Chapter 10: Characterization of attractors

Luc Jaulin,

ENSTA-Bretagne, Brest, France

http://www.ensta-bretagne.fr/jaulin/

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$$

Its solution is $\phi\left(t, \mathbf{x}_0\right)$. The *limit set* is

$$\mathbb{X}^{\infty} = \left\{\mathbf{x} \in \mathbb{R}^n \mid \forall \varepsilon > 0,\ \exists t > 1,\ \left\|\phi\left(t, \mathbf{x}\right) - \mathbf{x}\right\| < \varepsilon\right\}.$$

1 Quantization

Paving: is a collection $\Omega = \{\mathbb{X}_1, \ldots, \mathbb{X}_p\}$ of non-overlapping polytopes $\mathbb{X}_i$ s.t.

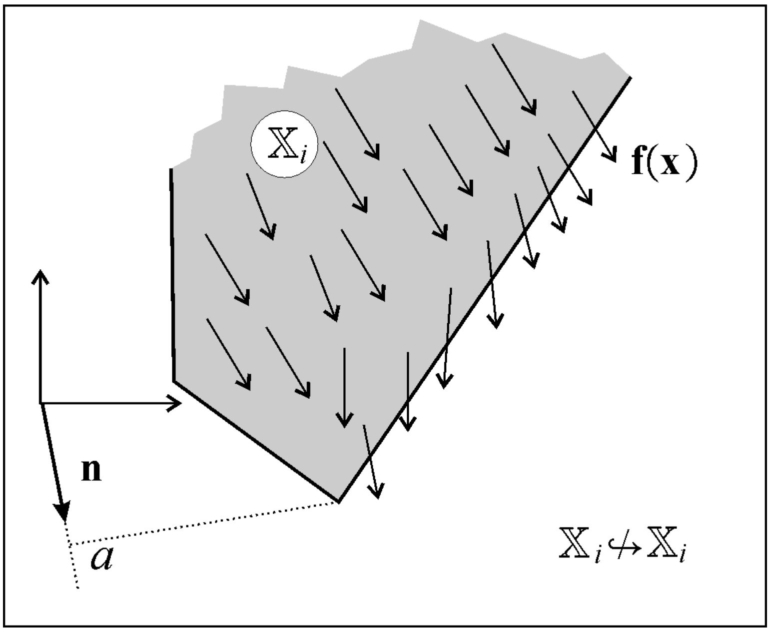
$$\bigcup_{i=1}^{p} \mathbb{X}_i = \mathbb{R}^n.$$

Relation.

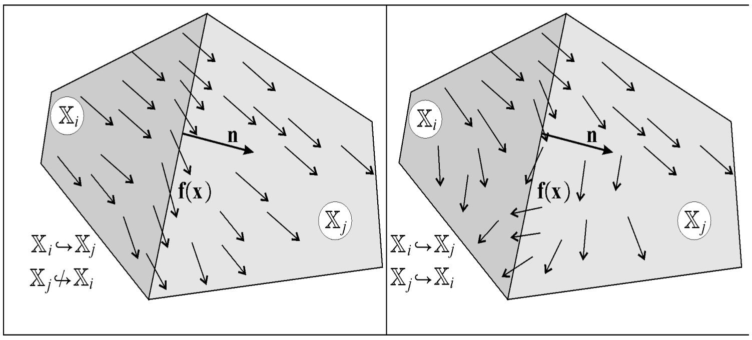
$$\eta_i(\mathbf{x}_0) = \inf_{t \geq 0} \{t \mid \phi(t, \mathbf{x}_0) \notin \mathbb{X}_i\}.$$

We define the relation $\hookrightarrow$ between $\mathbb{X}_i$ and $\mathbb{X}_j$

$$\left(\mathbb{X}_i \hookrightarrow \mathbb{X}_j\right) \Leftrightarrow \begin{cases} \text{(i)} & \exists \mathbf{x}_0 \in \mathbb{X}_i \mid \phi(\eta(\mathbf{x}_0), \mathbf{x}_0) \in \mathbb{X}_j \\ \text{(ii)} & \dim\left(\mathbb{X}_i \cap \mathbb{X}_j\right) \geq n - 1. \end{cases}$$



$\mathbb{X}_i$
$\mathbf{f}(\mathbf{x})$
$\mathbf{n}$
a
$\mathbb{X}_i \not\hookrightarrow \mathbb{X}_i$



