathscr{S}(\mathbb{A})$ is largest subset of $\partial\mathbb{A}$ such that the state can slide inside for a non degenerated interval of time.

If $\mathbb{A}$:$c(\mathbf{x}) \leq 0$, then

$$\begin{aligned}\mathbb{S}(\mathbb{A}) &= \partial\mathbb{A} \cap \left\{\mathbf{x} \mid \exists q, \mathbf{x} \in \overline{\mathbb{Q}^{-q}}, \underline{\mathscr{L}_a^c(\mathbf{x},q) \geq 0} \wedge \mathscr{L}_b^c(\mathbf{x},q) \leq 0\right\} \\ &= \partial\mathbb{A} \cap \bigcup_{q\in\{-1,1\}} \overline{\mathbb{Q}^{-q}} \cap \overline{\mathbb{L}_a^V(q)} \cap \mathbb{L}_b^V(q)\end{aligned}$$

Without the discrete variable q.

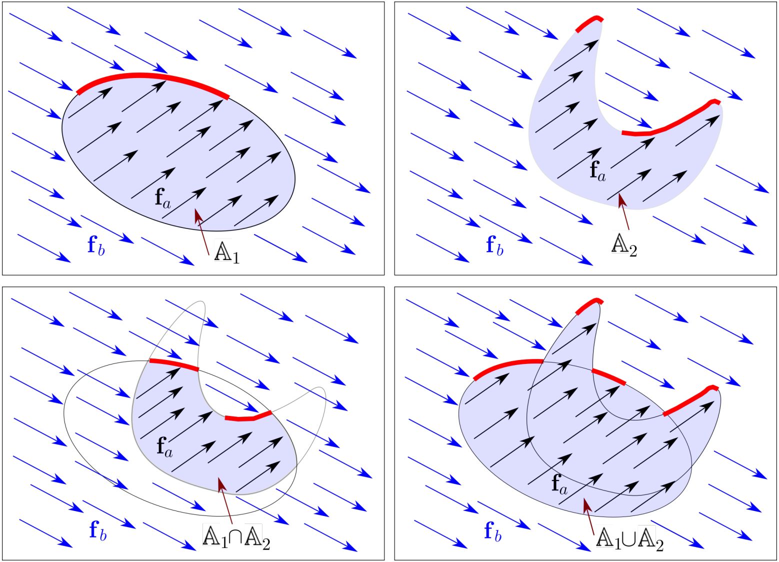
$$\mathbb{S}(\mathbb{A}) = \partial\mathbb{A} \cap \{\mathbf{x} \,|\, \mathscr{L}_a^c(\mathbf{x}) \geq 0 \wedge \mathscr{L}_b^c(\mathbf{x}) \leq 0\}.$$

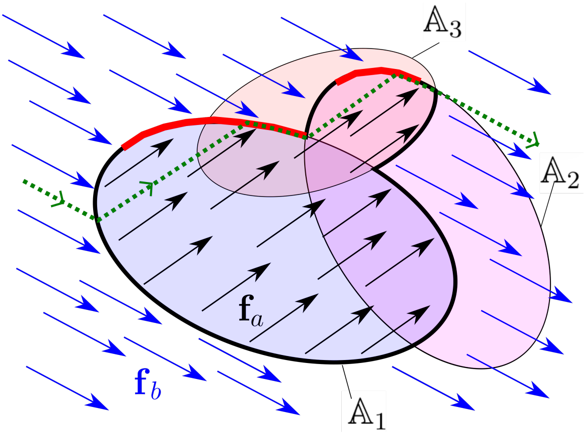


Sliding set $\mathbb{S}(\mathbb{A})$ (red) for $\mathbb{A} = \{\mathbf{x} | c(\mathbf{x}) \leq 0\}$

Proposition 3. If we have two closed sets $\mathbb{A}_1$ and $\mathbb{A}_2$. We have

$$\begin{array}{llcl} (i) & \mathbb{S}(\mathbb{A}_1 \cap \mathbb{A}_2) & = & (\mathbb{S}(\mathbb{A}_1) \cap \mathbb{A}_2) \cup (\mathbb{S}(\mathbb{A}_2) \cap \mathbb{A}_1) \\ (ii) & \mathbb{S}(\mathbb{A}_1 \cup \mathbb{A}_2) & = & \big(\mathbb{S}(\mathbb{A}_1) \cap \mathrm{clo}\overline{\mathbb{A}_2}\big) \cup \big(\mathbb{S}(\mathbb{A}_2) \cap \mathrm{clo}\overline{\mathbb{A}_1}\big) \end{array}$$