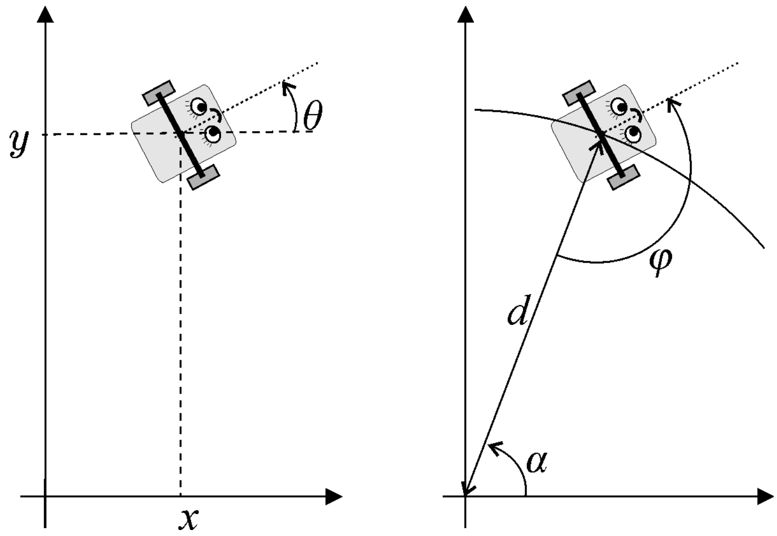
$\mathbb{X}_i$
$\mathbf{n}$
$\mathbf{f}(\mathbf{x})$
$\mathbb{X}_j$
$\mathbb{X}_i \hookrightarrow \mathbb{X}_j$
$\mathbb{X}_j \not\hookrightarrow \mathbb{X}_i$
$\mathbb{X}_i$
$\mathbf{n}$
$\mathbf{f}(\mathbf{x})$
$\mathbb{X}_j$
$\mathbb{X}_i \hookrightarrow \mathbb{X}_j$
$\mathbb{X}_j \hookrightarrow \mathbb{X}_i$

Quantization. It is a graph $\mathcal{G}$ defined by

$$\left(\mathbb{X}_i, \mathbb{X}_j\right) \in \mathcal{G} \Leftrightarrow \mathbb{X}_i \hookrightarrow \mathbb{X}_j.$$

2 Station keeping

$$
\begin{cases}
\dot{x} & = & \cos\theta \\
\dot{y} & = & \sin\theta \\
\dot{\theta} & = & u
\end{cases}
$$



y
θ
x
φ
d
α

The robot can be described by

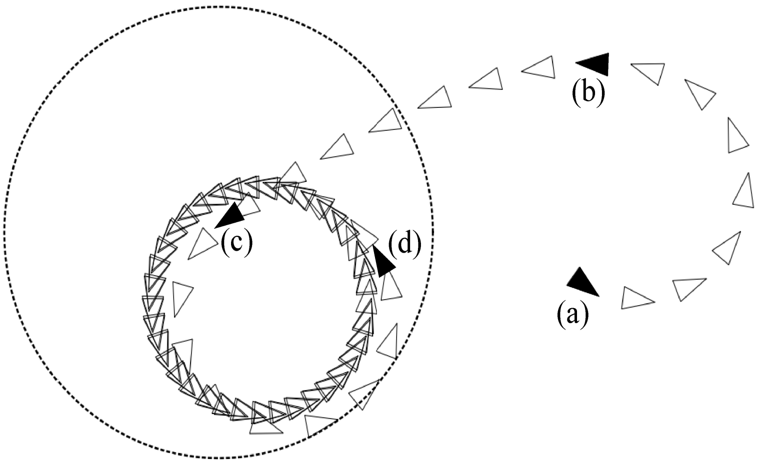
$$\begin{cases} \text{(i)} & \dot{\varphi} = \frac{\sin\varphi}{d} + u \\ \text{(ii)} & \dot{d} = -\cos\varphi. \\ \text{(iii)} & \dot{\alpha} = -\frac{\sin\varphi}{d}. \end{cases}$$

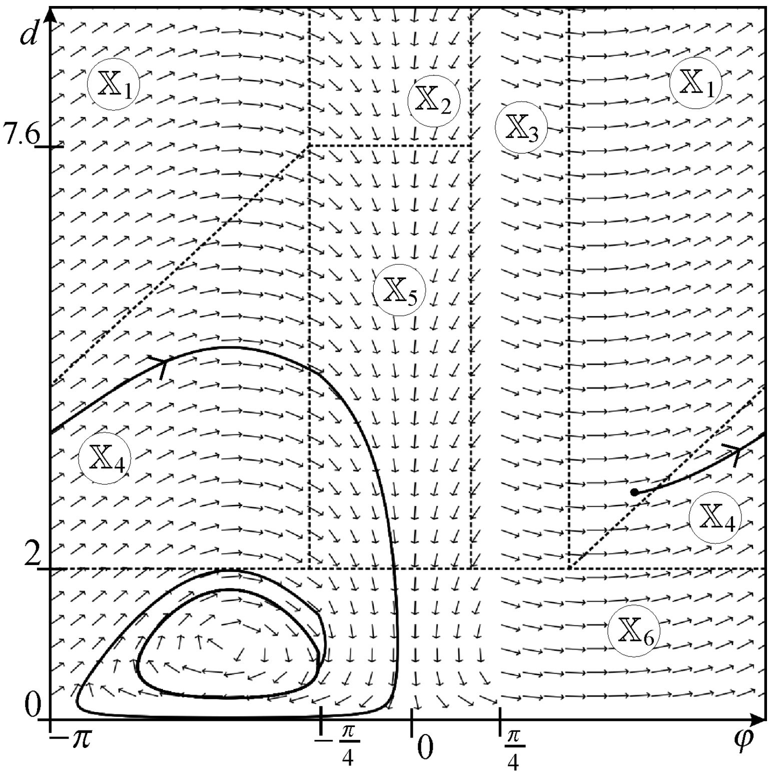
Control.

$$u = \begin{cases} +1 & \text{if } \cos\varphi \leq \frac{1}{\sqrt{2}} \quad \text{(the robot turns left)} \\ -\sin\varphi & \text{otherwise} \quad \text{(the robot goes toward zero)} \end{cases}$$

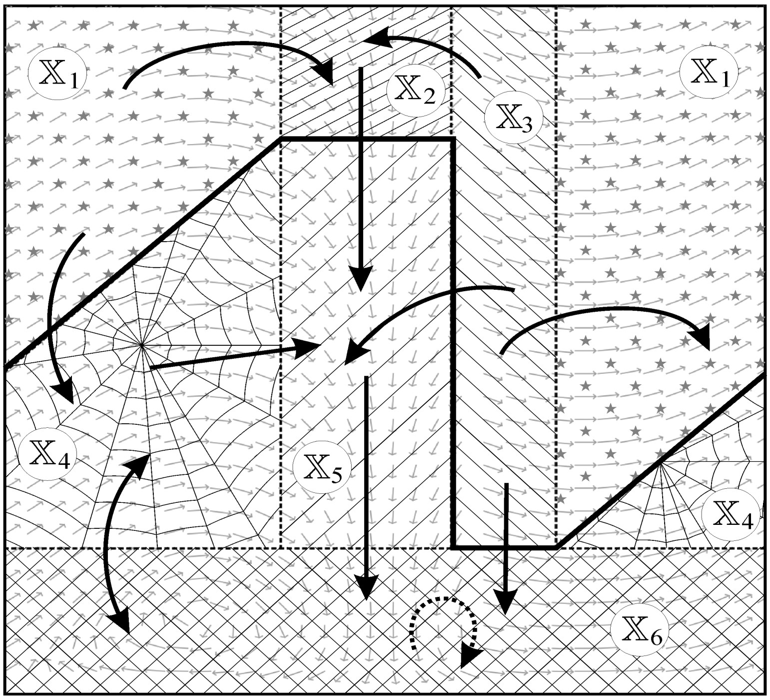
Closed loop system.

$$\begin{cases} \text{(i)} & \dot{\varphi} &= \begin{cases} \dfrac{\sin\varphi}{d} + 1 & \text{if } \cos\varphi \leq \dfrac{1}{\sqrt{2}} \\ \left(\dfrac{1}{d} - 1\right)\sin\varphi & \text{otherwise} \end{cases} \\ \text{(ii)} & \dot{d} &= -\cos\varphi. \end{cases}$$


(b)
(c)
(d)
(a)

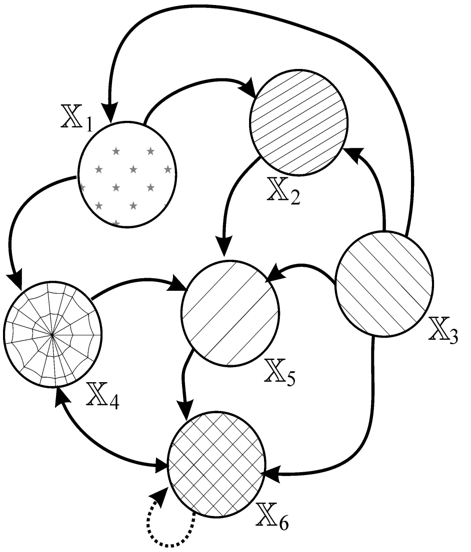


d
7.6
2
0
−π
−π/4
0
π/4
φ
𝕏1
𝕏2
𝕏3
𝕏1
𝕏5
𝕏4
𝕏4
𝕏6



$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_1$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_4$
$\mathbb{X}_6$

$$[\mathbf{G}] = \left[\underline{\mathbf{G}}, \overline{\mathbf{G}}\right] = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & [0,1] \end{pmatrix}$$



$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$

The transitive closure of the graph is

$$
\begin{aligned}
\left[\mathbf{G}^+\right] &= \left[\underline{\mathbf{G}}^+, \overline{\mathbf{G}}^+\right] = \left[\underline{\mathbf{G}} + \underline{\mathbf{G}}^2 + \underline{\mathbf{G}}^3 + \ldots,\ \overline{\mathbf{G}} + \overline{\mathbf{G}}^2 + \overline{\mathbf{G}}^3 + \ldots\right] \\
&= \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}
\end{aligned}
$$

The attractor of the graph is given by the ones in the diagonal. It corresponds to the set $\mathbb{X}_4 \cup \mathbb{X}_5 \cup \mathbb{X}_6$.

The attractor of the controlled robot satisfies

$$\mathbb{A} \subset (\mathbb{X}_4 \cup \mathbb{X}_5 \cup \mathbb{X}_6) \,.$$