$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_1$
$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_2$
$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_1 \cap \mathbb{A}_2$
$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_1 \cup \mathbb{A}_2$



$$\mathbb{S}(\mathbb{A}_1 \cup (\mathbb{A}_2 \cap \mathbb{A}_3))$$

For our boat, the sliding surface for $\mathbb{A}_i : c_i(\mathbf{x}) \leq 0$ is

$$
\begin{aligned}
\mathbb{S}(\mathbb{A}_i) &= \partial\mathbb{A}_i \cap \bigcup_{q\in\{-1,1\}} \overline{\mathbb{Q}^{-q}} \cap \overline{\mathbb{L}_a^i(q)} \cap \mathbb{L}_b^i \\
&= \partial\mathbb{A}_i \cap \mathbb{L}_b^i \cap \left(\overline{\mathbb{L}_a^i(1)} \cap \overline{\mathbb{Q}^-} \cup \overline{\mathbb{L}_a^i(-1)} \cap \overline{\mathbb{Q}^+}\right)
\end{aligned}
$$

where

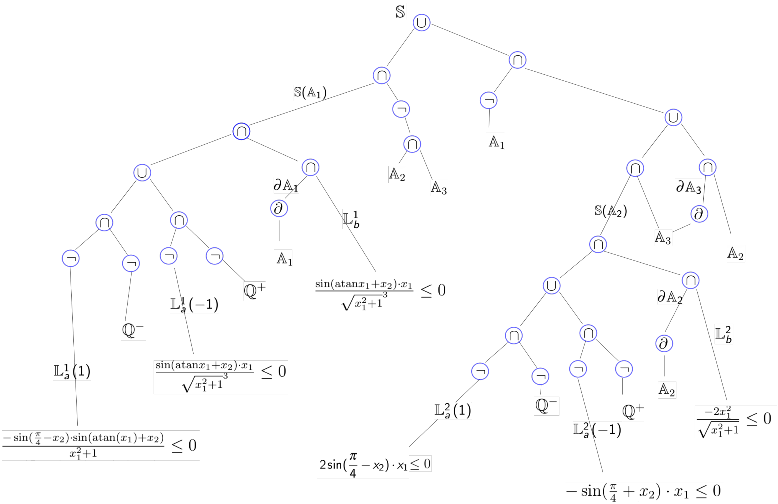
$$
\begin{aligned}
\mathbb{L}_a^i(q) &= \{\mathbf{x} | \mathscr{L}_a^{c_i}(\mathbf{x}, q) \leq 0\} \\
\mathbb{L}_b^i &= \left\{\mathbf{x} | \mathscr{L}_b^{c_i}(\mathbf{x}) \leq 0\right\}
\end{aligned}
$$

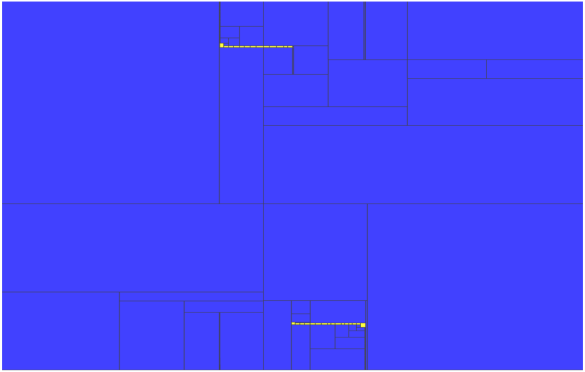
$$
\begin{aligned}
\mathscr{L}_a^{c_1}(\mathbf{x},q) &= \frac{dc_1}{d\mathbf{x}}(\mathbf{x})\cdot \mathbf{f}_a(\mathbf{x},q) &&= \frac{-\sin(\frac{q\pi}{4}-x_2)\cdot\sin(\operatorname{atan}(x_1)+x_2)}{x_1^2+1}\\
\mathscr{L}_b^{c_1}(\mathbf{x}) &= \frac{dc_1}{d\mathbf{x}}(\mathbf{x})\cdot \mathbf{f}_b(\mathbf{x},q) &&= \frac{\sin(\operatorname{atan}x_1+x_2)\cdot x_1}{\sqrt{x_1^2+1}^3}\\
\mathscr{L}_a^{c_2}(\mathbf{x},q) &= \frac{dc_2}{d\mathbf{x}}(\mathbf{x})\cdot \mathbf{f}_a(\mathbf{x}) &&= 2\sin(\frac{q\pi}{4}-x_2)\cdot x_1\\
\mathscr{L}_b^{c_2}(\mathbf{x}) &= \frac{dc_2}{d\mathbf{x}}(\mathbf{x})\cdot \mathbf{f}_b(\mathbf{x},q) &&= \frac{-2x_1^2}{\sqrt{x_1^2+1}}\\
\mathscr{L}_a^{c_3}(\mathbf{x},q) &= \frac{dc_3}{d\mathbf{x}}(\mathbf{x})\cdot \mathbf{f}_a(\mathbf{x},q) &&= 0\\
\mathscr{L}_b^{c_3}(\mathbf{x}) &= \frac{dc_3}{d\mathbf{x}}(\mathbf{x})\cdot \mathbf{f}_b(\mathbf{x},q) &&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{S}(\mathbb{A}_1) &= \partial\mathbb{A}_1 \cap \mathbb{L}_b^1 \cap \left(\overline{\mathbb{L}_a^1(1)} \cap \overline{\mathbb{Q}^-} \cup \overline{\mathbb{L}_a^1(-1)} \cap \overline{\mathbb{Q}^+}\right)\\
\mathbb{S}(\mathbb{A}_2) &= \partial\mathbb{A}_2 \cap \mathbb{L}_b^2 \cap \left(\overline{\mathbb{L}_a^2(1)} \cap \overline{\mathbb{Q}^-} \cup \overline{\mathbb{L}_a^2(-1)} \cap \overline{\mathbb{Q}^+}\right)\\
\mathbb{S}(\mathbb{A}_3) &= \partial\mathbb{A}_3
\end{aligned}
$$

```
La1Q=Function("x1","x2","-sin(asin(1)/2-x2)*sin(atan(x1)
La1R=Function("x1","x2","sin(asin(1)/2+x2)*sin(atan(x1)+
Lb1=Function("x1","x2","sin(atan(x1)+x2)*x1/sqrt((x1^2+1
La2Q=Function("x1","x2","2*sin(asin(1)/2-x2)*x1")
La2R=Function("x1","x2","-2*sin(asin(1)/2-x2)*x1")
Lb2=Function("x1","x2","-2*x1^2/sqrt(x1^2+1)")
dA1=A1&~A1
SLa1Q=SepFwdBwd(La1Q,[-oo,0])
SLa1R=SepFwdBwd(La1R,[-oo,0])
SLb1=SepFwdBwd(Lb1,[-oo,0])
S1=dA1 & SLb1 & ((~SLa1Q)&~R | (~SLa1R)&~Q)
dA2=A2&~A2
SLa2Q=SepFwdBwd(La2Q,[-oo,0])
SLa2R=SepFwdBwd(La2R,[-oo,0])
SLb2=SepFwdBwd(Lb2,[-oo,0])
S2=dA2 & SLb2 & ((~SLa2Q)&~R | (~SLa2R)&~Q)
```

```
dA3=A3&~A3
S23=S2&A3|dA3&A2
S=S1&~(A2&A3) | S23&~A1
```


$\mathbb{S}$
$\mathbb{S}(\mathbb{A}_1)$
$\partial\mathbb{A}_1$
$\mathbb{A}_1$
$\mathbb{L}_b^1$
$\mathbb{A}_2$
$\mathbb{A}_3$
$\mathbb{A}_1$
$\partial\mathbb{A}_3$
$\mathbb{S}(\mathbb{A}_2)$
$\mathbb{A}_3$
$\mathbb{A}_2$
$\mathbb{Q}^+$
$\mathbb{Q}^-$
$\mathbb{L}_a^1(-1)$
$\frac{\sin(\mathrm{atan}x_1+x_2)\cdot x_1}{\sqrt{x_1^2+1}^3} \leq 0$
$\partial\mathbb{A}_2$
$\mathbb{L}_b^2$
$\mathbb{L}_a^1(1)$
$\frac{\sin(\mathrm{atan}x_1+x_2)\cdot x_1}{\sqrt{x_1^2+1}^3} \leq 0$
$\mathbb{A}_2$
$\mathbb{L}_a^2(1)$
$\mathbb{Q}^-$
$\mathbb{Q}^+$
$\frac{-2x_1^2}{\sqrt{x_1^2+1}} \leq 0$
$\mathbb{L}_a^2(-1)$
$\frac{-\sin(\frac{\pi}{4}-x_2)\cdot\sin(\mathrm{atan}(x_1)+x_2)}{x_1^2+1} \leq 0$
$2\sin(\frac{\pi}{4}-x_2)\cdot x_1 \leq 0$
$-\sin(\frac{\pi}{4}+x_2)\cdot x_1 \leq 0$



L. Jaulin and F. Le Bars.
An Interval Approach for Stability Analysis; Application to Sailboat Robotics.
IEEE Transaction on Robotics, 27(5), 2012.

L. Jaulin and F. Le Bars.
Characterizing sliding surfaces of cyber-physical systems.
Acta Cybernetica (submitted), 2019.